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Characterizing Quasi-Friendly Divisors

C. A. Holdener
Grinnell College
Grinnell, IA 50112
USA
holdener@grinnell.edu

J. A. Holdener
Department of Mathematics and Statistics
Kenyon College
Gambier, OH 43022
USA
holdenerj@kenyon.edu

Abstract

Abundancy ratios are rational numbers $\frac{k}{m}$ satisfying $\frac{\sigma(N)}{N} = \frac{k}{m}$ for some $N \in \mathbb{Z}_{\geq 1}$, where σ is the sum-of-divisors function. In this paper we examine abundancy ratios of the form $\frac{\sigma(m)+1}{m}$, where $\gcd(m, \sigma(m)+1) = 1$. Defining D to be a *quasi-friendly divisor* of N if $\frac{\sigma(N)}{N} = \frac{\sigma(D)+1}{D}$ with $D|N$, our main results characterize all possible quasi-friendly divisors D having two or more distinct prime divisors and satisfying $\gcd(D, \sigma(D)+1) = 1$. In fact, we prove that no such quasi-friendly divisor can have more than two distinct prime divisors.

1 Introduction

The *abundancy index* of a positive integer n is defined to be the ratio $I(n) = \frac{\sigma(n)}{n}$, where σ is the sum-of-divisors function defined by $\sigma(n) = \sum_{d|n} d$. If $I(n)$ is an integer, then n is said to be *multiply perfect* and in the case where $I(n) = 2$, the integer n is said to be *perfect*. All known perfect numbers are even, having the form $2^{p-1}(2^p - 1)$, where $2^p - 1$

is a Mersenne prime (meaning p must be prime, as well). Whether or not there exists an odd perfect number is a long-standing open question, although there are a number of results restricting the form and size of an odd perfect number should one exist. For example, if an odd perfect number exists, then it must be larger than 10^{1500} and have at least 101 (not necessarily distinct) prime divisors [7, 3]. It must have at least 10 distinct prime divisors [6]. Probably the most well-known result about odd perfect numbers is Euler's characterization which states that an odd perfect number must have the form $p^{\alpha}m^2$, where p, m and α are positive integers, p is a prime not dividing m, and $p \equiv \alpha \equiv 1 \pmod 4$.

In fact, the existence of an odd perfect number is equivalent to the existence of a positive integer N satisfying $\sigma(N)/N = (2p^{\alpha}(p-1))/(p^{\alpha+1}-1)$, where, again, N, p, and α are positive integers, p is a prime not dividing m, and $p \equiv \alpha \equiv 1 \pmod 4$ [4]. Hence, the existence of odd perfect numbers can be better understood with a greater understanding of the map $I : \mathbb{Z}_{\geq 1} \to \mathbb{Q}$ defined by $I(n) = \sigma(n)/n$. It is generally very difficult to determine whether a given rational number $k/m > 1$ is in the image of the map I, meaning k/m is an *abundancy ratio*, or whether $k/m \notin \text{Image}(I)$, in which case k/m is said to be an *abundancy outlaw* or *outlaw*. However, there are partial results that can be used to characterize certain families of abundancy outlaws within the rationals [8, 10]. For example, C. W. Anderson proved that if $k, m \in \mathbb{Z}_{\geq 1}$ satisfy $\gcd(k, m) = 1$ with $m < k < \sigma(m)$, then $\frac{k}{m}$ is an outlaw, see [1] or [11]. Hence, the object of our study, $\frac{\sigma(m)+1}{m}$, is the smallest ratio for a fixed denominator m where the abundancy status is unknown. In the early 1970's Anderson conjectured that the set $\text{Image}(I)$ is a recursive set, meaning that there exists a recursive algorithm that can be employed to determine whether or not a given rational number k/m is an outlaw [2]. This remains an open problem today.

A well-known open problem related to perfect numbers is whether or not $\frac{5}{3} = \frac{\sigma(3)+1}{3}$ is an abundancy ratio. The question is interesting because if $\frac{\sigma(d)}{d} = \frac{5}{3}$, then $5d$ is an odd perfect number. (It is not hard to show that if such a d were to exist, then $5 \nmid d$, meaning $\frac{\sigma(5d)}{5d} = \frac{\sigma(5)}{5} \cdot \frac{\sigma(d)}{d} = \frac{6}{5} \cdot \frac{5}{3} = 2$. For a complete proof, see [11].) More generally, it is not yet known whether $\frac{p+2}{p} = \frac{\sigma(p)+1}{p}$ is an abundancy ratio for *any* odd prime p.

In this paper we report on our efforts to understand abundancy ratios of the form $\frac{\sigma(N)}{N} = \frac{\sigma(D)+1}{D}$, where D is a positive integer and $\gcd(D, \sigma(D)+1) = 1$. The symbol D was chosen here to emphasize the fact that if $\frac{\sigma(N)}{N} = \frac{\sigma(D)+1}{D}$ for two integers N and D where D and $\sigma(D)+1$ are relatively prime, then D must be a divisor of N. (This follows from Property 12 in Section 3).

Definition 1. For $M, N \in \mathbb{Z}_{\geq 1}$, we say M is a *quasi-friend* to N if

$$\frac{\sigma(M)+1}{M} = \frac{\sigma(N)}{N}.$$

If, additionally, $M|N$ then we say M is a *quasi-friendly divisor* of N.

