Journal of Integer Sequences, Vol. 23 (2020),
Article 20.2.6

Two New Explicit Formulas for the Even-Indexed Bernoulli Numbers

Sumit Kumar Jha
International Institute of Information Technology
Hyderabad-500 032
India
kumarjha.sumit@research.iiit.ac.in

Abstract

We give two new explicit formulas for the even-indexed Bernoulli numbers in terms of the Stirling numbers of the second kind.

1 Introduction

Definition 1. The *Bernoulli numbers* B_n can be defined by the following generating function:

$$\frac{t}{e^t - 1} = \sum_{n \geq 0} \frac{B_n t^n}{n!},$$

where $|t| < 2\pi$.

Definition 2. The *Stirling number of the second kind*, denoted by $S(n, m)$, is the number of ways of partitioning a set of n elements into m nonempty sets.

There are many explicit formulas known for the Bernoulli numbers [1, 9, 4, 5, 6]. For example, all of the formulas below express the Bernoulli numbers explicitly in terms of the

Stirling numbers of the second kind:

$$B_r = \sum_{k=1}^{r} (-1)^k \cdot k! \cdot \frac{S(r,k)}{k+1},$$

$$B_r = \sum_{k=1}^{r} (-1)^{k-1} \cdot (k-1)! \cdot \frac{S(r,k)}{k+1},$$

$$B_{r+1} = \sum_{k=1}^{r} \frac{(-1)^{k-1}\, k!\, S(r,k)}{(k+1)(k+2)},$$

$$B_r = \frac{r}{1-2^r} \sum_{k=1}^{r-1} \frac{(-1)^k\, k!\, S(r-1,k)}{2^{k+1}},$$

$$B_{r+1} = \frac{(-1)^r \cdot (r+1) \cdot 2^{r-1}}{2^{r+1}-1} \sum_{k=1}^{r} \frac{(-1)^k\, k!\, S(r,k)}{k+1} \cdot 2^{-2k} \binom{2k}{k},$$

$$B_r = \sum_{i=0}^{r} (-1)^i \frac{\binom{r+1}{i+1}}{\binom{r+i}{i}} S(r+i,i),$$

$$B_{r+1} = -\frac{r+1}{4(1+2^{-(r+1)}(1-2^{-r}))} \left(\sum_{k=1}^{r} (-1)^k \cdot \frac{S(r,k)}{k+1} \cdot \left(\frac{3}{4}\right)^{(k)} + 4^{-r} E_r \right), \tag{1}$$

where (E_r) denotes the *Euler numbers* defined by the following generating function:

$$\frac{1}{\cosh t} = \frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} \frac{E_n}{n!} \cdot t^n.$$

In the following section, we derive two new explicit formulas for the even-indexed Bernoulli numbers in terms of the Stirling numbers of the second kind.

2 Main results

Our main results are the following.

Theorem 3. *For all positive integers r we have*

$$B_{2r} = \frac{-4r}{3(3-3^{1-2r})} \sum_{k=1}^{2r-1} (-1)^k \frac{S(2r-1,k)}{k+1} \left(\frac{2}{3}\right)^{(k)}, \tag{2}$$

and

$$B_{2r} = \frac{2r}{3((1-2^{1-2r})(1-3^{1-2r})-2)} \sum_{k=1}^{2r-1} (-1)^k \frac{S(2r-1,k)}{k+1} \left(\frac{5}{6}\right)^{(k)}, \tag{3}$$

where $S(2r-1,k)$ denotes the Stirling numbers of the second kind, and

$$x^{(n)} = x(x+1)(x+2)\cdots(x+n-1)$$

denotes the rising factorial.

