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Derangements and Alternating Sum of Permutations by Integration

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Abstract

Let $P(n,j)$ denote the number of j-permutations of n objects. In this paper we obtain the generating function for the alternating sequence $(-1)^j P(n,j)$. Our method gives an integral representation for the difference $D_n - \frac{n!}{e}$, where D_n denotes the number of derangements on n objects. Using this integral representation, we compute the moments of this difference, and we also get an asymptotic expansion for D_n with coefficients in terms of the Bell numbers B_n. We also give a simple proof of the irrationality of e.

1 Introduction and summary of results

Let $C(n,j)$ denote the number of j-combinations of n objects, and $P(n,j)$ denote the number of j-permutations of n objects, counting the number of ways to choose an ordered selection of j items from a set of n items. Many summation identities concerning $C(n,j)$ can be found in the literature. For example, see [4, Section 0.15], [11, Section 2.3.4], and [12, pp. 343–355] for a list of 334 identities.

In comparison, there are fewer summation identities concerning $P(n,j)$ in the literature. Our first theorem provides an integral representation for the alternating sum over $P(n,j)$. The relation in this theorem is equivalent to one given by Askey and Ismail [2] and Kayll [9]. Our simple proof differs from their and is included for completeness.

Theorem 1. *Let $a \geq 1$ be a fixed real. For any positive integer n let*

$$L_n(a) = \int_1^a \log^n t \, dt. \tag{1}$$

Then, for any integer $n \geq 1$ and for $x \geq 0$,

$$\sum_{j=0}^{n} (-1)^j P(n,j) x^{n-j} = \frac{(-1)^n n! + L_n(e^x)}{e^x}. \tag{2}$$

More precisely, by letting $x = 1$ in (2) we obtain

$$\sum_{j=0}^{n} (-1)^j P(n,j) = \frac{(-1)^n n! + L_n(e)}{e}. \tag{3}$$

As an application of Theorem 1, we continue our study [6] of D_n, the number of derangements on a set of cardinality n. We observe that the alternating sum at the left hand side of (3) and D_n are related as follows:

$$D_n = n! \sum_{j=0}^{n} \frac{(-1)^j}{j!} = (-1)^n n! \sum_{j=0}^{n} \frac{(-1)^{n-j}}{(n-j)!} = (-1)^n \sum_{j=0}^{n} (-1)^j P(n,j).$$

Thus, by considering the relation (3), we obtain

$$D_n = \frac{n!}{e} + (-1)^n \frac{L_n(e)}{e}, \tag{4}$$

for each integer $n \geq 1$. The relation (4) is true for $n = 0$, too. This relation provides an explicit integral representation for the difference

$$D_n - \frac{n!}{e}.$$

Using this integral representation we compute the moments of this difference, as follows:

Theorem 2. *We have*

$$\sum_{n=1}^{\infty} \left(D_n - \frac{n!}{e} \right) = -1 + \frac{1}{e} + \frac{\mathrm{Ei}(2) - \mathrm{Ei}(1)}{e^2} \approxeq -0.218114, \tag{5}$$

where Ei *denotes the exponential integral function defined by the Cauchy principal value of the integral*

$$\mathrm{Ei}(x) = -\int_{-x}^{\infty} \frac{e^{-z}}{z} \, dz,$$

and

$$\sum_{n=1}^{\infty}\left(D_n-\frac{n!}{e}\right)^2=-\frac{(e-1)^2}{e^2}+\frac{4}{e^2}\int_0^{\frac{1}{2}}h(z)\,dz\approxeq 0.433113, \tag{6}$$

where

$$h(z)=\frac{e^{2z}}{\sqrt{1-z^2}}\arctan\frac{z}{\sqrt{1-z^2}}+\frac{e^{2-2z}}{\sqrt{2z-z^2}}\arctan\frac{z}{\sqrt{2z-z^2}}.$$

Moreover, for each integer $k\geq 1$ the following multiple integral representation holds:

$$\sum_{n=1}^{\infty}\left(D_n-\frac{n!}{e}\right)^k=-\frac{(e-1)^k}{e^k}+\frac{1}{e^k}\int_0^1\cdots\int_0^1\frac{e^{x_1+\cdots+x_k}}{1-(-1)^k x_1\cdots x_k}\,d\mathbf{X}, \tag{7}$$

where $\mathbf{X}$ represents the k-tuple $(x_1,\ldots,x_k)$.