Remark 2. The term "quasi-friend" was inspired by the notion of *friendly numbers*, which is the term applied to a pair of distinct integers M and N satisfying $\frac{\sigma(M)}{M} = \frac{\sigma(N)}{N}$.[1] Hence, if $\frac{\sigma(M)+1}{M} = \frac{\sigma(N)}{N}$, then in some sense M and N are almost friendly. We decided to avoid the word "almost" here because an *almost perfect number* is a positive integer n satisfying $\sigma(n) = 2n - 1$, falling short of perfection by 1. It seemed more fitting to use the combining form "quasi" because a *quasiperfect number* is a positive integer n exceeding perfection, with $\sigma(n) = 2n + 1$.

Remark 3. A pair of integers M, N is required when defining a quasi-friendly divisor. A quasi-friendly divisor M of N is not necessarily a quasi-friendly divisor of other multiples of M. For example, $M = 1$ divides every integer, but it is a quasi-friendly divisor of only the perfect numbers (since N perfect implies $\frac{\sigma(N)}{N} = 2 = \frac{\sigma(1)+1}{1}$). More generally, it is easy to check that every power of 2 smaller than 2^p is a quasi-friendly divisor of an even perfect number $2^{p-1}(2^p - 1)$, but $D = 2^0 = 1$ is the only quasi-friendly divisor where $\frac{\sigma(D)+1}{D}$ is reduced.

Table 1 of Section 2 presents all $N < 10^{10}$ having a quasi-friendly divisor D with $\frac{\sigma(D)+1}{D}$ reduced. The data reveals families of even and odd quasi-friendly divisors. Motivated by the data, we prove there are infinitely many abundancy ratios of the form $\frac{\sigma(D)+1}{D}$. In Section 3 we present some preliminary properties and propositions relating to the σ function used in proving the results in Sections 4 and 5. Our main results characterize all possible pairs (D, N), where D has at least two distinct prime divisors and D is a quasi-friendly divisor of N with $\frac{\sigma(D)+1}{D}$ reduced. Our results confirm what the data reveals. All instances of D appearing in the table have at most two distinct prime divisors, and we prove that if D has three or more distinct prime divisors and $\gcd(D, \sigma(D)+1) = 1$, then $\frac{\sigma(D)+1}{D}$ is an abundancy outlaw. We also completely characterize all quasi-friendly divisors D of an integer N when D has precisely two distinct prime divisors. The only other case—where D is the power of a single prime—remains unresolved. We do characterize the even-powered primes, $D = q^{2l}$, appearing in Table 1; they all arise from values of N satisfying $\gcd(D, N/D) = 1$. In Section 5 we present some partial results regarding values of N and D, where $D = q^b$ for some $b \in \mathbb{Z}_{\geq 1}$ with $\gcd(D, N/D) > 1$. Of course, it is this last unresolved case that initially motivated our study.

2 What the data reveals

Before presenting our results, it will be useful to look at some data. Table 1 lists all values of N less than 10^{10} satisfying $\frac{\sigma(N)}{N} = \frac{\sigma(D)+1}{D}$ for some $D \in \mathbb{Z}_{\geq 1}$, with $\gcd(D, \sigma(D)+1) = 1$.[2] The values of such N make up sequence A240991 from the *On-Line Encyclopedia of Integer*

[1] If $M \neq N$ and $\frac{\sigma(M)}{M} = \frac{\sigma(N)}{N}$, then $\gcd(M, \sigma(M)) \neq 1$ and $\gcd(N, \sigma(N)) \neq 1$. Any N satisfying $\gcd(N, \sigma(N)) = 1$ is necessarily *solitary*, meaning N is the only integer having abundancy ratio equal to $\frac{\sigma(N)}{N}$. For example, all primes are solitary by Proposition 14.

[2] The entries in the table were identified by searching all $N \in \mathbb{Z}_{\geq 1}$ up to 10^{10} satisfying $\text{numerator}\big(\text{reduced}\big(\frac{\sigma(N)}{N}\big)\big) = \sigma\big(\text{denominator}\big(\text{reduced}\big(\frac{\sigma(N)}{N}\big)\big)\big) + 1$.

Sequences (OEIS) [9], although it does not appear that anyone had studied the form of these integers prior to this paper.

6, 18, 28, 117, 162, 196, 496, 775, 1458, 8128, 9604, 13122, 15376, 19773, 24025, 88723, 118098, 257049, 470596, 744775, 796797, 1032256, 1062882, 2896363, 6725201, 9565938, 12326221, 14776336, 23059204, 25774633, 27237961, 33550336,... (OEIS A240991)

A close examination of the table reveals clear patterns relating to the factorizations of both N and their quasi-friendly divisors, D. All values of $D > 1$ have the form $D = q^b p^a$, where $p = \sigma(q^b)$ is prime and $b, a \in \mathbb{Z}_{\geq 0}$. In such instances, $N = q^b p^{a+1}$. (In the case where $q = 2$, this means $p = \sigma(2^b) = 2^{b+1} - 1$ is a Mersenne prime.) If $D = 1$, then N is a perfect number. In Theorem 4 below we confirm that integers N of the aforementioned forms do satisfy $\frac{\sigma(N)}{N} = \frac{\sigma(D)+1}{D}$ and then use the result to prove there are infinitely many abundancy ratios of the form $\frac{\sigma(D)+1}{D}$.

