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A Summation Involving the Divisor and GCD Functions

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Abstract

Let N be a positive integer. Dudek asked for an asymptotic formula for the sum of $\tau(\gcd(a,b))$ for all a and b with $ab \le N$. We give an asymptotic result. The approach is partly geometric and differs from the approach used in many recent gcd-sum results.

1 Introduction

Euclid's lemma states that if p is a prime number, and p divides ab, then p divides a or p divides b (see, for example, Hardy and Wright [4, Theorem 3]). In 2017, Dudek [3] quantified the lemma's truth when the prime number requirement is relaxed. He suggested it would be interesting to see an asymptotic formula for

$$S(N) := \sum_{ab \le N} \tau\left(\gcd(a,b)\right).$$

Let N be a positive integer throughout. An asymptotic formula does indeed exist, as we show in our theorem as follows:

Theorem 1. *Let N be a positive integer. Then*

$$S(N) = \zeta(2) N \log N + ((2\gamma - 1)\zeta(2) - 2\theta)N + O\left(\sqrt{N}\right),$$

where

$$\theta = \sum_{d<\infty} \frac{\log d}{d^2}.$$

There has been considerable interest in gcd-sum functions (see Tóth [6] and Haukkanen [5] for surveys). We note that results for a related summation,

$$\sum_{a\leq N} \left(\sum_{b=1}^{N} \tau(\gcd(b, N)) \right),$$

can be inferred from the work of Bordellès [2]. Our theorem is proven differently from the works of Bordellès [2], Haukkanen [5], and Tóth [6] in that we use geometric techniques.

2 Notation and preparatory lemmas

We use the cartesian plane with the normal x and y axes. Throughout the term 'on and under the curve' will be above but not including the x-axis, and to the right but not including the y-axis.

For any integer $n \geq 1$, we let $\tau(n)$ denote the number of divisors of n. As usual the Riemann zeta function is given by

$$\zeta(s) = \sum_{j=1}^{\infty} \frac{1}{j^s},$$

for all complex numbers s whose real part is greater than 1. We recall that the notation $f(x) = O(g(x))$ is equivalent to the assertion that there exists a constant $c > 0$ such that $|f(x)| \leq c|g(x)|$ for all x. Finally, we use $|A|$ to denote the cardinality of a set A.

We will require the following lemmas:

Lemma 2. *Let a and b be positive integers with $ab \leq N$. Then*

$$\tau(\gcd(a, b)) = \sum_{d\leq\sqrt{N}} \sum_{\substack{d|a \\ d|b}} 1. \tag{1}$$

Proof. Let $\tau(\gcd(a, b)) = k$ for some positive integer k. So

$$\tau(\gcd(a, b)) = |\{d_1, \ldots, d_k : d_i | \gcd(a, b)\}|.$$

If $d_i | \gcd(a, b)$ then, by the properties of the greatest common divisor, $d|a$ and $d|b$. Therefore $\{d_i, i = 1, \ldots, k : d_i | \gcd(a, b)\} \subseteq \{d : d|a, d|b\}$ and so

$$\tau(\gcd(a, b)) \leq \sum_{d\leq\sqrt{N}} \sum_{\substack{d|a \\ d|b}} 1. \tag{2}$$

Conversely, suppose that $d_i \in \{d : d|a, d|b\}$. Then $d|a$ and $d|b$. So d divides $\gcd(a,b)$ from the definition of the greatest common divisor. So

$$\{d_i, \dots d_k : d_i | \gcd(a,b)\} \supseteq \{d : d|a, d|b\}.$$

Therefore

$$\tau(\gcd(a,b)) \geq \sum_{d \leq \sqrt{N}} \sum_{\substack{d|a \\ d|b}} 1,$$

which proves the lemma. □

The divisor summatory function have been well studied. For our purposes it will suffice to use the following (see, for example, Hardy and Wright [4, Notes to Chapter XVIII]):

Lemma 3.

$$\sum_{x \leq N} \tau(x) = N \log N + (2\gamma - 1)N + O\left(N^{\kappa}\right),$$

where $1/4 \leq \kappa < 1/2$ *for sufficiently large* N*.*

The final lemma formalizes the key insight; every point (x, y) on and under the curve $xy \leq N$ contributes exactly 1 to the right hand side of (1).

