Journal of Integer Sequences, Vol. 23 (2020),
Article 20.7.4

Generalized Rascal Triangles

Philip K. Hotchkiss
Mathematics Department
Westfield State University
Westfield, MA 01086
USA
photchkiss@westfield.ma.edu

Abstract

The "rascal triangle" was introduced by three middle school students in 2010. In this paper we describe number triangles that are generalizations of the rascal triangle, and show that these generalized rascal triangles are characterized by arithmetic sequences on all diagonals, as well as rascal-like multiplication and addition rules.

1 Introduction

In 2010, three middle school students—A. Anggaro, E. Liu, and A. Tulloch [1]—were asked to determine the next row of numbers in the following triangular array:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
? & & ? & & ? & & ? & & ?
\end{array}
$$

Figure 1: A triangular array.

Instead of providing the expected answer

$$1 \quad 4 \quad 6 \quad 4 \quad 1$$

from Pascal's triangle (sequence A007318 in the *On-Line Encyclopedia of Integer Sequences* (OEIS) [4]), they produced the row

$$1 \qquad 4 \qquad 5 \qquad 4 \qquad 1.$$

They did this by using the rule that the outside numbers are 1s and the inside numbers are determined by the **diamond formula**

$$\textbf{South} = \frac{\textbf{East} \cdot \textbf{West} + 1}{\textbf{North}}$$

where **North**, **South**, **East**, and **West** form a diamond in the triangular array as in Fig. 2.

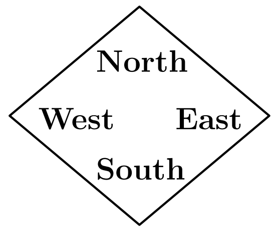


Figure 2: **North**, **South**, **East**, and **West** entries in a triangular array.

Continuing with this rule Anggaro, Liu, and Tulloch created a number triangle they called the *rascal triangle* (sequence A077028 in the OEIS).

1
1 1
1 2 1
1 3 3 1
1 4 5 4 1
1 5 7 7 5 1
1 6 9 10 9 6 1
1 7 11 13 13 11 7 1
1 8 13 16 17 16 13 8 1

Figure 3: The first nine rows of the rascal triangle.

Because the diamond formula involves division, their instructor challenged Anggaro, Liu, and Tulloch to prove that their formula would always result in an integer. They did this by observing that the diagonals running from right to left in the rascal triangle formed arithmetic sequences whose constant differences increased by one as they moved from one diagonal to the next, as illustrated in Table 1 below.

First diagonal:	$1, 1, 1, 1, \ldots$
Second diagonal:	$1, 2, 3, 4, \ldots$
Third diagonal:	$1, 3, 5, 7, \ldots$
Fourth diagonal:	$1, 4, 7, 10, \ldots$

Table 1: The first four diagonals of the rascal triangle: sequences A000012, A000027, A005408, and A016777 in the OEIS.

In particular Anggaro, Liu, and Tulloch recognized that the k^{th} entry on the r^{th} diagonal running from right to left is given by $1+rk$, where $r=0$ corresponds to the outside diagonal consisting of all 1s, and $k=0$ corresponds to the first entry on each diagonal. Thus in any diamond as in Fig. 2, if **South** $=1+rk$, then **East** $=1+r(k-1)$, **West** $=1+(r-1)k$, and **North** $=1+(r-1)(k-1)$.

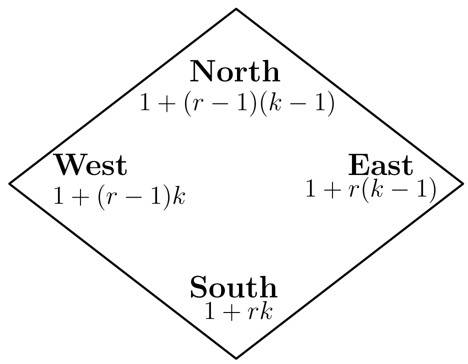


Figure 4: Algebraic Representation of **North**, **East**, **West**, and **South**.

A straightforward calculation then verifies that

$$\textbf{South} = \frac{\textbf{East}\cdot\textbf{West}+1}{\textbf{North}}.$$

In the spring 2015 semester a mathematics for liberal arts class taught by my colleague, Julian Fleron, discovered that the rascal triangle can also be generated by the rule that the outside numbers are 1s and the inside numbers are determined by the formula

$$\textbf{South} = \textbf{East}+\textbf{West}-\textbf{North}+1.$$

This formula also follows from the arithmetic sequences along the diagonals [2]. Thus, the rascal triangle has the property that for any diamond as in Fig. 2, the **South** entry satisfies two equations:

$$\textbf{South} = \frac{\textbf{East}\cdot\textbf{West}+1}{\textbf{North}}, \tag{1}$$

$$\textbf{South} = \textbf{East}+\textbf{West}-\textbf{North}+1. \tag{2}$$

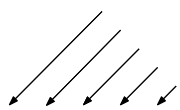
The fact that both Eqs. (1) and (2) can be used to generate the rascal triangle was intriguing to me, and I assumed that the rascal triangle was uniquely defined by either one of the two equations; and so I began trying to prove that Eq. (2) implied Eq. (1) or vice versa. In addition, I followed Fleron's lead and had some of my mathematics for liberal arts classes look for patterns in the rascal triangle, and to my delight they also made some original discoveries [3]. During the following summer, while exploring the patterns found by my students, I realized that there were other number triangles for which one equation held for the interior entries but the other did not, as will be shown below. This led me to the realization that there was a larger class of number triangles, of which the rascal triangle was but one example.

This paper is organized as follows. In Section 2 we look at some illustrative examples and define generalized rascal triangles. In Section 3 we show that generalized rascal triangles are characterized by the existence of modifications of both Eqs. (1) and (2). In Section 4 we conclude with proofs of some properties of generalized rascal triangles, including several discovered by my students.

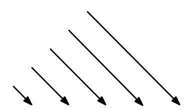
2 Motivating examples and generalized rascal triangles

Throughout this paper we will use the following conventions.

Definition 1. For any number triangle, the diagonals running from right to left will be called the *major* diagonals while the diagonals running from left to right will be called the *minor* diagonals.



(a) major diagonals.



(b) Minor diagonals.

We begin by looking at several examples of number triangles that motivated my exploration, and eventual definition, of generalized rascal triangles.

Example 2. One of the first number triangles I created was the number triangle T in Fig. 6 (sequence A309555 in the OEIS) where the outside numbers are 3s and I used Eq. (2) to find the interior numbers.

3
3 3
3 4 3
3 5 5 3
3 6 7 6 3
3 7 9 9 7 3

Figure 6: The first six rows of T.

This triangle is a "translation" of the rascal triangle, in that every entry in T is 2 more than the corresponding entry in the rascal triangle. At this point I had begun to suspect that there were number triangles other than the rascal triangle that satisfied both Eqs. (1) and (2) so I expected that the interior numbers of T would also satisfy Eq. (1), and was surprised to discover that they did not. However, I soon realized that they did satisfy a modified version of Eq. (1):

$$\textbf{South} = \frac{\textbf{West} \cdot \textbf{East} + 3}{\textbf{North}}.$$

Example 3. Soon thereafter I created the number triangle S in Fig. 7 (sequence A309557 in the OEIS) where the outside major diagonal is the sequence $2, 5, 8, 11, \ldots$ (sequence A016789 in the OEIS), the outside minor diagonal is the sequence $2, 3, 4, 5, \ldots$ (sequence A000027 in the OEIS), and this time I used Eq. (1) to find the interior numbers.

$$\begin{array}{ccccccccccc} &&&&& 2 &&&&& \\ &&&& 5 && 3 &&&& \\ &&& 8 && 8 && 4 &&& \\ && 11 && 13 && 11 && 5 && \\ & 14 && 18 && 18 && 14 && 6 & \\ 17 && 23 && 25 && 23 && 17 && 7 \end{array}$$

Figure 7: The first six rows of S.

I now correctly suspected the interior numbers would not satisfy Eq. (2), but quickly observed that they did satisfy the modified version:

$$\textbf{South} = \textbf{West} + \textbf{East} + 2 - \textbf{North}.$$

Further explorations with number triangles initially suggested to me that if the interior numbers of a number triangle satisfied a modified version of one of Eq. (1) or (2) then they also satisfied a modification of the other, as in Example 4.

Example 4. Let W be the number triangle in Fig. 8 (sequence A332790 in the OEIS).

$$\begin{array}{ccccccccccc} &&&&& 1 &&&&& \\ &&&& 3 && 4 &&&& \\ &&& 5 && 11 && 7 &&& \\ && 7 && 18 && 19 && 10 && \\ & 9 && 25 && 31 && 27 && 13 & \\ 11 && 32 && 43 && 44 && 35 && 16 \end{array}$$

Figure 8: The first six rows of W.

The outside major diagonal is the sequence $1, 3, 5, 7, \ldots$ (sequence A005408 in the OEIS), the outside minor diagonal is the sequence $1, 4, 7, 10, \ldots$ (sequence A016777 in the OEIS), and interior numbers satisfy neither Eqs. (1) nor (2). However, there are modifications of both equations that work for all interior numbers,

$$\textbf{South} = \textbf{West} + \textbf{East} + 5 - \textbf{North}$$

and

$$\textbf{South} = \frac{\textbf{West} \cdot \textbf{East} - 1}{\textbf{North}}.$$

I subsequently discovered that it is possible for the interior numbers in a number triangle satisfy one of Eq. (1) or (2) but no modification of the other as illustrated in Examples 5 and 6 below.

Example 5. Let U be the number triangle in Fig. 9 (sequence A309559 in the OEIS) where the outside major diagonal is the sequence $1, 2, 3, 4, \ldots$) (sequence A000027 in the OEIS), the outside minor diagonal is the sequence $1, 2, 4, 7, \ldots$ (sequence A000124 in the OEIS), and the interior numbers were generated using Eq. (2).

1
2 2
3 4 4
4 6 7 7
5 8 10 11 11
6 10 13 15 16 16

Figure 9: The first six rows of U.

In this case there is no modification of Eq. (1) that works for all interior numbers. To see this, consider the two diamonds in Fig. 10.

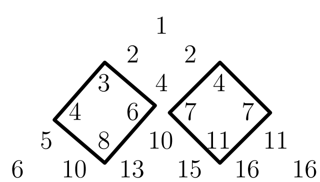


Figure 10: Different diamonds in U.

