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Two Remarks on the Largest Prime Factors of n and $n+1$

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Abstract

Let $P(n)$ be the largest prime factor of n. We give an alternative proof of the existence of infinitely many n such that $P(n) > P(n+1) > P(n+2)$. Further, we prove that the set $\{P(n+1)/P(n)\}_{n\in\mathbb{N}}$ has infinitely many limit points $\{0, x_n, 1, y_n\}_{n\in\mathbb{N}}$ with $0 < x_n < 1 < y_n$ and $\lim x_n = \lim y_n = 1$.

1 Introduction

Let $n \geq 2$ be a positive integer. Let $P(n)$ denote denote the largest prime factor of n. Erdős and Pomerance [4] proved that the number of $n \leq x$ such that $P(n) < P(n+1)$ is at least $0.0099x$, and the same holds for $P(n) > P(n+1)$. This lower density 0.0099 was subsequently improved by several authors (0.05544 by de la Bretèche, Pomerance and Tenenbaum [2], 0.1063 by Z. Wang [14]). The current record holders are Lü and Wang [7], who proved that the lower density is at least 0.2017.

Erdős and Pomerance [4] also noted that the three patterns $P(n) < P(n+1) > P(n+2)$, $P(n) > P(n+1) < P(n+2)$, and $P(n) < P(n+1) < P(n+2)$ occur infinitely often. They presented a simple proof for the infinitude of the third pattern. Namely, they took

$$n = p^{2^m} - 1, \ n+1 = p^{2^m}, \text{ and } n+2 = p^{2^m} + 1,$$

where p is prime and $m = \inf\{k | P(p^{2^k}+1) > p\}$. They left the infinitude of the fourth pattern $P(n) > P(n+1) > P(n+2)$ as an open problem.

This problem was later solved by Balog [1], who showed that the number of occurrence of this pattern for $n \le x$ is $\gg \sqrt{x}$. Building on earlier results by Matomäki, Radziwiłł, and Tao [10], and Teräväinen [12], Tao and Teräväinen [13] proved that the following sets have positive lower density:

$$\{n \in \mathbb{N} \mid P(n) < P(n+1) < P(n+2) > P(n+3)\} \text{ and}$$

$$\{n \in \mathbb{N} \mid P(n) > P(n+1) > P(n+2) < P(n+3)\}.$$

Using the Maynard-Tao theorem [11], in this paper we provide a simple alternative proof of the infinitude of the patterns $P(n) < P(n+1) < P(n+2)$ and $P(n) > P(n+1) > P(n+2)$. We prove that both patterns occur for $\gg x/(\log x)^{50}$ values of $n \le x$. The result is weaker than Tao and Teräväinen's, and stronger than Balog's.

Theorem 1. *For sufficiently large x, we have*

$$\#\{n \le x \mid P(n) < P(n+1) < P(n+2)\} \gg \frac{x}{(\log x)^{50}} \text{ and}$$

$$\#\{n \le x \mid P(n) > P(n+1) > P(n+2)\} \gg \frac{x}{(\log x)^{50}}.$$

Erdős and Pomerance [4, Theorem 1] proved that for any $\epsilon > 0$, there is $\delta > 0$ such that the number of $n \le x$ with

$$x^{-\delta} < \frac{P(n+1)}{P(n)} < x^{\delta}$$

is less than ϵx. They remarked that this means $P(n)$ and $P(n+1)$ are usually not close. In the opposite direction, we prove that this ratio can approach arbitrarily close to 1 from both sides.

Theorem 2. *For any $\epsilon > 0$, we have*

$$\#\left\{n \le x \,\middle|\, 1 \le \frac{P(n+1)}{P(n)} < 1+\epsilon\right\} \gg_\epsilon \frac{x}{(\log x)^{50}} \text{ and}$$

$$\#\left\{n \le x \,\middle|\, 1-\epsilon < \frac{P(n+1)}{P(n)} \le 1\right\} \gg_\epsilon \frac{x}{(\log x)^{50}}.$$

Let $R := \{\frac{P(n+1)}{P(n)} \mid n \in \mathbb{N}\}$. As a direct consequence of Theorem 2, we obtain that $1 \in \overline{R}$. From the proof of Theorem 2, we obtain a finite set $\{a_1, \ldots, a_{50}\} \subseteq \mathbb{N}$ with $1 \le a_i < a_j$ for each $1 \le i < j \le 50$ such that $a_{j_1}/a_{i_1} \in \overline{R} \cap (1,\infty)$ for some $1 \le i_1 < j_1 \le 50$, and $a_{i_2}/a_{j_2} \in \overline{R} \cap (0,1)$ for some $1 \le i_2 < j_2 \le 50$. Changing a_i with ϵ, we obtain a sequence rational numbers with $0 < x_n < 1 < y_n$ such that $\lim x_n = \lim y_n = 1$ and $\{x_n, y_n\}_{n\in\mathbb{N}} \subseteq \overline{R}$.