To prove the above result, we first recall the following fact

Theorem 4. *[10] The nth Bernoulli polynomial, $B_n(t)$, defined by*

$$B_n(t) = \sum_{j=0}^{n} \binom{n}{j} B_{n-j} t^j$$

takes the following 'special' values at certain rational numbers with small denominators

$$B_n(1) = B_n(0) = B_n \qquad \text{for } n \geq 1,$$

$$B_n\left(\frac{1}{2}\right) = (2^{1-n}-1)B_n,$$

$$B_{2n}\left(\frac{1}{4}\right) = B_{2n}\left(\frac{3}{4}\right) = \frac{2-2^{2n}}{4^{2n}}B_{2n}, \tag{4}$$

$$B_{2n}\left(\frac{1}{3}\right) = B_{2n}\left(\frac{2}{3}\right) = \frac{3-3^{2n}}{2\cdot 3^{2n}}B_{2n}, \tag{5}$$

$$B_{2n}\left(\frac{1}{6}\right) = B_{2n}\left(\frac{5}{6}\right) = \frac{(2-2^{2n})(3-3^{2n})}{2\cdot 6^{2n}}B_{2n}. \tag{6}$$

Remark 5. Granville and Sun [2] noted that, "It is not known if $B_n(a/q)$ has as simple a 'closed form' for any other rational a/q with $1 \leq a \leq q-1$ and $(a,q)=1$, though this has long been considered an interesting question."

Our proof also requires the following result.

Lemma 6. *For all $0 < t < 1$ and $l \in \mathbb{N}$ we have*

$$\int_0^\infty x^{t-1}\frac{\mathrm{Li}_{-l}(-x)}{1+x}\,dx = \frac{\pi}{\sin t\pi}\left(\frac{B_{l+1}(1-t)-B_{l+1}}{l+1}\right), \tag{7}$$

where $\mathrm{Li}_{-l}(-x)$ is the negative polylogarithm function, and $B_r(1-t)$ denotes the Bernoulli polynomial.

Proof. Consider the following generating function [8]

$$\frac{\mathrm{Li}_{-r}(-x)}{1+x} = \sum_{n=0}^{\infty} S_r(n)(-x)^n, \tag{8}$$

where $S_r(n) = \sum_{k=1}^n k^r$ for $n \geq 1$, and $S_r(0) = 0$.

We use Ramanujan's master theorem (RMT) from [3] that states that

$$\int_0^\infty x^{t-1}\{\phi(0) - x\phi(1) + x^2\phi(2) - \cdots\}\, dx = \frac{\pi}{\sin t\pi}\phi(-t),$$

where the integral is convergent for $0 < \Re(t) < 1$, and after certain conditions are satisfied by ϕ.

Using RMT with Eq. (8) gives us Eq. (7). □

Proof of Theorem 3. Substituting $t = 2/3$ and $l = 2r - 1$ in Eq. (7) gives us

$$\int_0^\infty x^{-1/3}\frac{\mathrm{Li}_{1-2r}(-x)}{1+x}\, dx = \frac{2\pi}{\sqrt{3}}\left(\frac{B_{2r}(1/3) - B_{2r}}{2r}\right).$$

We use the following representation from the note [7]

$$\mathrm{Li}_{1-2r}(-x) = \sum_{k=1}^{2r-1} k!S(2r-1,k)\left(\frac{1}{1+x}\right)^{k+1}(-x)^k \tag{9}$$

to conclude that

$$\begin{aligned}\int_0^\infty x^{-1/3}\frac{\mathrm{Li}_{1-2r}(-x)}{1+x}\, dx &= \sum_{k=1}^{2r-1}(-1)^k\cdot k!\, S(2r-1,k)\int_0^\infty \frac{x^{k-1/3}}{(1+x)^{k+2}}\, dx\\ &= \sum_{k=1}^{2r-1}(-1)^k\cdot k!\, S(2r-1,k)\frac{\Gamma(k+2/3)\Gamma(4/3)}{\Gamma(k+2)}\\ &= \sum_{k=1}^{2r-1}(-1)^k\cdot\frac{S(2r-1,k)}{k+1}\left(\frac{2}{3}\right)^{(k)}\cdot\frac{2\pi}{3\sqrt{3}},\end{aligned}$$

where $\Gamma(\cdot)$ denotes the Gamma function. Recalling Eq. (5) we have

$$B_{2r}(1/3) - B_{2r} = -\left(3 - 3^{1-2r}\right)\frac{B_{2r}}{2}.$$

Now we can readily conclude Eq. (2).