For the diamond on the left, the modification of Eq. (1) would need to be

$$\textbf{South} = \frac{\textbf{West} \cdot \textbf{East} + 0}{\textbf{North}}$$

while for the diamond on the right, the modification of Eq. (1) would need to be

$$\textbf{South} = \frac{\textbf{West} \cdot \textbf{East} - 5}{\textbf{North}}.$$

Example 6. Let V be the number triangle in Fig. 11 (sequence A332963 in the OEIS) in which the outside diagonals are the alternating sequence $1, 2, 1, 2, \ldots$ (sequence A000034 in the OEIS) and the interior numbers were generated by Eq. (1).

1
2 2
1 5 1
2 3 3 2
1 7 2 7 1
2 4 5 5 4 2

Figure 11: The first six rows of V.

Here there is no modification of Eq. (2) that works for all interior numbers. To see this, consider the two diamonds in Fig. 12.

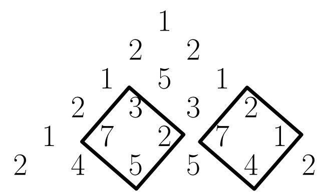


Figure 12: Different diamonds in V.

For the diamond on the left, the modification of Eq. (2) would need to be

$$\textbf{South} = \textbf{West} + \textbf{East} - 1 - \textbf{North}$$

while for the diamond on the right, the modification of Eq. (2) would need to be

$$\textbf{South} = \textbf{West} + \textbf{East} - 2 - \textbf{North}.$$

Note that for T, S, and W all the major and minor diagonals are arithmetic sequences and the interior numbers satisfy equations similar to Eqs. (1) and (2). Whereas in U, some of the diagonals are not arithmetic sequences and although the interior entries in U satisfy Eq. (2), there is no modification of Eq. (1) that will work for all of the interior entries. While in V, none of the diagonals are arithmetic sequences and although the interior entries in V satisfy Eq. (1) there is no modification of Eq. (2) that will work for all of the interior entries. This suggests that for number triangles with arithmetic sequences on both the major and minor diagonals, the interior numbers satisfy two equations of the form Eqs. (1) and (2). This motivates Definitions 8 and 9 below.

Notation 7. For a number triangle T we will use $T_{r,k}$ to denote the k^{th} entry on the r^{th} major diagonal with $r = 0$ corresponding to the outside major diagonal and $k = 0$ corresponding to the first entry on each major diagonal. With this notation, $T_{0,0}$ corresponds to the top number of T. Note that on the minor diagonals, $T_{r,k}$ denotes the r^{th} entry on the k^{th} minor diagonal with $k = 0$ corresponding to the outside minor diagonal on the right and $r = 0$ corresponding to the first entry on each minor diagonal.

Definition 8. Let $c, d, d_1, d_2 \in \mathbb{Z}$. A number triangle T is called a *generalized rascal triangle* if

$$T_{r,k} = c + kd_1 + rd_2 + rkd \tag{3}$$

for all $r, k \geq 0$. We will write $T(c, d, d_1, d_2)$ for the generalized rascal triangle determined by the constants c, d, d_1, d_2.

In a generalized rascal triangle $T(c, d, d_1, d_2)$ we have that $c = T_{0,0}$, the top entry, $d_1 = T_{0,k+1} - T_{0,k}$, the arithmetic difference along the outside major diagonal, $d_2 = T_{k+1,0} - T_{k,0}$, the arithmetic difference along the outside minor diagonal, and

$$d = (T_{r+1,k+1} - T_{r+1,k}) - (T_{r,k+1} - T_{r,k}) = (T_{r+1,k+1} - T_{r,k+1}) - (T_{r+1,k} - T_{r,k}),$$

the change in the arithmetic differences as we move from one major diagonal to the next or move from one minor diagonal to the next. In particular, the r^{th} major diagonal is the arithmetic sequence

$$M_r(k) = (c + rd_2) + k(d_1 + rd) \tag{4}$$

and the k^{th} minor diagonal is the arithmetic sequence

$$m_k(r) = (c + kd_1) + r(d_2 + kd). \tag{5}$$

We now generalize Eqs. (1) and (2).

Definition 9. Let $d, D \in \mathbb{Z}$ and let T be a number triangle. If the interior numbers satisfy the equation

$$T_{r,k} = \frac{T_{r-1,k} \cdot T_{r,k-1} + D}{T_{r-1,k-1}} \tag{6}$$

we call this a *rascal-like multiplication rule* with multiplicative constant D; and if the interior numbers satisfy

$$T_{r,k} = T_{r-1,k} + T_{r,k-1} + d - T_{r-1,k-1} \tag{7}$$

that will be called a *rascal-like addition rule* with additive constant d.

Example 10. The generalized rascal triangle $T(1, 1, 0, 0)$, which is defined by the equation

$$T_{r,k} = 1 + 0k + 0r + rk = 1 + rk,$$

has a constant sequence of 1s on the outside diagonals, and the arithmetic differences increase by 1 as we move from one major (resp., minor) diagonal to the next. This is, of course, the rascal triangle (sequence A077028 in the OEIS).

$$\begin{array}{ccccccccccccccccc}
&&&&&&&&1&&&&&&&&\\
&&&&&&&1&&1&&&&&&&\\
&&&&&&1&&2&&1&&&&&&\\
&&&&&1&&3&&3&&1&&&&&\\
&&&&1&&4&&5&&4&&1&&&&\\
&&&1&&5&&7&&7&&5&&1&&&\\
&&1&&6&&9&&10&&9&&6&&1&&\\
&1&&7&&11&&13&&13&&11&&7&&1&\\
1&&8&&13&&16&&17&&16&&13&&8&&1
\end{array}$$

Figure 13: The first nine rows of the rascal triangle.

We will denote this triangle by R, and its equation by $R_{r,k} = 1 + rk$. As was mentioned earlier, the interior numbers satisfy both the rascal-addition rule

$$R_{r,k} = R_{r-1,k} + R_{r,k-1} + 1 - R_{r-1,k-1},$$

and the rascal-multiplication rule

$$R_{r,k} = \frac{R_{r-1,k} \cdot R_{r,k-1} + 1}{R_{r-1,k-1}}.$$

Example 11. The generalized rascal triangle $T(2,2,3,1)$, is the number triangle determined by the equation

$$T_{r,k} = 2 + 3k + r + 2rk.$$

This number triangle has a top entry of $c = 2$, the outside major diagonal has an arithmetic difference of $d_1 = 3$ which gives the arithmetic sequence $2, 5, 8, 11, \ldots$ (sequence A016789 in the OEIS), and the outside minor diagonal has an arithmetic difference of $d_2 = 1$, which gives the arithmetic sequence $2, 3, 4, 5, \ldots$ (sequence A000027 in the OEIS). Moreover, the arithmetic differences change by $d = 2$ as we move from one major (resp., minor) diagonal to the next. This results in the number triangle S from Example 3 (sequence A309557 in the OEIS).

$$\begin{array}{ccccccccccc} & & & & & 2 & & & & & \\ & & & & 5 & & 3 & & & & \\ & & & 8 & & 8 & & 4 & & & \\ & & 11 & & 13 & & 11 & & 5 & & \\ & 14 & & 18 & & 18 & & 14 & & 6 & \\ 17 & & 23 & & 25 & & 23 & & 17 & & 7 \end{array}$$

Figure 14: The first six rows of $S = T(2,2,3,1)$.

The rascal-like addition rule for S is

$$T_{r,k} = T_{r-1,k} + T_{r,k-1} + 2 - T_{r-1,k-1}$$

and the rascal-like multiplication rule for S is

$$T_{r,k} = \frac{T_{r-1,k} \cdot T_{r,k-1} + 1}{T_{r-1,k-1}}.$$

3 Addition and multiplication rules for generalized rascal triangles

It is easy to show that every generalized rascal triangle has a rascal-like addition rule (7), and a rascal-like multiplication rule (6) (see Proposition 14 below). My initial (naive) assumption was that if a number triangle T had a rascal-like addition or rascal-like multiplication rule, it was a generalized rascal triangle. However, as illustrated in Examples 5 and 6 above, the existence of just a rascal-like addition rule or just a rascal-like multiplication rule for the interior numbers is not sufficient for a number triangle to be a generalized rascal triangle. Nevertheless, as we will show below, if the outside major and minor diagonals are arithmetic sequences, then the existence of either a rascal-like addition rule or a rascal-like multiplication rule for the interior numbers is both necessary and sufficient for a number triangle to be a generalized rascal triangle.

We begin with two lemmas whose proofs are left to the reader.

Lemma 12. *Let T be a number triangle. If $T_{r,k}$ is on the n^{th} row, then $r+k=n$. (See Fig. 15.)*

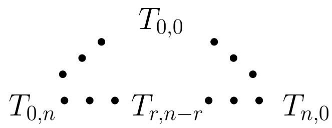


Figure 15: Row n.

Lemma 13. *Let $c, d_1, d_2, r, k, d \in \mathbb{Z}$. Then*

$$\Big(c+(k-1)d_1+rd_2+r(k-1)d\Big)\Big(c+kd_1+(r-1)d_2+(r-1)kd\Big)+cd-d_1d_2$$
$$=\Big(c+kd_1+rd_2+rkd\Big)\Big(c+(k-1)d_1+(r-1)d_2+(r-1)(k-1)d\Big).$$

We first show that generalized rascal triangles satisfy rascal-like addition and rascal-like multiplication rules.

Proposition 14. *Let $c, d, d_1, d_2 \in \mathbb{Z}$ and $T(c, d, d_1, d_2)$ be the associated generalized rascal triangle. Then for $r, k \geq 1$*

$$T_{r,k} = T_{r,k-1} + T_{r-1,k} + d - T_{r-1,k-1}$$

and whenever $T_{r-1,k-1} \neq 0$

$$T_{r,k} = \frac{T_{r,k-1} \cdot T_{r-1,k} + D}{T_{r-1,k-1}}$$

where $D = cd - d_1d_2$.