Lemma 4. *Fix both* N *and* $d \leq N$ *positive numbers. Then*

$$\sum_{ab \leq N} \sum_{\substack{d|a \\ d|b}} 1 = \sum_{c \leq N/d^2} \tau(c).$$

Proof. Let

$$J = \{(a, b) : ab \leq N, d|a, d|b\}$$

and

$$K = \{r : r|c \text{ for some } c \leq N/d^2\}.$$

It suffices to show that $|J| = |K|$. Suppose $(a, b) \in J$. So $a = rd$ and $b = sd$ where r and s and both positive integers. Since $ab \leq N$ we have $1 \leq rsd^2 \leq N$ and so $1 \leq rs \leq N/d^2$ (noting that rs is an integer and so $1/d^2 \leq rs$ implies $1 \leq rs$). So the point (r, s) is an integer point on and under the curve $xy = N/d^2$. Thus r divides some c with $1 \leq c \leq N/d^2$. So $r \in K$ from which it follows that $|J| \leq |K|$. The argument can be reversed. This proves the lemma. □

3 Proof of Theorem 1

Using Lemma 2 we have

$$\begin{aligned} S(N) &= \sum_{ab\le N}\sum_{d\le\sqrt{N}}\sum_{\substack{d\mid a\\ d\mid b}} 1 \\ &= \sum_{d\le\sqrt{N}}\sum_{ab\le N}\sum_{\substack{d\mid a\\ d\mid b}} 1. \end{aligned} \tag{3}$$

From Lemma 4 we have, for a fixed d, that

$$\sum_{ab\le N}\sum_{\substack{d\mid a\\ d\mid b}} 1 = \sum_{c\le N/d^2}\tau(c).$$

Substituting into (3) and then using Lemma 3, we obtain

$$\begin{aligned} S(N) &= \sum_{d\le\sqrt{N}}\sum_{c\le N/d^2}\tau(c) \\ &= \sum_{d\le\sqrt{N}}\left(\frac{N}{d^2}\log\left(\frac{N}{d^2}\right)+\frac{(2\gamma-1)N}{d^2}+O\left(\left(\frac{N}{d^2}\right)^{\kappa}\right)\right) \\ &= (N\log N+(2\gamma-1)N)\sum_{d\le\sqrt{N}}\frac{1}{d^2}-2N\sum_{d\le\sqrt{N}}\frac{\log d}{d^2}+\sum_{d\le\sqrt{N}}O\left(\left(\frac{N}{d^2}\right)^{\kappa}\right). \end{aligned} \tag{4}$$

We point out that we have introduced some inefficiency here. We would expect the actual error terms to average out over the summation. But we have resorted to summing upper bounds. Next (see, for example, Apostol [1, Theorem 3.2(b)]) we have

$$\sum_{d\le\sqrt{N}}\frac{1}{d^2}=\zeta(2)-\frac{1}{\sqrt{N}}+O\left(\frac{1}{N}\right).$$

So

$$\begin{aligned} (N\log N+(2\gamma-1)N)\sum_{d\le N}\frac{1}{d^2} &= (N\log N+(2\gamma-1)N)\left(\zeta(2)-\frac{1}{\sqrt{N}}+O\left(\frac{1}{N}\right)\right) \\ &= \zeta(2)N\log N-\sqrt{N}\log N+(2\gamma-1)\zeta(2)N+O\left(\log N\right). \end{aligned} \tag{5}$$

Next, since we have absolute convergence,

$$-2N\sum_{d\le\sqrt{N}}\frac{\log d}{d^2}=-2N\left(\sum_{d<\infty}\frac{\log d}{d^2}-\sum_{d>\sqrt{N}}\frac{\log d}{d^2}\right).$$

Recall that

$$\theta = \sum_{d<\infty} \frac{\log d}{d^2} \text{ throughout.}$$

Then, using Euler's summation formula, we have

$$\sum_{d>\sqrt{N}} \frac{\log d}{d^2} = \frac{\log N}{2\sqrt{N}} + \frac{1}{\sqrt{N}} + O\left(\frac{\log N}{N}\right).$$

So

$$-2N \sum_{d\leq\sqrt{N}} \frac{\log d}{d^2} = -2N\theta + \sqrt{N}\log N + 2\sqrt{N} + O(\log N).$$

Finally, using Apostol [1, Theorem 3.2], we have

$$\begin{aligned} \sum_{d\leq\sqrt{N}} O\left(\left(\frac{N}{d^2}\right)^{\kappa}\right) &= N^{\kappa} O\left(\sum_{d\leq\sqrt{N}} \frac{1}{d^{2\kappa}}\right) \\ &= N^{\kappa} O\left(N^{1/2-\kappa}\right) \\ &= O\left(\sqrt{N}\right). \end{aligned} \tag{6}$$

Substituting (5) and (6) into (4) completes the proof.

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