In fact, there is an elementary proof of $1 \in \overline{R}$. This elementary proof is based on a solution ($\epsilon = 1$ in the following argument) that appeared in Mathematics Stack Exchange [3] by Barry Cipra. For any $0 < \epsilon \le 1$, take primes p and q satisfying $p < q < (1+\epsilon)p$ so that

$$\frac{1}{1+\epsilon} < \frac{p}{q} < 1 < \frac{q}{p} < 1+\epsilon.$$

By the extended Euclidean algorithm, there exist integers u and v with $0 < u < q$, $0 < v < p$ and $pu - qv = 1$. Let $U = q - u$, $V = p - v$. Then $qV - pU = 1$. The integers qu and qV have q as largest prime factor. Since $u + U = q < (1+\epsilon)p$, at least one of $u \le p$ or $U \le p$ holds. If $u \le p$, then p is the largest prime factor of pu. If $U \le p$, then p is also the largest prime factor of pU. Thus, either one of the following holds:

$$n = qv,\ n+1 = pu,\ \frac{P(n+1)}{P(n)} = \frac{p}{q},$$

or

$$n = pU,\ n+1 = qV,\ \frac{P(n+1)}{P(n)} = \frac{q}{p}.$$

Therefore, $1 \in \overline{R}$ follows. From this argument the number of $n \le x$ with $\frac{1}{1+\epsilon} < \frac{P(n+1)}{P(n)} < 1+\epsilon$ is $\gg_\epsilon x/(\log x)^2$. Slightly modifying this argument, we have for any $x \in [1,2]$, either x or $1/x$ is in $\overline{R}$. However, this argument does not determine whether a limit point is in $[0,1)$ or $(1,\infty)$.

By Dirichlet's theorem on primes in arithmetic progressions, it is easy to see that 0 is also a limit point of R. For if we take a prime $n = ar - 1$, with a large, then $P(n) = ar - 1$ and $P(n+1) \le \max(a,r)$. Assuming the prime k-tuples conjecture (Conjecture 4), we prove that all nonnegative real numbers are limit points of R.

Theorem 3 (Conditional)**.** *Assuming the prime k-tuples conjecture (Conjecture 4), we have* $\overline{R} = [0,\infty)$.

2 Estimates on the numbers of prime k-tuples

A set of k-tuple of linear forms $\{a_1x + b_1, \dots a_kx + b_k\}$ is said to be *admissible* if for any prime p there is $x_p \in \mathbb{Z}$ such that $p \nmid \prod_{i=1}^{k}(a_ix_p + b_i)$. We consider the tuples with

$$\prod_i a_i \ne 0 \quad \text{and} \quad \prod_{i<j}(a_ib_j - a_jb_i) \ne 0.$$

The following is a special case of Bateman-Horn conjecture (a quantitative estimate on Dickson's prime k-tuples conjecture).

Conjecture 4 (Bateman-Horn)**.** Let $k \geq 2$ and $A_k = \{a_1x+b_1, \ldots, a_kx+b_k\}$ be an admissible set of linear forms. Then for sufficiently large x, the number $R_k(x)$ of $r \leq x$ such that a_ix+b_i, $1 \leq i \leq k$ are all prime satisfies

$$R_k(x) \gg_{A_k} \frac{x}{(\log x)^k}.$$

Substantial progress toward this conjecture begin with Zhang's result [15] on bounded gaps in primes. Subsequently, Maynard [8] and Polymath8b ([11] led by Tao) improved upon Zhang's result. We state a quantitative form of the Maynard-Tao theorem for admissible sets of linear forms. The proof requires slight modifications of [11] and the stated lower bound can be found in [11, Remark 32]. Note that the following is unconditional.