Proof. Since T is a generalized rascal triangle, $T_{r,k} = c+kd_1+rd_2+rkd$. Thus, when $r, k \geq 1$

$$\begin{aligned}
T_{r,k-1} + T_{r-1,k} + d - T_{r-1,k-1} &= (c+(k-1)d_1+rd_2+r(k-1)d) \\
&\quad + (c+kd_1+(r-1)d_2+(r-1)kd)+d \\
&\quad - (c+(k-1)d_1+(r-1)d_2+(r-1)(k-1)d) \\
&= c+kd_1+rd_2+r(k-1)d+(r-1)kd+d-(r-1)(k-1)d \\
&= c+kd_1+rd_2+rkd-rd+rkd-kd+d-rkd+rd+kd-d \\
&= c+kd_1+rd_2+rkd \\
&= T_{r,k}.
\end{aligned}$$

To show that T has a rascal-like multiplication rule, let $D = cd - d_1d_2$ and suppose $r, k \geq 1$. Then by Lemma 13

$$\begin{aligned}T_{r-1,k} \cdot T_{r,k-1} + D &= \Big(c + (k-1)d_1 + rd_2 + r(k-1)d\Big)\Big(c + kd_1 + (r-1)d_2 + (r-1)kd\Big) \\ &\quad + cd - d_1d_2 \\ &= \Big(c + kd_1 + rd_2 + rkd\Big)\Big(c + (k-1)d_1 + (r-1)d_2 + (r-1)(k-1)d\Big) \\ &= T_{r,k}T_{r-1,k-1}.\end{aligned}$$

Thus, when $T_{r-1,k-1} \neq 0$

$$T_{r,k} = \frac{T_{r-1,k} \cdot T_{r,k-1} + D}{T_{r-1,k-1}}.$$

□

Note that the additive constant d for the rascal-like addition rule is the same as the d in the definition of the generalized rascal triangle.

We now prove that a number triangle that has arithmetic sequences on the outside diagonals and satisfies either a rascal-like addition rule or a rascal-like multiplication rule for the interior numbers must be a generalized rascal triangle.

Proposition 15. *Let $d_1, d_2, d \in \mathbb{Z}$ and T be a number triangle with $T_{r,0} = T_{0,0} + rd_2$, $T_{0,k} = T_{0,0} + kd_1$, and $T_{r,k} = T_{r,k-1} + T_{r-1,k} + d - T_{r-1,k-1}$. Then there exists a constant $c \in \mathbb{Z}$ such that $T = T(c, d, d_1, d_2)$.*

Proof. Let $c = T_{0,0}$. To show that $T = T(c, d, d_1, d_2)$, we first note that

$$T_{r,0} = c + rd_2 = c + 0d_1 + rd_2 + r \cdot 0d$$

and

$$T_{0,k} = c + kd_1 = c + kd_1 + 0d_2 + 0 \cdot kd,$$

so on the exterior diagonals, $T_{r,k} = c + kd_1 + rd_2 + rkd$. For the interior numbers $T_{r,k}$ with $r, k \geq 1$, we prove that $T_{r,k} = ckd_1 + rd_2 + rkd$ by induction on the row number n for $n \geq 2$. Note that we start with $n = 2$ since the rows $n = 0$ and $n = 1$ have no interior numbers.

For the case $n = 2$ note that by Lemma 12 we have that $n = r + k$ for each entry $T_{r,k}$ on the n^{th} row; so suppose $r, k \in \mathbb{N}$ with $r + k = 2$. Since $r, k \geq 1$, $r + k = 2$ means $r = 1$ and $k = 1$ and so $T_{r,k-1} = T_{1,0} = c + d_2$, $T_{r-1,k} = T_{0,1} = c + d_1$, and $T_{r-1,k-1} = T_{0,0} = c$. Since

$$T_{1,1} = T_{0,1} + T_{1,0} + d - T_{0,0}$$

we have

$$T_{1,1} = c + d_1 + c + d_2 + d - c = c + d_1 + d_2 + d.$$

Now inductively suppose that whenever $r, k \in \mathbb{N}$ with $2 \leq r + k \leq n - 1$, we have that $T_{r,k} = c + kd_1 + rd_2 + rkd$. Then for $r + k = n$ we have $(r-1)+k = n-1$ and $r+(k-1) = n-1$. Using the addition rule and the induction hypothesis we get that

$$\begin{aligned} T_{r,k} &= T_{r,k-1} + T_{r-1,k} + d - T_{r-1,k-1} \\ &= \big(c + (k-1)d_1 + rd_2 + r(k-1)d\big) + \big(c + kd_1 + (r-1)d_2 + (r-1)kd\big) + d \\ &\quad - \big(c + (k-1)d_1 + (r-1)d_2 + (r-1)(k-1)d\big) \\ &= c + kd_1 + rd_2 + rkd. \end{aligned}$$

Thus, $T_{r,k} = c + kd_1 + rd_2 + rkd$ for $r, k \geq 0$, and so T is the generalized rascal triangle $T(c, d, d_1, d_2)$. □

Proposition 16. *Let $D, d_1, d_2 \in \mathbb{Z}$ and T be a number triangle with $T_{r,0} = T_{0,0} + rd_2$, $T_{0,k} = T_{0,0} + kd_1$, $T_{r,k} \neq 0$ for all $r, k \geq 0$ and $T_{r,k} = \dfrac{T_{r,k-1} \cdot T_{r-1,k} + D}{T_{r-1,k-1}}$. Then there exist constants $c, d \in \mathbb{Z}$ such that $T = T(c, d, d_1, d_2)$ and $D = cd - d_1d_2$.*

Note that since we are assuming all the interior numbers satisfy a rascal-like multiplication rule, we require that $T_{r,k} \neq 0$ for all $r, k \geq 0$.

Proof. We first determine the constants c and d and establish the relation $D = cd - d_1d_2$. Let $c = T_{0,0}$ and $d = T_{1,1} - T_{0,1} - T_{1,0} + T_{0,0}$. Next note that since T has a rascal-like multiplication rule, we have that

$$T_{1,1} = \frac{T_{1,0} \cdot T_{0,1} + D}{T_{0,0}}$$

which means

$$T_{0,0} \cdot T_{1,1} = T_{1,0} \cdot T_{0,1} + D.$$

Moreover because $T_{r,k} \neq 0$ for all $r, k \geq 0$, we have that $c = T_{0,0} \neq 0$ and so

$$\begin{aligned} d &= \frac{dT_{0,0}}{T_{0,0}} \\ &= \frac{T_{0,0}(T_{1,1} - T_{0,1} - T_{1,0} + T_{0,0})}{T_{0,0}} \\ &= \frac{T_{0,0} \cdot T_{1,1} - T_{0,0} \cdot T_{0,1} - T_{0,0} \cdot T_{1,0} + T_{0,0}^2}{T_{0,0}} \\ &= \frac{(T_{0,1} \cdot T_{1,0} + D) - T_{0,0} \cdot T_{0,1} - T_{0,0} \cdot T_{1,0} + T_{0,0}^2}{T_{0,0}} \\ &= \frac{D + (T_{0,1} \cdot T_{1,0} - T_{0,0} \cdot T_{0,1} - T_{0,0} \cdot T_{1,0} + T_{0,0}^2)}{T_{0,0}} \\ &= \frac{D + (T_{0,1} - T_{0,0})(T_{1,0} - T_{0,0})}{T_{0,0}} \\ &= \frac{D + d_1 d_2}{c}. \end{aligned}$$

Thus,

$$D = cd - d_1 d_2.$$

To prove that $T = T(c, d, d_1, d_2)$, we first note that

$$T_{r,0} = c + rd_2 = c + 0d_1 + rd_2 + r \cdot 0d$$

and

$$T_{0,k} = c + kd_1 = c + kd_1 + 0d_2 + 0 \cdot kd,$$

so on the exterior diagonals, $T_{r,k} = c + kd_1 + rd_2 + rkd$. For the interior numbers $T_{r,k}$ with $r, k \geq 1$, we prove $T_{r,k} = ckd_1 + rd_2 + rkd$ by induction on the row number n for $n \geq 2$. Note that we start with $n = 2$ since the rows $n = 0$ and $n = 1$ have no interior numbers.

For the case $n = 2$ note that by Lemma 12 we have that $n = r + k$ for each entry $T_{r,k}$ on the n^{th} row; so suppose $r, k \in \mathbb{N}$ with $r + k = 2$. Since $r, k \geq 1$, $r + k = 2$ means $r = 1$ and $k = 1$ and so $T_{r,k-1} = T_{1,0} = c + d_2$, $T_{r-1,k} = T_{0,1} = c + d_1$, and $T_{r-1,k-1} = T_{0,0} = c$. Since

$$d = T_{1,1} - T_{0,1} - T_{1,0} + T_{0,0},$$

we have

$$T_{1,1} = T_{0,1} + T_{1,0} - T_{0,0} + d = c + d_1 + d_2 + d.$$

Now inductively suppose that whenever $r, k \in \mathbb{N}$ with $2 \leq r + k \leq n - 1$ we have that $T_{r,k} = c + kd_1 + rd_2 + rkd$. Then for $r + k = n$ we have $(r-1) + k = n-1$ and $r + (k-1) = n-1$.

Using the multiplication rule, the induction hypothesis, and Lemma 13 we have that

$$\begin{aligned}
T_{r,k} &= \frac{T_{r,k-1} \cdot T_{r-1,k} + D}{T_{r-1,k-1}} \\
&= \frac{(c + (k-1)d_1 + rd_2 + r(k-1)d)(c + kd_1 + (r-1)d_2 + (r-1)kd) + cd - d_1 d_2}{(c + (k-1)d_1 + (r-1)d_2 + (r-1)(k-1)d)} \\
&= \frac{(c + kd_1 + rd_2 + rkd)(c + (k-1)d_1 + (r-1)d_2 + (r-1)(k-1)d)}{(c + (k-1)d_1 + (r-1)d_2 + (r-1)(k-1)d)} \\
&= c + kd_1 + rd_2 + rkd.
\end{aligned}$$

Thus, $T_{r,k} = c + kd_1 + rd_2 + rkd$ for $r, k \geq 0$, and so T is the generalized rascal triangle $T(c, d, d_1, d_2)$. □

Combining Propositions 14, 15 and 16 gives us the following theorem:

Theorem 17. *Let $c, d, d_1, d_2 \in \mathbb{Z}$ and T be a number triangle with the arithmetic sequences $T_{0,k} = c + kd_1$ and $T_{r,0} = c + rd_2$ on the exterior diagonals. Then T is the generalized rascal triangle $T(c, d, d_1, d_2)$ and if and only if*

$$\begin{aligned}
T_{r,k} &= c + kd_1 + rd_2 + rkd, \\
T_{r,k} &= T_{r,k-1} + T_{r-1,k} + d - T_{r-1,k-1},
\end{aligned}$$

and whenever $T_{r-1,k-1} \neq 0$

$$T_{r,k} = \frac{T_{r-1,k} \cdot T_{r,k-1} + D}{T_{r-1,k-1}}$$

where $D = cd - d_1 d_2$.