Theorem 4. *Let p, q be primes and $a, b \in \mathbb{Z}_{\geq 0}$. If $\sigma(q^b) = p$, then $D = q^b p^a$ is a quasi-friendly divisor of $N = q^b p^{a+1}$.*

Proof. Assuming the hypotheses in the theorem and using the fact that $p\sigma(p^a) + 1 = \sigma(p^{a+1})$ for every prime p, we find

$$\frac{\sigma(N)}{N} = \frac{\sigma(q^b)\sigma(p^{a+1})}{q^b p^{a+1}} = \frac{p\sigma(p^{a+1})}{q^b p^{a+1}} = \frac{p\sigma(p^a) + 1}{q^b p^a} = \frac{\sigma(q^b)\sigma(p^a) + 1}{q^b p^a} = \frac{\sigma(D) + 1}{D}. \qquad \square$$

Definition 5. If $D = q^b p^a$ is a quasi-friendly divisor of $N = q^b p^{a+1}$, where $a, b \in \mathbb{Z}_{\geq 1}$ and $q, p = \sigma(q^b)$ are both primes, then we say that D is a *standard* quasi-friendly divisor of N.

The following proposition (which is not new) characterizes those integers having a prime-valued sum of divisors. The result ensures that the exponent b in $D = q^b p^a$ must be one less than a prime number.

Proposition 6. *For $n \in \mathbb{Z}_{\geq 1}$, if $\sigma(n)$ is prime, then $n = q^b$ for some prime q and some positive integer b, with $b + 1$ prime.*

Proof. If n has two or more distinct prime divisors then $\sigma(n)$ is composite because σ is a multiplicative function. Thus $n = q^b$ is a power of a prime q. Clearly $b \geq 1$ (since $\sigma(1) = 1$ is not prime). Now, if $b + 1$ is composite, say $b + 1 = rs$ with $r, s \in \mathbb{Z}_{\geq 2}$, then

$$\sigma(q^b) = \frac{q^{rs} - 1}{q - 1} = \frac{q^r - 1}{q - 1}\left(q^{r(s-1)} + q^{r(s-2)} + \cdots + q^r + 1\right)$$

is composite. Therefore, $n = q^b$ where $b + 1$ is prime. $\square$

Given a standard quasi-friendly divisor D of N, we now characterize precisely when the ratio $\frac{\sigma(D)+1}{D}$ is reduced.

N	D
$2 \cdot 3$	1
$2 \cdot 3^2$	$2 \cdot 3$
$2^2 \cdot 7$	1
$3^2 \cdot 13$	3^2
$2 \cdot 3^4$	$2 \cdot 3^3$
$2^2 \cdot 7^2$	$2^2 \cdot 7$
$2^4 \cdot 31$	1
$5^2 \cdot 31$	5^2
$2 \cdot 3^6$	$2 \cdot 3^5$
$2^6 \cdot 127$	1
$2^2 \cdot 7^4$	$2^2 \cdot 7^3$
$2 \cdot 3^8$	$2 \cdot 3^7$
$2^4 \cdot 31^2$	$2^4 \cdot 31$
$3^2 \cdot 13^3$	$3^2 \cdot 13^2$
$5^2 \cdot 31^2$	$5^2 \cdot 31$
$17^2 \cdot 307$	17^2
$2 \cdot 3^{10}$	$2 \cdot 3^9$
$3^2 \cdot 13^4$	$3^2 \cdot 13^3$
$2^2 \cdot 7^6$	$2^2 \cdot 7^5$
$5^2 \cdot 31^3$	$5^2 \cdot 31^2$
$3^6 \cdot 1093$	3^6
$2^6 \cdot 127^2$	$2^6 \cdot 127$
$2 \cdot 3^{12}$	$2 \cdot 3^{11}$
$41^2 \cdot 1723$	41^2
$7^4 \cdot 2801$	7^4
$2 \cdot 3^{14}$	$2 \cdot 3^{13}$

N	D
$59^2 \cdot 3541$	59^2
$2^4 \cdot 31^4$	$2^4 \cdot 31^3$
$2^2 \cdot 7^8$	$2^2 \cdot 7^7$
$71^2 \cdot 5113$	71^2
$17^2 \cdot 307^2$	$17^2 \cdot 307$
$2^{12} \cdot 8191$	1
$3^2 \cdot 13^6$	$3^2 \cdot 13^5$
$89^2 \cdot 8011$	89^2
$2 \cdot 3^{16}$	$2 \cdot 3^{15}$
$101^2 \cdot 10303$	101^2
$131^2 \cdot 17293$	131^2
$5^6 \cdot 19531$	5^6
$3^2 \cdot 13^7$	$3^2 \cdot 13^6$
$5^2 \cdot 31^5$	$5^2 \cdot 31^4$
$2 \cdot 3^{18}$	$2 \cdot 3^{17}$
$167^2 \cdot 28057$	167^2
$13^4 \cdot 30941$	13^4
$173^2 \cdot 30103$	173^2
$2^2 \cdot 7^{10}$	$2^2 \cdot 7^9$
$41^2 \cdot 1723^2$	$41^2 \cdot 1723$
$2 \cdot 3^{20}$	$2 \cdot 3^{19}$
$293^2 \cdot 86143$	293^2
$17^4 \cdot 88741$	17^4
$17^2 \cdot 307^3$	$17^2 \cdot 307^2$
$2^{16} \cdot 131071$	1

Table 1: Values of N and D satisfying $\frac{\sigma(N)}{N} = \frac{\sigma(D)+1}{D}$, where $\gcd(\sigma(D)+1, D) = 1$ and $N \leq 10^{10}$ (left: $1 \leq N \leq 10^7$ and right: $10^7 \leq N \leq 10^{10}$)