To prove Eq. (3) we substitute $t = 5/6$ and $l = 2r - 1$ in the Eq. (7) to get

$$\int_0^\infty x^{-1/6}\frac{\mathrm{Li}_{1-2r}(-x)}{1+x}\, dx = 2\pi\left(\frac{B_{2r}(1/6) - B_{2r}}{2r}\right).$$

The representation (9) also lets us to conclude that

$$\begin{aligned}\int_0^\infty x^{-1/6}\frac{\mathrm{Li}_{1-2r}(-x)}{1+x}\,dx &= \sum_{k=1}^{2r-1}(-1)^k\cdot k!\,S(2r-1,k)\int_0^\infty \frac{x^{k-1/6}}{(1+x)^{k+2}}\,dx\\ &= \sum_{k=1}^{2r-1}(-1)^k\cdot k!\,S(2r-1,k)\frac{\Gamma(k+5/6)\Gamma(7/6)}{\Gamma(k+2)}\\ &= \sum_{k=1}^{2r-1}(-1)^k\cdot\frac{S(2r-1,k)}{k+1}\left(\frac{5}{6}\right)^{(k)}\cdot\frac{\pi}{3}.\end{aligned}$$

Recalling Eq. (6) we have

$$B_{2r}(1/6)-B_{2r}=\frac{B_{2r}}{2}((1-2^{1-2r})(1-3^{1-2r})-2).$$

Now we can readily conclude Eq. (3). □

Remark 7. Substituting $t=3/4$ and $l=2r-1$ in Eq. (7), and using Eq. (4) we can obtain

$$B_{2r}=-\frac{2r}{4(1+2^{-2r}(1-2^{-(2r-1)}))}\sum_{k=1}^{2r-1}(-1)^k\cdot\frac{S(2r-1,k)}{k+1}\cdot\left(\frac{3}{4}\right)^{(k)}.$$

The above is just a special case of Eq. (1) which was obtained in [6].

References

[1] H. W. Gould, Explicit formulas for Bernoulli numbers, *Amer. Math. Monthly* **79** (1972), 44–51.

[2] A. Granville and Z. W. Sun, Values of Bernoulli polynomials, *Pacific J. Math.* **172** (1996), 117–137.

[3] G. H. Hardy, *Ramanujan. Twelve Lectures on Subjects Suggested by His Life and Work*, 3rd edition, Chelsea, 1978.

[4] S. K. Jha, Two new explicit formulas for the Bernoulli numbers, preprint, 2019. Available at https://arxiv.org/abs/1905.11216.

[5] S. K. Jha, A new explicit formula for the Bernoulli numbers in terms of the Stirling numbers of the second kind, preprint, 2019. Available at https://osf.io/95gkw.

[6] S. K. Jha, A new explicit formula for Bernoulli numbers involving the Euler number, *Mosc. J. Comb. Number Theory* **8** (2019), 385–387.

[7] S. E. Landsburg, Stirling numbers and polylogarithms, preprint, 2009. Available at http://www.landsburg.com/query.pdf.

[8] L. Lewin, *Polylogarithms and Associated Functions*, North-Holland, 1981.

[9] J. Quaintance and H. W. Gould, *Combinatorial Identities For Stirling Numbers: The Unpublished Notes Of H. W. Gould*, World Scientific, 2015.

[10] Z. H. Sun, Congruences involving Bernoulli and Euler numbers, *J. Number Theory* **128** (2008), 280–312.

2010 *Mathematics Subject Classification*: Primary 11B68.
Keywords: Bernoulli number, Stirling number of the second kind, polylogarithm function, Ramanujan's master theorem.

(Concerned with sequences A000367, A002445, A008277, A027641, and A027642.)

Received November 6 2019; revised versions received November 7 2019; January 19 2020; January 20 2020; January 23 2020; January 27 2020; February 10 2020; February 20 2020. Published in *Journal of Integer Sequences*, February 20 2020.

Return to Journal of Integer Sequences home page.