We should note that generalized rascal triangles can contain entries that are zero. Since they have both a rascal-like addition and a rascal-like multiplication, we will, in general, use the rascal-like addition instead of the rascal-like multiplication.

4 Properties of generalized rascal triangles

4.1 Arithmetic diagonals implies a generalized rascal triangle

As we observed in Eqs. (4) and (5), if $c, d, d_1, d_2 \in \mathbb{Z}$, then one consequence of the definition of the generalized rascal triangle $T(c, d, d_1, d_2)$ is that all major and minor diagonals are arithmetic sequences. Not surprisingly, the converse of this observation is true. If all the major and minor diagonals of a number triangle T are arithmetic sequences, then T is a generalized rascal triangle.

We start by showing that if all the diagonals of a number triangle are arithmetic sequences then the constant differences for these sequences change by a fixed amount as we move from one diagonal to the next.

Lemma 18. *Let T be a number triangle with arithmetic sequences on all major and minor diagonals and let $M_r(k) = T_{r,0} + k\alpha_r$ and $m_k(r) = T_{0,k} + r\beta_k$ denote the arithmetic sequences on the major and minor diagonals respectively. Then there exists a constant $d \in \mathbb{Z}$ such that $d = \alpha_r - \alpha_{r-1} = \beta_k - \beta_{k-1}$ for all $r, k \geq 1$.*

Proof. Since $T_{r,k} = M_r(k) = m_k(r)$, $T_{r,k} = T_{r,k-1} + \alpha_r = T_{r-1,k} + \beta_k$ $\forall r, k \geq 1$. Let $r, k \geq 1$ be arbitrary. Then,

$$\begin{aligned} T_{r,k} &= T_{r,k-1} + \alpha_r = T_{r-1,k-1} + \beta_{k-1} + \alpha_r \\ &= T_{r-1,k} + \beta_k = T_{r-1,k-1} + \alpha_{r-1} + \beta_k \end{aligned}$$

which means

$$\beta_{k-1} + \alpha_r = \alpha_{r-1} + \beta_k,$$

if and only if

$$\alpha_r - \alpha_{r-1} = \beta_k - \beta_{k-1}.$$

Since this is true for all $r, k \geq 1$, if we let $d = \alpha_1 - \alpha_0 = \beta_1 - \beta_0$ we get that $d = \alpha_r - \alpha_{r-1} = \beta_k - \beta_{k-1}$ for all $r, k \geq 1$. □

Proposition 19. *Let T be a number triangle with arithmetic sequences on all major and minor diagonals. Then T is a generalized rascal triangle.*

Proof. We must show that there exist constants $c, d, d_1, d_2 \in \mathbb{Z}$ such that $T = T(c, d, d_1, d_2)$. Let $c = T_{0,0}$, and let $M_r(k) = T_{r,0} + k\alpha_r$ and $m_k(r) = T_{0,k} + r\beta_k$ denote the arithmetic sequences on the major and minor diagonals respectively. By Lemma 18 there is a constant $d \in \mathbb{Z}$ so that $d = \alpha_r - \alpha_{r-1} = \beta_k - \beta_{k-1}$ for all $r, k \geq 1$. Let $d_1 = T_{1,0} - T_{0,0} = \alpha_0$ and $d_2 = T_{0,1} - T_{0,0} = \beta_0$. Then

$$\begin{aligned} T_{r,k} &= M_r(k) = T_{r,0} + k\alpha_r = m_0(r) + k\alpha_r = T_{0,0} + r\beta_0 + k\alpha_r \\ &= c + rd_2 + k(\alpha_{r-1} + d) = \cdots = c + rd_2 + k(\alpha_0 + rd) \\ &= c + rd_2 + kd_1 + rkd. \end{aligned}$$

□

Example 20. Consider the number triangle W (sequence A332790 in the OEIS) from Example 4 in Section 2 (see Fig. 8). This number triangle has $c = 1$ and arithmetic sequences on all diagonals. The first three major diagonals in W are the arithmetic sequences A005408, A017029, and A017605 in the OEIS.

$$\begin{gathered} 1, 3, 5, 7, \ldots \\ 4, 11, 18, 25, \ldots \\ 7, 19, 31, 43, \ldots \\ \vdots \end{gathered}$$

Since the constant difference on the outside major diagonal is 2, $d_1 = 2$.

The first three minor diagonals in W are the arithmetic sequences A016777, A017101, and A154609 in the OEIS.

$$\begin{gathered} 1, 4, 7, 10, \ldots \\ 3, 11, 19, 27, \ldots \\ 5, 18, 31, 44, \ldots \\ \vdots \end{gathered}$$

Since the constant difference on the outside minor diagonal is 3, $d_2 = 3$.

Furthermore the differences for the arithmetic sequences change by 5 each time which means $d = 5$ and so $T = T(1, 5, 2, 3)$. Thus,

$$T_{r,k} = 1 + 3k + 2r + 5rk,$$

and the rascal-like addition rule is

$$T_{r,k} = T_{r-1,k} + T_{r,k-1} + 5 - T_{r-1,k-1}.$$

For the rascal-like multiplication rule

$$D = cd - d_1 d_2 = 1 \cdot 5 - 2 \cdot 3 = -1,$$

which means

$$T_{r,k} = \frac{T_{r-1,k} \cdot T_{r,k-1} - 1}{T_{r-1,k-1}}$$

whenever, $T_{r-1,k-1} \neq 0$.

4.2 Uniqueness of the rascal triangle

While we have seen that the rascal triangle is not the only number triangle that is generated by both a rascal-like multiplication rule and a rascal-like addition rule, the rascal triangle is unique in the sense that if T is a generalized rascal triangle with $d = 1$ and $c, d_1, d_2 \in \mathbb{Z}$ with $D = cd - d_1 d_2 = c - d_1 d_2 = 1$, then T sits inside the rascal triangle as a sub-triangle.

Definition 21. A number triangle T' is called a *sub-triangle* of a number triangle T if there exists $r_0, k_0 \in \mathbb{N}$ such that $T'_{r,k} = T_{r_0+r,k_0+k}$, as illustrated in Fig. 16.

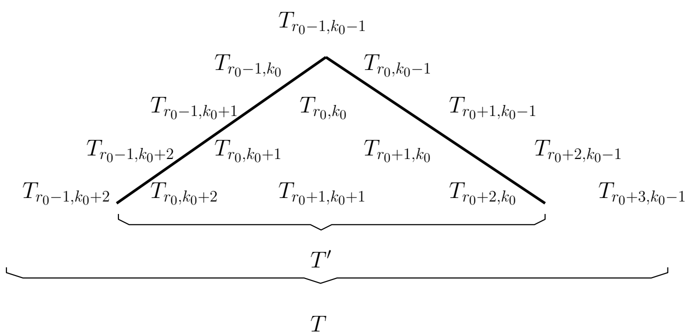


Figure 16: Sub-triangle T' starting at T_{r_0,k_0}.

Corollary 22. *Let $c, d_1, d_2 \in \mathbb{Z}$, such that $c - d_1 d_2 = 1$ and let T be the generalized rascal triangle $T(c, 1, d_1, d_2)$. Then T is a sub-triangle of the rascal triangle R.*

Proof. We have $c = 1 + d_1 d_2$, so $T_{0,0} = c = 1 + d_1 d_2 = R_{d_1,d_2}$ and

$$\begin{aligned} T_{r,k} &= c + kd_1 + rd_2 + rk \\ &= 1 + d_1 d_2 + kd_1 + rd_2 + rk \\ &= 1 + (d_1 + r)(d_2 + k) \\ &= R_{d_1+r,d_2+k}. \end{aligned}$$

Thus T is the sub-triangle of R starting at R_{d_1,d_2}. □

Definition 23. A number triangle T' is called a *multiple* of a number triangle T if there exists a constant m such that for $r, k \geq 0$, $T'_{r,k} = mT_{r,k}$.

The original rascal triangle R corresponds to the generalized rascal triangle $T(1, 1, 0, 0)$. If we take $c = d$ and $d_1 = d_2 = 0$, then $T(c, c, 0, 0)$ is a multiple of R.

Corollary 24. *Let $c \in \mathbb{Z}$ and let T be the generalized rascal triangle $T(c, c, 0, 0)$. Then $T = cR$.*

The proof of this is left to the reader.

4.3 Student discovered properties in generalized rascal triangles.

Over several semesters, I challenged students in my mathematics for liberal arts classes to find patterns in the rascal triangle. To my delight, they discovered several properties that, as far as I can tell, were unknown at the time [3]. Further investigations showed that these properties were also present in generalized rascal triangles; we conclude by presenting proofs of several of these properties, as well as some others, for generalized rascal triangles.

The first property about row sums, was discovered by Evan, who observed that the row sums in the rascal triangle $R = T(1, 1, 0, 0)$ had constant third differences. That is, they exhibited cubic growth.

Proposition 25. *Let $c, d, d_1, d_2 \in \mathbb{Z}$ and $T(c, d, d_1, d_2)$ be the associated generalized rascal triangle; then the row sum s_n for the n^{th} row is*

$$s_n = \frac{d}{6}n^3 + \left(\frac{d_1 + d_2}{2}\right) n^2 + \left(c + \frac{d_1 + d_2}{2} - \frac{d}{6}\right) n + c.$$

Proof. Since T is a generalized rascal triangle, $T_{r,k} = c + kd_1 + rd_2 + rkd$. By Lemma 12 we have $k = n - r$ for every entry $T_{r,k}$ on the n^{th} row. Thus

$$\begin{aligned}
s_n &= \sum_{r=0}^{n} (c + (n-r)d_1 + rd_2 + r(n-r)d) \\
&= \sum_{r=0}^{n} c + \sum_{r=0}^{n} n\, d_1 - \sum_{r=0}^{n} d_1 r + \sum_{r=0}^{n} d_2 r + \sum_{r=0}^{n} n\, d\, r - \sum_{r=0}^{n} d\, r^2 \\
&= (n+1)c + (n^2+n)d_1 - \left(\frac{n^2+n}{2}\right) d_1 + \left(\frac{n^2+n}{2}\right) d_2 + \left(\frac{n^3+n^2}{2}\right) d \\
&\quad - \left(\frac{2n^3 + 3n^2 + n}{6}\right) d \\
&= \frac{d}{6}n^3 + \left(\frac{d_1+d_2}{2}\right) n^2 + \left(c + \frac{d_1+d_2}{2} - \frac{d}{6}\right) n + c.
\end{aligned}$$

□

For the next patterns, we need the following definition.