Proposition 7. *Let $D = q^b p^a$ be a standard quasi-friendly divisor of $N = q^b p^{a+1}$. Then* $\gcd(D, \sigma(D)+1) = 1$ *(or, in other words, the fraction $\frac{\sigma(D)+1}{D}$ is written in lowest terms) if and only if $q \nmid (a+2)$.*

Proof. Since $\sigma(D) + 1 = \sigma(q^b)\sigma(p^a) + 1 = p\sigma(p^a) + 1$ is relatively prime to p, it suffices to characterize when $\sigma(D) + 1$ is divisible by q. Since

$$p = \sigma(q^b) = q^b + q^{b-1} + \cdots + q + 1 \equiv 1 \pmod{q}$$

we see that

$$\sigma(D) + 1 = p\sigma(p^a) + 1 = \sum_{i=0}^{a+1} p^i \equiv \sum_{i=0}^{a+1} 1^i \equiv a + 2 \pmod{q}.$$

Thus $\gcd(D, \sigma(D) + 1) = 1$ if and only if $q \nmid (a + 2)$. □

Corollary 8. *There are infinitely many abundancy ratios of the form $\frac{\sigma(D)+1}{D}$ (in reduced form).*

Proof. Take $D = 2^1 \cdot 3^a$ where a is odd. Then $a + 2$ is also odd and therefore $2 \nmid a + 2$. Applying Proposition 7, we conclude that $\frac{\sigma(D)+1}{D}$ is an abundancy ratio in reduced form. If one desires D to be odd, take $D = 3^2 \cdot 13^a$ where $3 \nmid (a + 2)$. □

In the rest of this paper, we present results discovered while trying to determine if there are any other integers N (that is, not described in Theorem 4) having a quasi-friendly divisor. We will need some preliminaries first.

3 Preliminaries

The range of the map $I : \mathbb{Z}_{\geq 1} \to (1, \infty)$ is a complicated set. In fact, I is known to be a dense map [5], and the set of abundancy outlaws is also dense in $(1, \infty)$ [2]. Existing results characterizing families of rational numbers as abundancy outlaws make use of some basic properties relating to the sum-of-divisors function. We provide the properties and propositions needed for the current study below.

Property 9. Given $n \in \mathbb{Z}_{\geq 1}$ with n odd, $\sigma(n)$ is odd if and only if n is a square.

Property 10. If $k, N \in \mathbb{Z}_{\geq 1}$ then $\frac{\sigma(kN)}{kN} \geq \frac{\sigma(N)}{N}$, with strict inequality when $k \geq 2$.

Property 11. If p is a prime and e is a positive integer, then

$$\frac{p+1}{p} \leq \frac{\sigma(p^e)}{p^e} < \frac{p}{p-1}.$$

Property 12. If $\frac{\sigma(N)}{N} = \frac{K}{M}$ with $\gcd(K, M) = 1$, then $M | N$.

For the first property observe that if $n = \prod_{i=1}^{k} p_i^{e_i}$, with $p_i > 2$, then $\sigma(n) \equiv \prod_{i=1}^{k}(e_i + 1) \pmod 2$, and $\prod_{i=1}^{k}(e_i + 1) \equiv 1 \pmod 2$ iff $e_i \equiv 0 \pmod 2$ for all $1 \leq i \leq k$, iff n is a square. Property 10 is true because $\frac{\sigma(kN)}{kN} = \frac{\sum_{d|kN} d}{kN} \geq \frac{\sum_{d|N}(kd)}{kN} = \frac{\sum_{d|N} d}{N} = \frac{\sigma(N)}{N}$ for all $k \in \mathbb{Z}_{\geq 1}$, and Property 11 follows from the formula $\frac{\sigma(p^e)}{p^e} = \frac{p^{e+1}-1}{p^e(p-1)}$. Certainly, $\frac{p^{e+1}-1}{p^e(p-1)} < \frac{p^{e+1}}{p^e(p-1)} = \frac{p}{p-1}$, and $\frac{p+1}{p} = \frac{\sigma(p)}{p} \leq \frac{\sigma(p^e)}{p^e}$. Finally, Property 12 follows quickly from the fact that $M|(NK)$ whenever $\sigma(N)/N = K/M$. Since K and M are relatively prime, $M|N$.

While it is seemingly impossible to classify many rational numbers as abundancy ratios or outlaws, there are certain cases where the classification is simple. Proposition 13, which first appeared in [1] but also appears in [11], is particularly helpful.

Proposition 13. *If $k, m \in \mathbb{Z}_{\geq 1}$ are relatively prime and $m < k < \sigma(m)$, then $\frac{k}{m}$ is an abundancy outlaw.*

Proof. Assume $\frac{\sigma(N)}{N} = \frac{k}{m}$, for $k, m, N \in \mathbb{Z}_{\geq 1}$, where k and m are relatively prime. By Property 12, $m|N$, and therefore $\frac{\sigma(N)}{N} \geq \frac{\sigma(m)}{m}$ by Property 10. Hence $\frac{k}{m} \geq \frac{\sigma(m)}{m}$, meaning $k \geq \sigma(m)$. Clearly, if $m < k < \sigma(m)$, then $\frac{k}{m}$ is an outlaw. □

To see Proposition 13 at work, consider the rational numbers $\frac{10}{9}$ and $\frac{11}{9}$. Since $\sigma(9) = 13$ and $9 < 10 < 11 < 13$, both rationals are easily classified as abundancy outlaws. More generally, $\frac{m+1}{m}$ is an outlaw for every composite number m. Taking $k = 12$ and $m = 9$ illustrates that k and m really do need to be relatively prime as $9 < 12 < 13$ while $\frac{12}{9} = \frac{4}{3} = \frac{\sigma(3)}{3}$ is not an outlaw.