Lemma 5 (Maynard-Tao-Polymath8b)**.** *Let $A = \{a_1r + b_1, \ldots, a_{50}r + b_{50}\}$ be an admissible set of linear forms. Then for sufficiently large x, the number $R(A, x)$ of $r \leq (x - \max_i b_i)/\max_i a_i$ such that at least two of the linear forms are primes satisfies*

$$R(A, x) \gg_A \frac{x}{(\log x)^{50}}.$$

We will apply the above lemma in the following two special cases:

Case 1. $0 < a_1 < \cdots < a_k$ and $b_i = 1$ for all $i = 1, \ldots, k$.

Case 2. $0 < a_1 < \cdots < a_k$ and $b_i = -1$ for all $i = 1, \ldots, k$.

The set of linear forms in these cases is always admissible.

3 The main lemma

We construct a special sequence $\{a_i\}$ by the following inductive process.

Lemma 6 (The main lemma)**.** *Let $k \geq 2$ and $e_k = 1$. For each $0 \leq j \leq k-2$, assume that $\{e_{k-j}, \ldots, e_k\}$ satisfies*

$$\sum_{s<i\leq t} e_i \;\Bigg|\; \sum_{k-j\leq i\leq s} e_i \quad \textit{for any} \quad k-j \leq s < t \leq k.$$

Let e_{k-j-1} be a multiple of

$$\operatorname{lcm}\left\{ \sum_{s\leq i\leq t} e_i \;\middle|\; k-j \leq s < t \leq k \right\}.$$

Then $a_i = \sum_{m\leq i} e_m$ satisfies $0 < a_j - a_i \mid a_i$ for each $1 \leq i < j \leq k$.

Proof. The proof is clear from the inductive construction. □

We exhibit some sequences $\{a_i\}$ that can be produced by the main lemma.

Example 7. If $k = 2$, then let $\{e_1, e_2\} = \{1, 1\}$ and $\{a_1, a_2\} = \{1, 2\}$.

If $k = 3$, then let $\{e_1, e_2, e_3\} = \{2, 1, 1\}$ and $\{a_1, a_2, a_3\} = \{2, 3, 4\}$.

If $k = 4$, then let $\{e_1, e_2, e_3, e_4\} = \{12, 2, 1, 1\}$ and $\{a_1, a_2, a_3, a_4\} = \{12, 14, 15, 16\}$.

If $k = 5$, then let $\{e_1, e_2, e_3, e_4, e_5\} = \{1680, 12, 2, 1, 1\}$ and

$\{a_1, a_2, a_3, a_4, a_5\} = \{1680, 1692, 1694, 1695, 1696\}$.

Note that e_1 can be made arbitrarily large in the final inductive step. We will use the sequence $\{a_i\}_{1\le i\le 50}$.

Lemma 8. *Let $\{a_i\}_{1\le i\le 50}$ be a sequence produced in the main lemma. That is, $0 < a_j - a_i \mid a_i$ for each $1 \le i < j \le 50$. Suppose that $a_ir + 1$ and $a_jr + 1$ are primes. Then by taking*

$$n = \frac{a_i}{a_j - a_i}(a_jr + 1),\ n + 1 = \frac{a_j}{a_j - a_i}(a_ir + 1), \tag{1}$$

we have for sufficiently large r,

$$P(n) = a_jr + 1,\ P(n + 1) = a_ir + 1.$$

Now suppose that $a_ir - 1$ and $a_jr - 1$ are primes. Then by taking

$$n = \frac{a_j}{a_j - a_i}(a_ir - 1),\ n + 1 = \frac{a_i}{a_j - a_i}(a_jr - 1), \tag{2}$$

we have for sufficiently large r,

$$P(n) = a_ir - 1,\ P(n + 1) = a_jr - 1.$$

Proof. We take large enough r so that $a_1r - 1$ exceeds the largest prime factor of $\prod a_i$. □

Remark 9. The author recently learned that a sequence $\{a_i\}$ with the property $0 < a_j - a_i \mid a_i$ for each $1 \le i < j$ was obtained earlier by Heath-Brown [6, Lemma 1], and such a sequence is used in an unpublished work of Maynard and Ford [5, Theorem 7.18]. Using such a sequence and Lemma 8 (1), Maynard and Ford proved that there is a constant $B > 0$ so that $P(n) \ge n/B$ and $P(n + 1) \ge (n + 1)/B$ for infinitely many n.