Definition 26. Let T denote a number triangle. For $n \geq 1$, an *n-diamond* in T is the diamond whose sides are formed by the entries $T_{r,k}$ to $T_{r+n-1,k}$ on the k^{th} minor diagonal, $T_{r+n-1,k}$ to $T_{r+n-1,k+n-1}$ on the $(r+n-1)^{\text{st}}$ major diagonal, $T_{r+n-1,k+n-1}$ to $T_{r,k+n-1}$ on the $(k+n-1)^{\text{st}}$ minor diagonal, and $T_{r,k+n-1}$ to $T_{r,k}$ on the r^{th} major diagonal (see Fig. 17). We call $T_{r,k}$ the top number of the diamond.

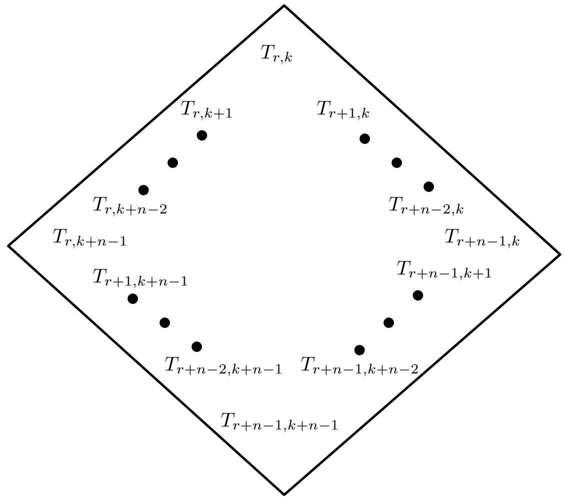


Figure 17: An n-diamond.

The following property was named after John, who discovered the original patterns about nested diamonds in the rascal triangle $R = T(1, 1, 0, 0)$.

Proposition 27 (John's odd/even diamond patterns)**.** *Let* $c, d, d_1, d_2 \in \mathbb{Z}$ *and* $T(c, d, d_1, d_2)$ *be the associated generalized rascal triangle. Then* T *has the following diamond patterns:*

i. (Odd diamond pattern.) *Let* D *be a* $(2n + 1)$*-diamond in* T *whose top number is* $T_{r,k}$ *and whose center number is* $T_{r+n,k+n}$ *as shown in Fig. 18.*

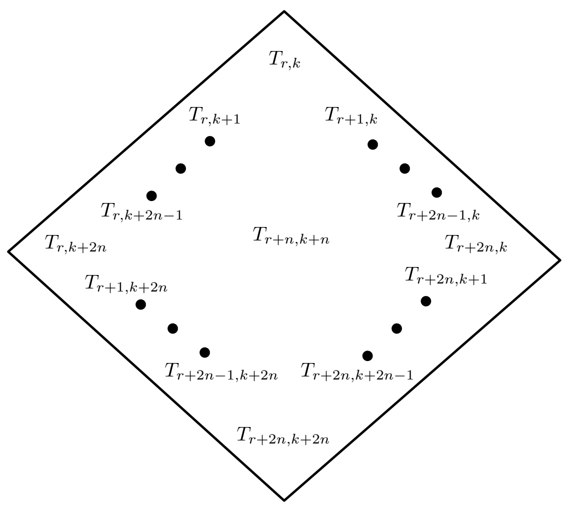


Figure 18: $(2n + 1)$-diamond

Then the average of the $8n$ *entries along the edge of the diamond is* $T_{r+n,k+n}$*. That is,*

$$\frac{1}{8n}\left(\sum_{i=0}^{2n-1} T_{r+i,k} + \sum_{i=0}^{2n-1} T_{r+2n,k+i} + \sum_{i=0}^{2n-1} T_{r+2n-i,k+2n} + \sum_{i=0}^{2n-1} T_{r,k+2n-i}\right) = T_{r+n,k+n}.$$

ii. (Even diamond pattern.) *Let D_1 be a 2-diamond whose top number is $T_{r,k}$, and for $n \leq \min\{r,k\}$ let D_n denote the $2n$-diamond whose top entry is $T_{r-n+1,k-n+1}$ as shown in Fig. 19.*

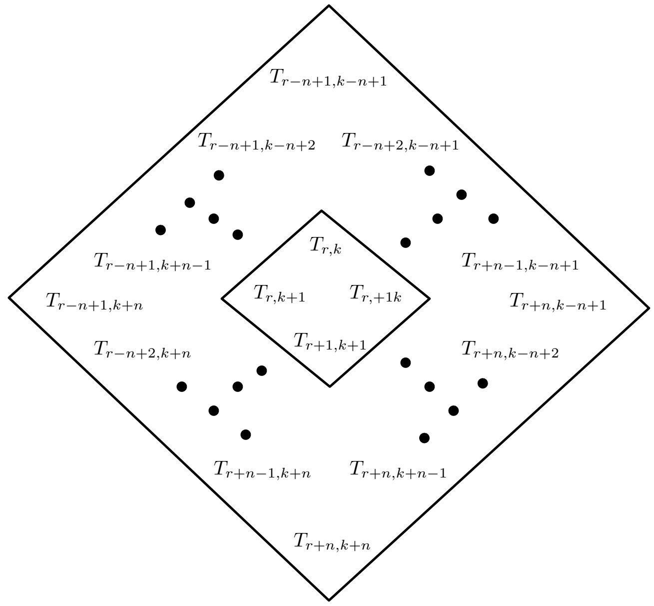


Figure 19: $2n$-diamond

Then the average of the entries along the edges of D_n is equal to the average of the four entries along the edges of D_1. That is,

$$\frac{1}{8n-4}\left(\sum_{i=0}^{2n-2} T_{r-n+1+i,k-n+1} + \sum_{i=0}^{2n-2} T_{r+n,k-n+1+i} + \sum_{i=0}^{2n-2} T_{r+n-i,k+n} + \sum_{i=0}^{2n-2} T_{r-n+1,k+n-i}\right)$$
$$= \frac{1}{4}\left(T_{r,k} + T_{r+1,k} + T_{r+1,k+1} + T_{r,k+1}\right).$$

Example 28. In the generalized rascal triangle $T(2,2,3,1)$ (sequence A309557 in the OEIS) in Fig. 20, the average of the 8 numbers along the edge of the red 3-diamond is equal to the center number 32.

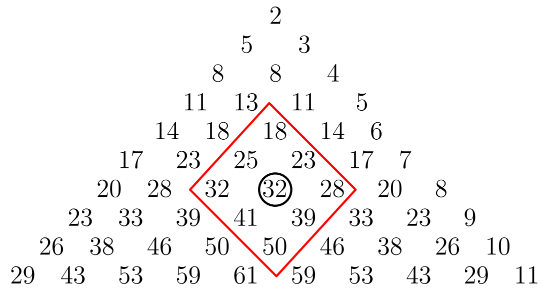


Figure 20: John's odd diamond pattern in $T(2, 2, 3, 1)$.

$$\frac{18 + 23 + 28 + 39 + 50 + 41 + 32 + 25}{8} = 32.$$

In the generalized rascal triangle $T(2, 2, 3, 1)$ (sequence A309557 in the OEIS) in Fig. 21, the average of the 12 numbers along the edge of the red 4-diamond is equal to the average of the 4 numbers in the black 2-diamond in the center.

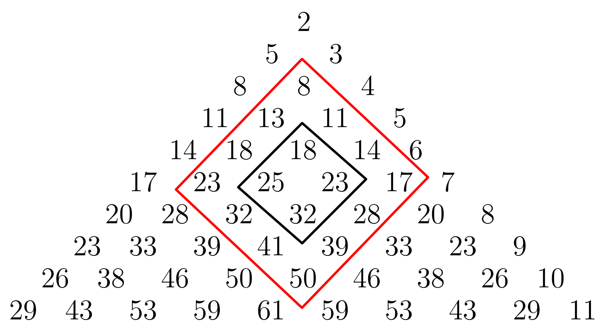


Figure 21: John's even diamond pattern in $T(2, 2, 3, 1)$.

$$\frac{8 + 11 + 14 + 17 + 28 + 39 + 50 + 41 + 32 + 23 + 18 + 13}{12} = \frac{18 + 23 + 32 + 25}{4} = 24.5.$$

Proof of Proposition 27. For the odd diamond pattern, we can regroup the terms in the numerator as follows

$$\sum_{i=0}^{2n-1} \Big(\big(T_{r+i,k} + T_{r+2n-i,k+2n} \big) + \big(T_{r+2n,k+i} + T_{r,k+2n-i} \big) \Big).$$

Since T is a generalized rascal triangle we have that $T_{r,k} = c + kd_1 + rd_2 + rkd$; so

$$\begin{aligned}
T_{r+i,k} + T_{r+2n-i,k+2n} &= (c + kd_1 + (r+i)(d_2 + kd)) + (c + (k+2n)d_1 \\
&\quad + (r + 2n - i)(d_2 + (k+2n)d)) \\
&= c + kd_1 + (r+i)(d_2 + kd) + c + kd_1 + 2nd_1 + (r-i)(d_2 + kd) \\
&\quad + (r-i)2nd + 2n(d_2 + kd) + 4n^2 d \\
&= 2c + 2kd_1 + 2rd_2 + rkd + id_2 + ikd + 2nd_1 + rkd - id_2 - ikd + 2nrd \\
&\quad - 2ndi + 2nd_2 + 2nkd + 4n^2 d \\
&= 2c + (2kd_1 + 2nd_1) + (2rd_2 + 2nd_2) + (2rkd + 2rnd + 2knd + 2n^2 d) + 2n^2 d \\
&\quad - 2ndi \\
&= 2(c + (k+n)d_1 + (r+n)d_2 + (r+n)(k+n)d) + 2n^2 d - 2ndi \\
&= 2T_{r+n,k+n} + 2n^2 d - 2ndi; \qquad (8)
\end{aligned}$$

and

$$\begin{aligned}
T_{r+2n,k+i} + T_{r,k+2n-i} &= (c + (r+2n)d_2 + (k+i)(d_1 + (r+2n)d)) \\
&\quad + (c + rd_2 + (k + 2n - i)(d_1 + rd)) \\
&= c + rd_2 + 2nd_2 + (k+i)(d_1 + rd) + (k+i)2nd + c + rd_2 + (k-i)(d_1 + rd) \\
&\quad + 2n(d_1 + rd) \\
&= c + rd_2 + 2nd_2 + kd_1 + rkd + id_1 + ird + 2knd + 2ndi + c + kd_1 + rd_2 + rkd \\
&\quad - id_1 - ird + 2nd_1 + 2nrd + 2n^2 d - 2n^2 d \\
&= 2c + (2rd_2 + 2nd_2) + (2kd_1 + 2nd_1) + (2rkd + 2rnd + 2knd + 2n^2 d) + 2ndi \\
&\quad - 2n^2 d \\
&= 2(c + (k+n)d_1 + (r+n)d_2 + (r+n)(k+n)d) + 2ndi - 2n^2 d \\
&= 2T_{r+n,k+n} + 2dni - 2n^2 d. \qquad (9)
\end{aligned}$$