Proposition 14. *If $\frac{\sigma(N)}{N} = \frac{m+1}{m}$ for $m, n \in \mathbb{Z}_{\geq 1}$, then either $m = 1$ and N is a perfect number or $N = m$ is prime.*

Proof. If $\frac{\sigma(N)}{N} = \frac{m+1}{m}$ and $m = 1$ then N is clearly perfect. Now assume $m > 1$. If m is composite, then $m = st$ for some $s, t \in \mathbb{Z}_{\geq 1}$ with s and t greater than one. Hence $1, s$ and m are distinct divisors of m, and $\sigma(m) \geq m + s + 1 > m + 1 > m$. Proposition 13 then implies $\frac{m+1}{m}$ is an outlaw, contradicting our assumption that $\frac{\sigma(N)}{N} = \frac{m+1}{m}$. We conclude that m must be prime. By Property 12, the prime m divides N, and therefore $m \leq N$. If $m < N$, then $\frac{\sigma(m)}{m} < \frac{\sigma(N)}{N}$ by Property 10. However $\frac{\sigma(m)}{m} = \frac{\sigma(N)}{N}$, so $N = m$. □

4 Main results

We are now ready to present our main results. Assume D is a quasi-friendly divisor of N, where $\frac{\sigma(D)+1}{D}$ is in reduced form. In the theorems to follow we characterize all possible D having two or more distinct prime divisors; we prove D must be a standard quasi-friendly divisor. (So D cannot have more than two distinct prime divisors.)

In Theorem 15, we also prove D is a standard quasi-friendly divisor of N whenever D and N/D are relatively prime.

Theorem 15. *Let D be a quasi-friendly divisor of N satisfying* $\gcd(D, \sigma(D)+1) = 1$ *and* $\gcd(N/D, D) = 1$. *Then $D = q^{2\ell}$ and $N = q^{2\ell}\sigma(q^{2\ell})$ for some odd prime q and $\ell \in \mathbb{Z}_{\geq 1}$ where $\sigma(q^{2\ell})$ is prime. (Hence D is a standard quasi-friendly divisor of N.)*

Proof. Let $D, N \in \mathbb{Z}_{\geq 1}$ with $\gcd(D, \sigma(D)+1) = \gcd(N/D, D) = 1$ and assume

$$\frac{\sigma(N)}{N} = \frac{\sigma(D)+1}{D}.$$

Since σ is multiplicative $\frac{\sigma(N)}{N} = \frac{\sigma(N/D)}{N/D}\frac{\sigma(D)}{D}$, and thus

$$\frac{\sigma(N/D)}{N/D} = \frac{D}{\sigma(D)}\frac{\sigma(D)+1}{D} = \frac{\sigma(D)+1}{\sigma(D)}.$$

By Proposition 14, $\sigma(D)$ is prime and $N/D = \sigma(D)$, and by Proposition 6, $D = q^b$ for some prime q and some positive integer b, with $b+1$ prime. Observe that q must be odd since $\gcd(2^b, \sigma(2^b)+1) = \gcd(2^b, 2^{b+1}) = 2^b > 1$. Since $q \neq 2$ and $\sigma(q^b)$ is prime, $b+1 \neq 2$, and therefore b is even. Hence $D = q^{2\ell}$ for some $\ell \in \mathbb{Z}_{\geq 1}$, meaning $N/D = \sigma(q^{2\ell})$ and $N = q^{2\ell}\sigma(q^{2\ell})$ as claimed. □

The data presented in Table 1 corroborates Theorem 15. For example, when $D = 17^2$, $N = 17^2\sigma(17^2) = 17^2 \cdot 307$, and 307 is prime. In fact, in all instances in Table 1 where D is the power of a single prime, q^b, b is even and N has the form $N = q^b\sigma(q^b)$ with $\sigma(q^b)$ prime.

Theorem 16. *Assume* $\frac{\sigma(N)}{N} = \frac{\sigma(D)+1}{D}$ *with* $\gcd(D, \sigma(D)+1) = 1$, *where D has two or more distinct prime divisors and* $\gcd(N/D, D)$ *is a positive power of a prime p. Then D is a standard quasi-friendly divisor of N.*

Proof. Assume $\frac{\sigma(N)}{N} = \frac{\sigma(D)+1}{D}$, where $D = kp^e$, $k > 1$, and $p \nmid k$. Given $N = mp^fD$, with $f \geq 1$ and $\gcd(m, D) = 1$, compute: $\frac{\sigma(N)}{N} = \frac{\sigma(m)\sigma(p^fD)}{mp^fD} = \frac{\sigma(D)+1}{D}$. So

$$\begin{aligned}\frac{\sigma(m)}{m} = \frac{p^f(\sigma(D)+1)}{\sigma(p^fD)} &= \frac{p^f\sigma(p^e)\sigma(k) + p^f}{\sigma(p^{e+f})\sigma(k)} \\ &= \frac{(\sigma(p^{e+f}) - \sigma(p^{f-1}))\sigma(k) + p^f}{\sigma(p^{e+f})\sigma(k)} \\ &= \frac{\sigma(p^{e+f})\sigma(k) + p^f - \sigma(p^{f-1})\sigma(k)}{\sigma(p^{e+f})\sigma(k)}.\end{aligned}$$

We consider the cases $m > 1$ and $m = 1$, in turn.