4 Proof of theorems

4.1 Proof of Theorem 1

Proof. Let $\{a_i\}_{1\le i\le 50}$ be a sequence produced in the main lemma. We apply (1) of Lemma 8. By letting $n+2$ divisible by $\frac{a_j}{a_j-a_i}+1$, we obtain $P(n) > P(n+1) > P(n+2)$ for n in (1). For this idea to work, we need to require r to be divisible by $\frac{a_j}{a_j-a_i}+1$ for any choice of $1\le i<j\le 50$. To see this, we let

$$M = \operatorname{lcm}\left\{ \frac{a_j}{a_j-a_i}+1 \;\middle|\; 1\le i<j\le 50 \right\}.$$

Then we work with the admissible set of linear forms $\{a_iMr+1\}_{1\le i\le 50}$. By Lemma 5 and the pigeonhole principle, there is a pair (i,j), $1\le i<j\le 50$ depending on x such that a_iMr+1 and a_jMr+1 are primes for $\gg x/(\log x)^{50}$ values of $r\le (x-a_{50})/(a_{50}^2M)$. For such $r\ge r_0$, we have $n=\frac{a_i}{a_j-a_i}(a_jMr+1)\le x$, $P(n)=a_jMr+1$, $P(n+1)=a_iMr+1$, and $P(n+2)\le (n+2)/(\frac{a_j}{a_j-a_i}+1)$. Thus, $P(n)>P(n+1)>P(n+2)$ is satisfied for such r.

To obtain an analogous result on $P(n)<P(n+1)<P(n+2)$, we apply Part (2) of Lemma 8. By letting $n-1$ divisible by $\frac{a_j}{a_j-a_i}+1$, we obtain $P(n-1)<P(n)<P(n+1)$ for n in (2). Then we work with the admissible set of linear forms $\{a_iMr-1\}_{1\le i\le 50}$. The rest of the argument is similar to the previous case. □

4.2 Proof of Theorem 2

Proof. Let $\epsilon>0$ be arbitrary. We show that the number of $n\le x$ with $1-\epsilon<\frac{P(n+1)}{P(n)}\le 1$ is $\gg_\epsilon \frac{x}{(\log x)^{50}}$. In the inductive process in Lemma 6, we let e_1 be large enough to have

$$1-\epsilon < \frac{a_i}{a_j} \le 1 \quad \text{for each } 1\le i<j\le 50.$$

Then we apply Lemma 8 (1) to conclude the existence of a pair (i,j), $1\le i<j\le 50$ depending on x such that a_ir+1 and a_jr+1 are primes for $\gg_\epsilon x/(\log x)^{50}$ values of $r\le (x-a_{50})/a_{50}^2$. It is clear that $\frac{a_ir+1}{a_jr+1}=\frac{a_i}{a_j}+\frac{a_j-a_i}{a_j(a_jr+1)}$. Since $P(n)=a_jr+1$ and $P(n+1)=a_ir+1$ for such r by Lemma 8 (1), we have

$$1 \ge \frac{P(n+1)}{P(n)} = \frac{a_ir+1}{a_jr+1} > \frac{a_i}{a_j} > 1-\epsilon.$$

The result now follows. □

To obtain an analogous result for $1\le \frac{P(n+1)}{P(n)} < 1+\epsilon$, we apply Lemma 8 (2).

4.3 Proof of Theorem 3

Proof. Let a_1 be an even positive integer, and a_2 be a positive integer with $(a_1, a_2) = 1$. By Bezout's identity, we can find positive integers b_1 and b_2 such that $a_1b_2 - a_2b_1 = (a_1, a_2) = 1$. The sets of linear forms $\{a_1r + b_1, a_2r + b_2\}$ and $\{a_1r - b_1, a_2r - b_2\}$ are admissible. By Conjecture 4, there are infinitely many r such that both of these forms are primes. We take

$$n = a_2(a_1r + b_1),\ n + 1 = a_1(a_2r + b_2)$$

or

$$n = a_1(a_2r - b_2),\ n + 1 = a_2(a_1r - b_1).$$

If we select r to exceed any prime factor of a_1a_2, then we see in both cases

$$\left\{\frac{a_1}{a_2}, \frac{a_2}{a_1}\right\} \subseteq \overline{R}.$$

Hence, it follows that any positive rational numbers with numerator and denominator of different parity are limit points of R, and consequently, $\overline{R} = [0, \infty)$. □

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