Combining Eqs. (8) and (9) we get

$$\begin{aligned}
&(T_{r+i,k} + T_{r+2n-i,k+2n}) + (T_{r+2n,k+i} + T_{r,k+2n-i}) \\
&= 2T_{r+n,k+n} + 2n^2 d - 2ndi + 2T_{r+n,k+n} + 2ndi - 2n^2 d \\
&= 4T_{r+n,k+n}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{1}{8n}\left(\sum_{i=0}^{2n-1} T_{r+i,k} + \sum_{i=0}^{2n-1} T_{r+2n,k+i} + \sum_{i=0}^{2n-1} T_{r+2n-i,k+2n} + \sum_{i=0}^{2n-1} T_{r,k+2n-i}\right) \\
&= \frac{1}{8n}\left(\sum_{i=0}^{2n-1}\left((T_{r+i,k} + T_{r+2n-i,k+2n}) + (T_{r+2n,k+i} + T_{r,k+2n-i})\right)\right) \\
&= \frac{1}{8n}\left(\sum_{i=0}^{2n-1} 4T_{r+n,k+n}\right) = \frac{1}{8n}\left(8n \cdot T_{r+n,k+n}\right) = T_{r+n,k+n}.
\end{aligned}$$

For the even diamond pattern, we can regroup the terms in the sum of the edges of a $2n$-diamond as follows:

$$\sum_{i=0}^{2n-2}\Big(T_{r-n+1+i,k-n+1}+T_{r+n,k-n+1+i}+T_{r+n-i,k+n}+T_{r-n+1,k+n-i}\Big).$$

Since T is a generalized rascal triangle we have that $T_{r,k}=c+kd_1+rd_2+rkd$. Thus,

$$\begin{aligned}
T_{r-n+1+i,k-n+1} &= c+(k-n+1)d_1+(r-n+1+i)d_2+(r-n+1+i)(k-n+1)d\\
&= c+(k+1)d_1-nd_1+(r+1)d_2-(n-i)d_2+(r+1)(k+1)d\\
&\quad -(k+1)(n-i)d-(r+1)nd+n(n-i)d\\
&= T_{r+1,k+1}-nd_1-(n-i)d_2-k(n-i)d-(n-i)d-rnd-nd+n(n-i)d, \qquad (10)\\
T_{r+n,k-n+1+i} &= c+(k-n+1+i)d_1+(r+n)d_2+(r+n)(k-n+1+i)d\\
&= c+(k+1)d_1-(n-i)d_1+rd_2+nd_2+r(k+1)d-r(n-i)d\\
&\quad +(k+1)nd-n(n-i)d\\
&= T_{r,k+1}-(n-i)d_1+nd_2-r(n-i)d+knd+nd-n(n-i)d, \qquad (11)\\
T_{r+n-i-i,k+n} &= c+(k+n)d_1+(r-n-i)d_2+(r+n-i)(k+n)d\\
&= c+kd_1+nd_1+rd_2+(n-i)d_2+rkd+rnd+k(n-i)d+n(n-i)d\\
&= T_{r,k}+nd_1+(n-i)d_2+rnd+k(n-i)d+n(n-i)d, \qquad (12)\\
T_{r-n+1,k+n-i} &= c+(k+n-i)d_1+(r-n+1)d_2+(r-n+1)(k+n-i)d\\
&= c+kd_1+(n-i)d_1+(r+1)d_2-nd_2+(r+1)kd+(r+1)(n-i)d\\
&\quad -nkd-n(n-i)d\\
&= T_{r+1,k}+(n-i)d_1-nd_2+(r+1)(n-i)d-nkd-n(n-i)d. \qquad (13)
\end{aligned}$$

Combining Eqs. (10)–(13) we get

$$\begin{aligned}
&T_{r-n+1+i,k-n+1}+T_{r+n,k-n+1+i}+T_{r+n-i,k+n}+T_{r-n+1,k+n-i}\\
&\quad = T_{r+1,k+1}-nd_1-(n-i)d_2-k(n-i)d-(n-i)d-rnd-nd+n(n-i)d\\
&\qquad +T_{r,k+1}-(n-i)d_1+nd_2-r(n-i)d+knd+nd-n(n-i)d\\
&\qquad +T_{r,k}+nd_1+(n-i)d_2+rnd+k(n-i)d+n(n-i)d\\
&\qquad +T_{r+1,k}+(n-i)d_1-nd_2+(r+1)(n-i)d-nkd-n(n-i)d\\
&\quad = T_{r+1,k+1}+T_{r,k+1}+T_{r,k}+T_{r+1,k}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{1}{8n-4}\left(\sum_{i=0}^{2n-2}(T_{r-n+1+i,k-n+1}+T_{r+n,k-n+1+i}+T_{r+n-i,k+n}+T_{r-n+1,k+n-i})\right)\\
&=\frac{1}{8n-4}\left(\sum_{i=0}^{2n-2}(T_{r+1,k+1}+T_{r,k+1}+T_{r,k}+T_{r+1,k})\right)\\
&=\frac{1}{8n-4}(2n-1)(T_{r+1,k+1}+T_{r,k+1}+T_{r,k}+T_{r+1,k})\\
&=\frac{1}{4}(T_{r+1,k+1}+T_{r,k+1}+T_{r,k}+T_{r+1,k}).
\end{aligned}$$

□

The remaining properties are recursive rules similar to the rascal-like addition rule in Eq. (7) which allow us to determine $T_{r,k}$ from entries in T that are in the rows above it.

The first three properties were named for Ashley, who discovered the original version for the rascal triangle $R = T(1, 1, 0, 0)$.

Proposition 29 (Ashley's rule)**.** *Let $c, d, d_1, d_2 \in \mathbb{Z}$ and $T(c, d, d_1, d_2)$ be the associated generalized rascal triangle. Then*

$$T_{r,k} = T_{r-1,k} + T_{r,k-1} - T_{r-2,k-1} + ((2-k)d - d_2) \tag{14}$$

for all $r \geq 2, k \geq 1$.

Note that the quantity $(2-k)d-d_2$ in Eq. (14) depends only on the minor diagonal k that contains $T_{r,k}$ since the quantities d and d_2 are fixed. When describing her original version of Proposition 29 for the Rascal Triangle $R = T(1, 1, 0, 0)$, Ashley called this quantity the *diagonal factor*.

Notation 30. In Figs. 22–25 in Examples 31–33 the generalized rascal triangle is $T(2, 2, 3, 1)$ (sequence A309557 in the OEIS). The blue circle corresponds to $T_{r,k}$, the black boxes correspond to the terms that are added and the red hexagons correspond to the terms that are subtracted. The bold red diagonal in Fig. 22 is the diagonal factor $(2-k)d-d_2$ in Eq. (14).

Example 31. If we let $T_{r,k} = 59$, then $r = 5$ and $k = 4$. Therefore $T_{r-1,k} = T_{4,4} = 50$, $T_{r,k-1} = T_{5,3} = 46$, $T_{r-2,k-1} = T_{3,3} = 32$, and the diagonal factor is $(2-k)d-d_2 = -5$, as illustrated in Fig. 22. Thus

$$T_{r-1,k} + T_{r,k-1} - T_{r-2,k-1} + ((2-k)d - d_2) = 50 + 46 - 32 - 5 = 59 = T_{r,k}.$$

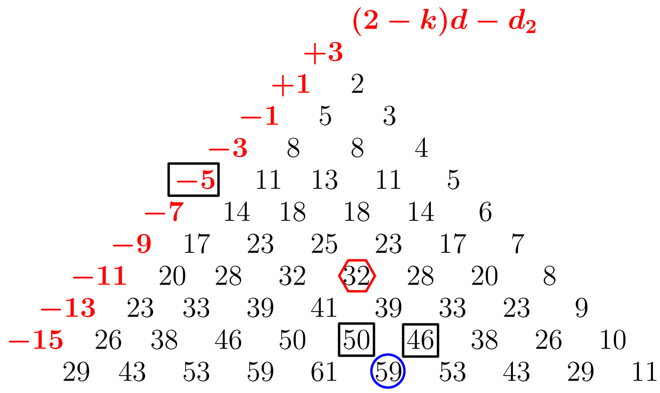


Figure 22: Ashley's rule for $T(2, 2, 3, 1)$.

Proof of Proposition 29. Recall from Eq. (5) that on the k^{th} minor diagonal, the entries form the arithmetic progression $c + kd_1 + r(kd + d_2)$ with common difference $kd + d_2$, hence $T_{r-1,k} + kd + d_2 = T_{r,k}$ for $r \geq 1, k \geq 0$. Thus

$$\begin{aligned}
&T_{r,k-1} + T_{r-1,k} - T_{r-2,k-1} + ((2-k)d - d_2) \\
&= T_{r,k-1} + T_{r-1,k} - T_{r-2,k-1} - (k-1)d - d_2 + d \\
&= T_{r,k-1} + T_{r-1,k} - (T_{r-2,k-1} + (k-1)d + d_2) + d \\
&= T_{r,k-1} + T_{r-1,k} - T_{r-1,k-1} + d \\
&= T_{r,k}
\end{aligned}$$

by Proposition 14. □

My colleague, J. Fleron and I subsequently discovered three ways of modifying Ashley's rule so that the diagonal factor $(2-k)d - d_2$ was not needed.