If $m > 1$, then $\frac{\sigma(m)}{m} > 1$ and $p^f - \sigma(p^{f-1})\sigma(k) > 0$. In this case, $\sigma(k) < \frac{p^f}{\sigma(p^{f-1})} < \frac{p^f}{p^{f-1}} = p$. Thus $\gcd(\sigma(k), p) = 1$, meaning $\gcd(\sigma(k), \sigma(p^{e+f})\sigma(k) + p^f - \sigma(p^{f-1})\sigma(k)) = 1$. Next let $c = \gcd(\sigma(p^{e+f}), p^f - \sigma(p^{f-1})\sigma(k))$. Then we can write $\frac{\sigma(m)}{m}$ in reduced form:

$$\frac{\sigma(m)}{m} = \frac{\frac{\sigma(p^{e+f})}{c}\sigma(k) + \frac{p^f - \sigma(p^{f-1})\sigma(k)}{c}}{\frac{\sigma(p^{e+f})}{c}\sigma(k)}.$$

Observe that because $k > 1$,

$$\sigma\big(\frac{\sigma(p^{e+f})}{c}\sigma(k)\big) > \frac{\sigma(p^{e+f})}{c}\sigma(k) + \frac{\sigma(p^{e+f})}{c} > \frac{\sigma(p^{e+f})}{c}\sigma(k) + \frac{p^f - \sigma(p^{f-1})\sigma(k)}{c}.$$

This inequality contradicts Proposition 13, so it must be the case that $m = 1$.

Since $m = 1$, $\frac{\sigma(m)}{m} = \frac{p^f(\sigma(D)+1)}{\sigma(p^f D)} = 1$, and so $p^f(\sigma(D) + 1) = \sigma(p^f D)$. Substituting $D = kp^e$ yields $p^f\sigma(p^e)\sigma(k) + p^f = \sigma(p^{e+f})\sigma(k) = (p^f\sigma(p^e) + \sigma(p^{f-1}))\sigma(k)$. Therefore $p^f = \sigma(p^{f-1})\sigma(k)$. Since $\gcd(p^f, \sigma(p^{f-1})) = 1$, $p^f|\sigma(k)$ and we conclude that $\sigma(k) = p^f$ and $\sigma(p^{f-1}) = 1$. Hence $f = 1$ and $\sigma(k) = p$, a prime. By Proposition 6, $k = q^b$ for some prime q and some positive integer b, with $b + 1$ prime. Thus $D = kp^e = q^b\sigma(q^b)^e$, $N = mp^f D = pD = q^b\sigma(q^b)^{e+1}$, and we conclude that D is a standard quasi-friendly divisor of N. □

Theorem 16 characterizes all quasi-friendly divisors D in which $\gcd(D, \sigma(D) + 1) = 1$, where D has more than one prime divisor while sharing only one of them with N/D. In fact, as we prove next, if $\gcd(N/D, D) = d > 1$, then D *must* share precisely one prime divisor with N/D. This means Theorem 16 characterizes all abundancy ratios $\frac{\sigma(D)+1}{D}$ when D has two or more prime divisors; D must be a standard quasi-friendly divisor. The only remaining case is when $D = q^b$ for some prime q and $b \in \mathbb{Z}_{\geq 1}$, where $\gcd(D, N/D)$ is also a power of q. This case remains unresolved, although we present some partial results in Section 5.

The following lemma will be used in the proof of Theorem 18. The divisor $\bar{d}$ in the lemma will play the role of the smallest multiple of d dividing N/D with $\gcd(N/(\bar{d}D), \bar{d}D) = 1$.

Lemma 17. *Let $\ell \in \mathbb{Z}_{\geq 1}$ and let $p_1, p_2, \ldots, p_\ell$ be distinct primes. Let $D = k\big(\prod_{i=1}^{\ell} p_i^{e_i}\big)$ and let $\bar{d} = \prod_{i=1}^{\ell} p_i^{f_i}$, where each $e_i, f_i, k \in \mathbb{Z}_{\geq 1}$, and $p_i \nmid k$. Then*

$$\sigma(\bar{d}D) - \bar{d}\sigma(D) \leq \bar{d}$$

if and only if $\ell = 1$ and $\sigma(k) \leq \frac{p_1^{f_1}}{\sigma(p_1^{f_1-1})}$.

Proof. First consider the case where $\ell > 1$. Order the primes as $p_1 > p_2 > \cdots > p_\ell$. Note that $\sigma(\bar{d}D)$ is the sum of all the divisors of $\bar{d}D$. On the other hand, $\bar{d}\sigma(D)$ is the sum of those divisors of $\bar{d}D$ that are divisible by $\bar{d}$. Hence $\sigma(\bar{d}D) - \bar{d}\sigma(D)$ is the sum of those divisors of $\bar{d}D$ not divisible by $\bar{d}$. Since $p_\ell^{f_\ell}|\bar{d}$, one such divisor is

$$d' = p_1^{f_1+1}p_2^{f_2}\cdots p_{\ell-1}^{f_{\ell-1}}p_\ell^{f_{\ell-1}},$$

(where $f_1 + 1$ is allowed since $e_1 \geq 1$). Now, since $p_1 > p_\ell$, we have $d' > \bar{d}$, and hence $\sigma(\bar{d}D) - \bar{d}\sigma(D) \geq d' > \bar{d}$.