Proposition 32 (Modified Ashley's rule)**.** *Let $c, d, d_1, d_2 \in \mathbb{Z}$ and $T(c, d, d_1, d_2)$ be the associated generalized rascal triangle; then for $r, k \geq 3$*

$$\begin{aligned}
T_{r,k} &= T_{r-1,k} + T_{r,k-1} - T_{r-2,k-1} - T_{r-2,k-2} + T_{r-3,k-2} && (15)\\
&= T_{r,k-1} + T_{r-1,k-1} - T_{r-2,k-2} - T_{r-2,k-3} + T_{r-3,k-3} && (16)\\
&= T_{r-1,k} + T_{r-1,k-1} - T_{r-2,k-2} - T_{r-3,k-2} + T_{r-3,k-3}. && (17)
\end{aligned}$$

Example 33. For the first modification, Eq. (15), if we let $T_{r,k} = 59$ then $r = 5$ and $k = 4$. Therefore $T_{r-1,k} = T_{4,4} = 50$, $T_{r,k-1} = T_{5,3} = 46$, $T_{r-2,k-1} = T_{3,3} = 32$, $T_{r-2,k-2} = T_{3,2} = 23$, and $T_{r-3,k-2} = T_{2,2} = 18$, as shown in Fig. 23. Hence

$$T_{r-1,k} + T_{r,k-1} - T_{r-2,k-1} - T_{r-2,k-2} + T_{r-3,k-2} = 50 + 46 - 32 - 23 + 18 = 59 = T_{r,k}.$$

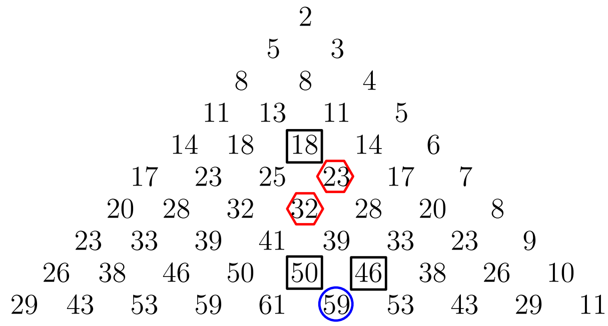


Figure 23: Modification 1 of Ashley's rule for $T(2,2,3,1)$.

For the second modification, Eq. (16), if we let $T_{r,k} = 59$ then $r = 5$ and $k = 4$. Therefore $T_{r,k-1} = T_{5,3} = 46$, $T_{r-1,k-1} = T_{4,3} = 39$, $T_{r-2,k-2} = T_{3,2} = 23$, $T_{r-2,k-3} = T_{3,1} = 14$, and $T_{r-3,k-3} = T_{2,1} = 11$, as illustrated in Fig. 24. Thus

$$T_{r,k-1} + T_{r-1,k-1} - T_{r-2,k-2} - T_{r-2,k-3} + T_{r-3,k-3} = 46 + 39 - 23 - 14 + 11 = 59 = T_{r,k}.$$

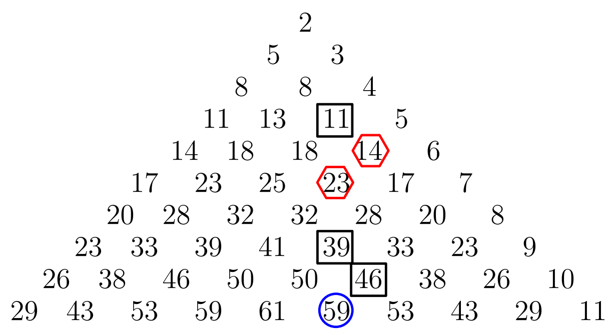


Figure 24: Modification 2 of Ashley's rule for $T(2,2,3,1)$.

For the third modification, Eq. (17), if we let $T_{r,k} = 59$ then $r = 5$ and $k = 4$. Therefore $T_{r-1,k} = T_{4,4} = 50$, $T_{r-1,k-1} = T_{4,3} = 39$, $T_{r-2,k-2} = T_{3,2} = 23$, $T_{r-3,k-2} = T_{2,2} = 18$, and $T_{r-3,k-3} = T_{2,1} = 11$, as shown in Fig. 25. Thus

$$T_{r-1,k} + T_{r-1,k-1} - T_{r-2,k-2} - T_{r-3,k-2} + T_{r-3,k-3} = 50 + 39 - 23 - 18 + 11 = 59 = T_{r,k}.$$

2
5 3
8 8 4
11 13 [11] 5
14 18 (18) 14 6
17 23 25 (23) 17 7
20 28 32 32 28 20 8
23 33 39 41 [39] 33 23 9
26 38 46 50 [50] 46 38 26 10
29 43 53 59 61 (59) 53 43 29 11

Figure 25: Modification 3 of Ashley's rule for $T(2,2,3,1)$.

Proof of Proposition 32. For Eq. (15):

$$\begin{aligned}
&T_{r-1,k} + T_{r,k-1} - T_{r-2,k-1} - T_{r-2,k-2} + T_{r-3,k-2} \\
&\quad = (c + kd_1 + (r-1)d_2 + (r-1)kd) \\
&\qquad + (c + (k-1)d_1 + rd_2 + r(k-1)d) \\
&\qquad - (c + (k-1)d_1 + (r-2)d_2 + (r-2)(k-1)d) \\
&\qquad - (c + (k-2)d_1 + (r-2)d_2 + (r-2)(k-2)d) \\
&\qquad + (c + (k-2)d_1 + (r-3)d_2 + (r-3)(k-2)d) \\
&\quad = c + kd_1 \phantom{{}-2d_1} + rd_2 - d_2 + rkd - kd \\
&\qquad + c + kd_1 - d_1 + rd_2 \phantom{{}-3d_2} + rkd \phantom{{}-3kd} - rd \\
&\qquad - c - kd_1 + d_1 - rd_2 + 2d_2 - rkd + 2kd + rd - 2d \\
&\qquad - c - kd_1 + 2d_1 - rd_2 + 2d_2 - rkd + 2kd + 2rd - 4d \\
&\qquad + c + kd_1 - 2d_1 + rd_2 - 3d_2 + rkd - 3kd - 2rd + 6d \\
&\quad = c + kd_1 + rd_2 + rkd = T_{r,k}.
\end{aligned}$$

For Eq. (16):

$$
\begin{aligned}
&T_{r,k-1} + T_{r-1,k-1} - T_{r-2,k-2} - T_{r-2,k-3} + T_{r-3,k-3} \\
&\quad = (c + (k-1)d_1 + rd_2 + r(k-1)d) \\
&\qquad + (c + (k-1)d_1 + (r-1)d_2 + (r-1)(k-1)d) \\
&\qquad - (c + (k-2)d_1 + (r-2)d_2 + (r-2)(k-2)d) \\
&\qquad - (c + (k-3)d_1 + (r-2)d_2 + (r-2)(k-3)d) \\
&\qquad + (c + (k-3)d_1 + (r-3)d_2 + (r-3)(k-3)d) \\
&\quad = \quad c + kd_1 - d_1 + rd_2 \qquad\quad + rkd - rd \\
&\qquad + c + kd_1 - d_1 + rd_2 - d_2 + rkd - rd - kd + d \\
&\qquad - c - kd_1 + 2d_1 - rd_2 + 2d_2 - rkd + 2rd + 2kd - 4d \\
&\qquad - c - kd_1 + 3d_1 - rd_2 + 2d_2 - rkd + 3rd + 2kd - 6d \\
&\qquad + c + kd_1 - 3d_1 + rd_2 - 3d_2 + rkd - 3rd - 3kd + 9d \\
&\quad = c + kd_1 + rd_2 + rkd = T_{r,k}.
\end{aligned}
$$

The proof for Eq. (17) follows immediately from Eq. (16) by replacing $T_{r,k-1}$ by $T_{r-1,k} + d_2 - d_1 + (k_0 - r_0 - 1)d$ and $T_{r-2,k-3}$ by $T_{r-3,k-2} + d_2 - d_1 + (k_0 - r_0 - 1)d$. These last steps are a consequence of applying Lemma 34 below to the two adjacent columns C_1 and C_2 of T which contain $T_{r-3,k-2}$ and $T_{r-1,k}$ and $T_{r,k-1}$ and $T_{r-2,k-3}$ respectively; and where T_{r_0,k_0} is the entry in column C_1 that is in the first row of T that contains entries from both C_1 and C_2 as illustrated in Fig. 26 where the boxed entries denote the columns C_1 and C_2. □

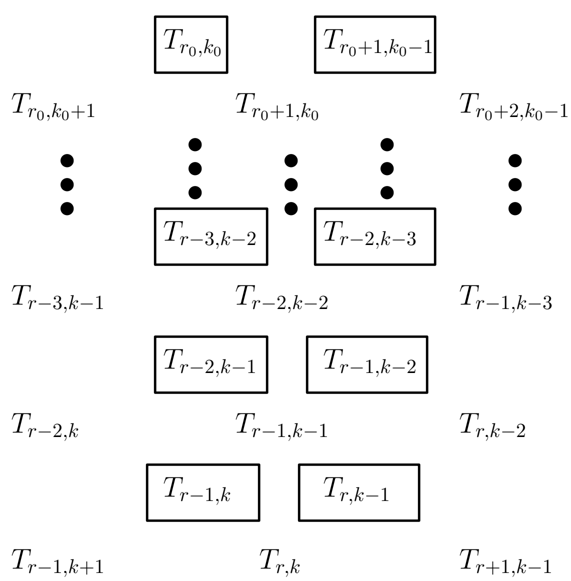


Figure 26: Columns of T for the proof of Eq. (17).

Lemma 34 (Constant difference between parallel columns)**.** *Let $c, d, d_1, d_2 \in \mathbb{Z}$ and let C_1 and C_2 be two columns in the associated generalized rascal triangle $T(c, d, d_1, d_2)$ with C_2 m spaces to the right of C_1. If n_0 is the first row of T containing entries in C_1 and C_2 and n is any row of T below n_0 containing entries in C_1 and C_2, then the difference between the entries of C_1 and C_2 in row n_0 is the same as the difference between the entries of C_1 and C_2 in row n. More precisely, if T_{r_0,k_0} and T_{r_1,k_1} are the entries in C_1 and C_2 respectively in row n_0 and T_{r_2,k_2} and T_{r_3,k_3} are the entries in C_1 and C_2 respectively in row n, then*

$$T_{r_3,k_3} - T_{r_2,k_2} = T_{r_1,k_1} - T_{r_0,k_0} = md_2 - md_1 + (k_0 - r_0 - m)md.$$

Example 35. Using the generalized rascal triangle $T(2, 2, 3, 1)$ (sequence A309557 in the OEIS), consider the two columns in T indicated by the boxed numbers in Fig. 27. Here $n_0 = 4$ with $T_{r_0,k_0} = T_{1,3} = 18$ and $m = 3$, so

$$md_2 - md_1 + (k_0 - r_0 - m)md = 3 \cdot 1 - 3 \cdot 3 + (3 - 1 - 3) \cdot 3 \cdot 2 = -12,$$

which is the difference between entries of the two columns that are in the same row.

$$6 - 18 = 20 - 32 = 38 - 50 = -12.$$

2
5 3
$\boxed{8}$ 8 4
11 13 11 5
14 $\boxed{18}$ 18 14 $\boxed{6}$
17 23 25 23 17 7
20 28 $\boxed{32}$ 32 28 $\boxed{20}$ 8
23 33 39 41 39 33 23 9
26 38 46 $\boxed{50}$ 50 46 $\boxed{38}$ 26 10
29 43 53 59 61 59 53 43 29 11

Figure 27: Constant Difference Between Parallel Columns for $T(2, 2, 3, 1)$.

Proof of Lemma 34. Since T_{r_0,k_0} is the entry in C_1 on row n_0 and C_2 is m spaces to the right of C_1, the entry in C_2 in row n_0 is $T_{r_1,k_1} = T_{r_0+m,k_0-m}$ by Lemma 12. Furthermore, because entries in a column occur in every two rows of T, there is an integer j so that $n = n_0 + 2j$. Thus the entries on the row $n_0 + 2j$ in the columns C_1 and C_2 are $T_{r_2,k_2} = T_{r_0+j,k_0+j}$ and $T_{r_3,k_3} = T_{r_0+m+j,k_0-m+j}$ respectively (see Fig. 28).

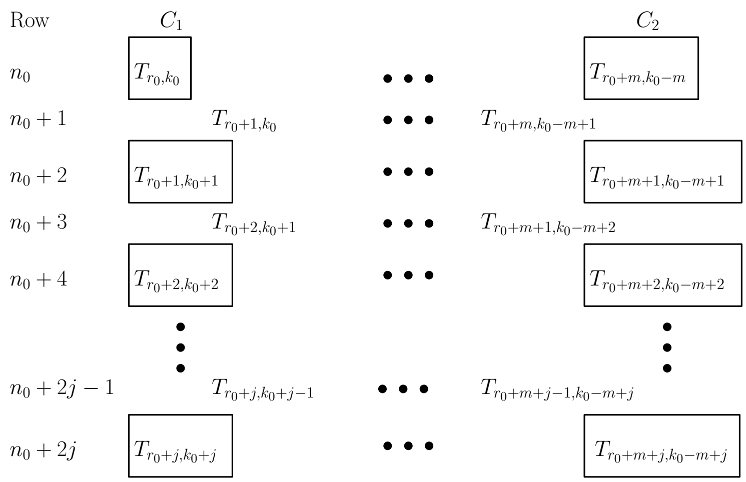


Figure 28: Entries in columns C_1 and C_2 in rows n_0 to n_0+2j.

Therefore,

$$\begin{aligned} T_{r_0+m,k_0-m} - T_{r_0,k_0} &= c + (k_0-m)d_1 + (r_0+m)d_2 + (r_0+m)(k_0-m)d \\ &\quad - (c + k_0 d_1 + r_0 d_2 + r_0 k_0 d) \\ &= md_2 - md_1 + (k_0 - r_0 - m)md, \end{aligned}$$

and

$$\begin{aligned} T_{r_0+m+j,k_0-m+j} - T_{r_0+j,k_0+j} &= c + (k_0-m+j)d_1 + (r_0+m+j)d_2 + (r_0+m+j)(k_0-m+j)d \\ &\quad - (c + (k_0+j)d_1 + (r_0+j)d_2 + (r_0+j)(k_0+j)d) \\ &= md_2 - md_1 + (k_0 - r_0 - m)md. \end{aligned}$$

□

The final property was named after Timothy and Meg who originally discovered the original version for the rascal triangle $R = T(1,1,0,0)$. Note that this property only applies to generalized rascal triangles with $d_1 = d_2 = 0$.

Proposition 36 (T-Meg rule)**.** *Let $c, d \in \mathbb{Z}$ and $T(c,d,0,0)$ be the associated generalized rascal triangle; then*

$$T_{r,k} = T_{0,r+k-2} + T_{1,r+k-3} + T_{r-1,k-1} + 2(d-c)$$

for $r \geq 1, k \geq 2$.

Notation 37. In Figs. 29–31 in Examples 38–40 the generalized rascal triangle is $T(3,1,0,0)$ (sequence A309555 in the OEIS). The blue circle corresponds to $T_{r,k}$, the black squares and diamonds correspond to the three terms from the row two above $T_{r,k}$ that are being added.

Example 38. If we let $T_{r,k} = T_{3,3} = 12$ then $r = 3$ and $k = 3$. Therefore $T_{0,r+k-2} = T_{0,4} = 3$, $T_{1,r+k-3} = T_{1,3} = 6$, $T_{r-1,k-1} = T_{2,2} = 7$, and $2(d-c) = -4$. Thus,

$$T_{0,r+k-2} + T_{1,r+k-3} + T_{r-1,k-1} + 2(d-c) = 12 = 3 + 6 + 7 - 4 = 12 = T_{r,k}.$$

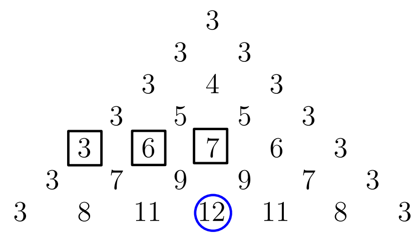


Figure 29: T-Meg rule for $T(3,1,0,0)$.

Note that Proposition 36 is true even when $T_{r,k}$ is the second or third entry in a row, i.e., $r = 1$ or 2. When $r = 1$, then $T_{r-1,k-1} = T_{0,k-1} = T_{0,r+k-2}$ and when $r = 2$, $T_{r-1,k-1} = T_{1,k-1} = T_{1,r+k-3}$ and so when $r = 1$ or 2 one of the one of the terms in the row two above $T_{r,k}$ is being added twice.

Example 39. If we let $T_{r,k} = T_{2,4} = 11$ then $r = 2$ and $k = 4$. Hence $T_{0,r+k-2} = T_{0,4} = 3$, $T_{1,r+k-3} = T_{r-1,k-1} = T_{1,3} = 6$ and $2(d-c) = -4$. Thus,

$$T_{0,r+k-2} + T_{1,r+k-3} + T_{r-1,k-1} + 2(d-c) = 3 + 6 + 6 - 4 = 11 = T_{r,k}.$$

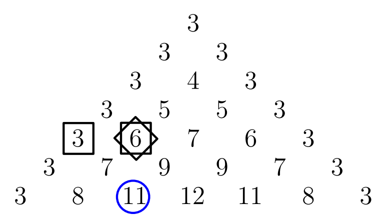


Figure 30: T-Meg rule for $T(3,1,0,0)$.

Example 40. If we let $T_{r,k} = T_{1,5} = 8$ then $r = 1$ and $k = 5$. Therefore $T_{0,r+k-2} = T_{r-1,k-1} = T_{0,4} = 3$, $T_{1,r+k-3} = 6$ and $2(d-c) = -4$. Thus,

$$T_{0,r+k-2} + T_{1,r+k-3} + T_{r-1,k-1} + 2(d-c) = 3 + 6 + 3 - 4 = 8 = T_{r,k}.$$

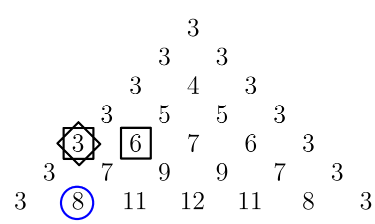


Figure 31: T-Meg rule for $T(3,1,0,0)$.

Proof of Proposition 36. Since $d_1 = d_2 = 0$ we have $T_{r,k} = c + rkd$. Thus,

$$\begin{aligned} &T_{0,r+k-2} + T_{1,r+k-3} + T_{r-1,k-1} + 2(d-c) \\ &\qquad = (c + 0(r+k-2)d) + (c + 1(r+k-3)d) + (c + (r-1)(k-1)d) + 2d - 2c \\ &\qquad = c + c + rd + kd - 3d + c + rkd - rd - kd + d + 2d - 2c \\ &\qquad = c + rkd = T_{r,k}. \end{aligned}$$

□

5 Conclusion

The results in this paper grew out of explorations by mathematics for liberal arts students looking for patterns in the rascal triangle (sequence A077028 in the OEIS). My students enthusiasm and insights inspired me to look more deeply at the structure of the rascal triangle and the roles that Eqs. (1) and (2) played in that structure, which led to the generalized rascal triangles.

As these triangles are closely related to the rascal triangle and Pascal's triangle (sequence A007318 in the OEIS) it is natural to ask if they are as rich mathematically as those triangles. In particular, what other relationships are there between the entries, and are there combinatorial interpretations of the entries for generalized rascal triangles? Brandt Kronholm and his student Jena Gregory, at the University of Texas, Rio Grande Valley, have begun investigating this last question using generating functions and an infinite sequence of number triangles whose limiting triangle is Pascal's triangle.

6 Acknowledgments

This work was an outgrowth of student investigations in Mathematics for Liberal Arts classes at Westfield State University. The author's work on the *Discovering the Art of Mathematics* project, which provides the basis for the curriculum in this course, was supported by NSF grants DUE-0836943, DUE-1225915, and a generous gift from Mr. Harry Lucas.

The author would like to acknowledge his students who observed the patterns discussed in this paper: Ashley Craig, John Coulombe, Timothy Schreiner, Meaghan Sparks, and Evan Wilson.

The author would also like to thank his colleague J. Fleron for his collaboration in exploring the student's patterns, as well as the comments and suggestions from him, the anonymous referee and the editor-in-chief that have improved this paper.

References

[1] A. Anggoro, E. Liu, and A. Tulloch, The rascal triangle, *College Math. J.* **41** (2010), 393–395.

[2] J. Fleron, Fresh perspectives bring discoveries, *Math Horiz.* **24** (2017), 15.

[3] P. Hotchkiss, Student inquiry and the rascal triangle, preprint, 2019. Available at http://www.arxiv.org/abs/1907.07749.

[4] N. J. A. Sloane et al., The On-Line Encyclopedia of Integer Sequences, 2020. Available at https://oeis.org.

2010 *Mathematics Subject Classification*: Primary 11B25; Secondary 11B99.
Keywords: rascal triangle, number triangle, arithmetic sequence.

(Concerned with sequences A000012, A000027, A000034, A000124, A005408, A007318, A016777, A016789, A017029, A017101, A017605, A077028, A154609, A309555, A309557, A309559, A332790, and A332963.)

Received August 7 2019; revised version received September 5 2019; March 27 2020; April 9 2020. Published in *Journal of Integer Sequences*, June 26 2020.

Return to Journal of Integer Sequences home page.