Now consider the case where $\ell = 1$. Then $D = kp_1^{e_1}$, $\bar{d} = p_1^{f_1}$, and

$$\begin{aligned}\sigma(\bar{d}D) - \bar{d}\sigma(D) &= \sigma(kp_1^{f_1+e_1}) - p_1^{f_1}\sigma(kp_1^{e_1}) \\ &= \sigma(k)\big(p_1^{f_1}\sigma(p_1^{e_1}) + \sigma(p_1^{f_1-1})\big) - p_1^{f_1}\sigma(k)\sigma(p_1^{e_1}) \\ &= \sigma(k)\sigma(p_1^{f_1-1}).\end{aligned}$$

Thus, $\sigma(\bar{d}D) - \bar{d}\sigma(D) \le \bar{d}$ if and only if $\sigma(k)\sigma(p_1^{f_1-1}) \le p_1^{f_1}$, or equivalently, $\sigma(k) \le \frac{p_1^{f_1}}{\sigma(p_1^{f_1-1})}$. □

Theorem 18. *If* $\frac{\sigma(N)}{N} = \frac{\sigma(D)+1}{D}$ *for* $D, N \in \mathbb{Z}_{\ge 1}$ *with* $\gcd(D, \sigma(D)+1) = 1$*, and* $\gcd(N/D, D) = d > 1$*, then* d *has one prime divisor. That is,* D *shares precisely one prime divisor with* N/D*.*

Proof. Assume $\frac{\sigma(N)}{N} = \frac{\sigma(D)+1}{D}$ where $\gcd(N/D, D) = d > 1$ and d has at least two distinct prime divisors. Write $N = m\bar{d}D$, where $\bar{d}$ is the smallest positive integer satisfying $d|\bar{d}$ and $\gcd(m, \bar{d}D) = 1$. Then

$$\frac{\sigma(N)}{N} = \frac{\sigma(m)}{m}\frac{\sigma(\bar{d}D)}{\bar{d}D} = \frac{\sigma(D)+1}{D}.$$

Solving for $\frac{\sigma(m)}{m}$ yields

$$\frac{\sigma(m)}{m} = \frac{\bar{d}\sigma(D) + \bar{d}}{\sigma(\bar{d}D)}.$$

Since $\frac{\sigma(m)}{m} \ge 1$, $\sigma(\bar{d}D) \le \bar{d}\sigma(D) + \bar{d}$ or $\sigma(\bar{d}D) - \bar{d}\sigma(D) \le \bar{d}$. This leads to a contradiction as Lemma 17 ensures $\sigma(\bar{d}D) - \bar{d}\sigma(D) > \bar{d}$ whenever D and N/D share at least two distinct prime divisors. We conclude that D shares precisely one prime divisor with N/D. □

Our results can be used to identify abundancy outlaws not captured by Proposition 13. For example, take $D = 55$. Using Corollary 20 we can identify $\frac{73}{55} = \frac{\sigma(55)+1}{55}$ as an outlaw. As reported in [2], Paul Erdős was aware of this particular outlaw in 1975.

Corollary 19. *If* $D \in \mathbb{N}$ *has three or more distinct prime divisors and* $\gcd(D, \sigma(D)+1) = 1$*, then* $\frac{\sigma(D)+1}{D}$ *is an abundancy outlaw.*

Corollary 20. *If* $D = p_1^{e_1}p_2^{e_2}$*, with* $p_1 < p_2$*, and* $p_2 \ne \sigma(p_1^{e_1})$ *and* $\gcd(D, \sigma(D)+1) = 1$*, then* $\frac{\sigma(D)+1}{D}$ *is an abundancy outlaw.*

5 The unresolved case

If $\frac{\sigma(N)}{N} = \frac{\sigma(q^b)+1}{q^b}$ where q is a prime not dividing N/q^b, then we know from Theorem 15 that $N = q^b\sigma(q^b)$, where $\sigma(q^b)$ is prime. So our interest now resides in the case where $\gcd(q^b, N/q^b) \ne 1$. We present a few partial results relating to the form of an integer N having a quasi-friendly divisor q^b. In particular, we prove that N is necessarily odd, and q must be the smallest prime divisor of N. If $b = 1$, then N must also be a square. Our results consider primes $q > 2$ because if $q = 2$, then $q^b = 2^b$ and $\sigma(q^b) + 1 = 2^{b+1}$ are not relatively prime. The standard quasi-friendly divisors $D = 2^b\sigma(2^b)^e = 2^b(2^{b+1} - 1)^e$ characterized in Theorem 16 exhaust all possibilities for an even quasi-friendly divisor satisfying $\gcd(D, \sigma(D)+1) = 1$.

Lemma 21. *If q is an odd prime and $N, b \in \mathbb{Z}_{\geq 1}$ satisfying $\frac{\sigma(N)}{N} = \frac{\sigma(q^b)+1}{q^b}$, then N is odd.*

Proof. Assume $\frac{\sigma(N)}{N} = \frac{\sigma(q^b)+1}{q^b}$, where q is an odd prime and N is even. Then $2|N$ and by Property 10, $\frac{\sigma(2)}{2} \leq \frac{\sigma(q^b)+1}{q^b}$. Applying Property 11, we conclude that

$$\frac{3}{2} \leq \frac{\sigma(q^b)+1}{q^b} = \frac{\sigma(q^b)}{q^b} + \frac{1}{q^b} < \frac{q}{q-1} + \frac{1}{q} < \frac{q+1}{q-1},$$

or equivalently, $3(q-1) < 2(q+1)$. Thus $q < 5$, and since q is odd $q = 3$. Applying Property 12, we conclude that $3|N$, and since N is even, $6|N$. Hence $\frac{\sigma(6)}{6} \leq \frac{\sigma(N)}{N}$, meaning $2 \leq \frac{\sigma(q^b)+1}{q^b} < \frac{q+1}{q-1}$ or $2(q-1) < q+1$. Thus, $q < 3$, which is a contradiction. We conclude that N must be odd. □

Proposition 22. *If q is an odd prime and $N, b \in \mathbb{Z}_{\geq 1}$ satisfying $\frac{\sigma(N)}{N} = \frac{\sigma(q^b)+1}{q^b}$, then N has at least two distinct prime divisors and q is the smallest prime divisor. Furthermore, if p is a prime divisor of N, $p \neq q$, then $p \geq \sigma(q^b)$.*

Proof. Assume q is an odd prime and $N, b \in \mathbb{Z}_{\geq 1}$ satisfying $\frac{\sigma(N)}{N} = \frac{\sigma(q^b)+1}{q^b}$. By Property 12, $q^b|N$ and $N = mq^c$ for some $c, m \in \mathbb{Z}_{\geq 1}$ with $c \geq b$ and $\gcd(q, m) = 1$. If N has just one prime divisor, then $N = q^c$ and

$$\frac{\sigma(N)}{N} = \frac{\sigma(q^c)}{q^c} = \frac{q^{c+1} - 1}{q^c(q-1)} = \frac{\sigma(q^b) + 1}{q^b}.$$

It is easy to check that $(\sigma(q^b) + 1)(q - 1) > q^{b+1}$ and hence $\frac{\sigma(q^b)+1}{q^b} > \frac{q}{q-1}$. However, by Property 11, $\frac{\sigma(q^c)}{q^c} < \frac{q}{q-1}$. We conclude that N has at least two distinct prime divisors. Next let p be a prime divisor of N distinct from q. Then $N = mq^cp^a$, for some $c, a, m \in \mathbb{Z}_{\geq 1}$ with $\gcd(q^cp^a, m) = 1$, and $\frac{\sigma(N)}{N} = \frac{\sigma(m)}{m}\frac{\sigma(q^c)}{q^c}\frac{\sigma(p^a)}{p^a} = \frac{\sigma(q^b)+1}{q^b}$. Observe $q^bp|q^cp^a$, so by Property 12, $\frac{\sigma(q^bp)}{q^bp} \leq \frac{\sigma(mq^cp^a)}{mq^cp^a}$ and

$$\frac{\sigma(q^b)}{q^b}\frac{p+1}{p} \leq \frac{\sigma(m)}{m}\frac{\sigma(q^c)}{q^c}\frac{\sigma(p^a)}{p^a} = \frac{\sigma(q^b)+1}{q^b}.$$

Thus $\frac{\sigma(q^b)(p+1)}{p} \leq \sigma(q^b) + 1$, meaning $\sigma(q^b) \leq p$. Since the prime divisor p was chosen arbitrarily, the result follows. □

Proposition 23. *If q is an odd prime and N is a positive integer satisfying $\frac{\sigma(N)}{N} = \frac{q+2}{q}$, then N is a square number having at least two distinct prime divisors, and q is the smallest prime divisor.*

Proof. Assume q is an odd prime and $\frac{\sigma(N)}{N} = \frac{\sigma(q)+1}{q} = \frac{q+2}{q}$ for some $N \in \mathbb{Z}_{\geq 1}$. To see that N is a square, observe that N is odd by Lemma 21, so $\sigma(N)q = N(q+2)$ implies that $\sigma(N)$ is odd. By Property 9 we conclude N is a square. The rest of the proposition follows from Proposition 22. □

While Proposition 23 examines the ratio $\frac{\sigma(q^b)+1}{p^b}$ when $b = 1$, in fact, a slight variation of the above proof shows N must be a square number whenever $\frac{\sigma(N)}{N} = \frac{\sigma(q^b)+1}{q^b}$ with b odd. To date, the only known quasi-friendly divisors of the form $D = q^b$ (with $\gcd(D, \sigma(D)+1) = 1$) are those characterized in Theorem 15. That is, all have the form $q^{2\ell}$ where $\ell \in \mathbb{Z}_{\geq 1}$ and $\sigma(q^{2\ell})$ is prime. Whether or not there are any quasi-friendly divisors of the form $q^{2\ell+1}$ is unknown.

6 Conclusion

To summarize our results relating to quasi-friendly divisors D where $\frac{\sigma(D)+1}{D}$ is reduced: we have described the form of all possible D with two distinct prime divisors, and we have demonstrated that D cannot have more than two distinct prime divisors. We've also characterized those integers N having a quasi-friendly divisor $D = q^b$, where q is a prime not dividing N/D. The unresolved case is where D is the power of a single prime and $\gcd(D, N/D) > 1$. Is it possible for a positive integer to have a prime quasi-friendly divisor? This is the question initially motivating our study, and the answer remains beyond our reach. We conjecture that the answer is no.

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