Ismail and Simeonov [8] derived the asymptotics of certain combinatorial numbers defined on multi-sets when the number of sets tends to infinity, but the sizes of the sets remain fixed. Their study includes the asymptotics of generalized derangements, numbers related to k-partite graphs, and exponentially weighted derangements. As another application of Theorem 1, by using the integral representation for the difference $D_n-\frac{n!}{e}$ we deduce a full asymptotic expansion for D_n with coefficients in terms of the Bell numbers B_n.

Theorem 3. *Given any positive integer r, for any integer $n\geq 1$ we have the asymptotic expansions*

$$\frac{L_n(e)}{e}=\sum_{k=1}^{r}(-1)^{k-1}\frac{B_k}{n^k}+O\left(\frac{1}{n^{r+1}}\right), \tag{8}$$

and

$$D_n=\frac{n!}{e}+\sum_{k=1}^{r}(-1)^{n+k-1}\frac{B_k}{n^k}+O\left(\frac{1}{n^{r+1}}\right), \tag{9}$$

where B_k denotes the k-th Bell number and the constant of O-term does not exceed B_{r+1} in both expansions.

2 Two remarks

Remark 4. The relation (3) is an analogue to the identity $\sum_{j=0}^{n}(-1)^jC(n,j)=0$. For an analogue to the identity $\sum_{j=0}^{n}C(n,j)=2^n$, we observe that

$$0<e-\sum_{j=0}^{n}\frac{1}{j!}=\sum_{j=1}^{\infty}\frac{1}{(n+j)!}=\frac{1}{n!}\sum_{j=1}^{\infty}\prod_{k=1}^{j}\frac{1}{n+k}<\frac{1}{n!}\sum_{j=1}^{\infty}\frac{1}{(n+1)^j}=\frac{1}{n\cdot n!}.$$

Thus, for each $n\geq 1$ we obtain

$$\sum_{j=0}^{n}P(n,j)=n!\sum_{j=0}^{n}\frac{1}{j!}=\lfloor e\,n!\rfloor. \tag{10}$$

The author previously introduced some enumerative applications of the relation (10) concerning the number of paths and cycles in complete graphs [5, 7].

Remark 5. The relation (3) enables us to provide a simple proof of the irrationality of the number e. We observe that $0 \leq \log t \leq 1$ for $1 \leq t \leq e$. Thus,

$$0 < L_n(e) \leq \int_1^e dt = e - 1 < e,$$

which implies $0 < \frac{L_n(e)}{e} < 1$ for each positive integer n. Now we assume that e is rational. Let $e = \frac{\alpha}{\beta}$ for some positive integers α and β. The relation (3) with $n = \alpha$ gives

$$\sum_{j=0}^{\alpha} (-1)^j P(n,j) = (-1)^{\alpha}(\alpha - 1)!\beta + \frac{L_\alpha(e)}{e},$$

implying that $\frac{L_\alpha(e)}{e}$ is an integer, a contradiction.

3 Proof of Theorem 1

Proof. Using integration by parts we obtain

$$\int \log^r t \, dt = t \log^r t - r \int \log^{r-1} t \, dt.$$

Thus, the recurrence $L_j(a) = a \log^j a - j L_{j-1}(a)$ holds for any integer $j \geq 1$. Multiplying both sides of this recurrence by $\frac{(-1)^j}{j!}$, we can rewrite it as

$$\frac{(-1)^j}{j!} L_j(a) - \frac{(-1)^{j-1}}{(j-1)!} L_{j-1}(a) = \frac{(-1)^j}{j!} a \log^j a.$$

Summing over $1 \leq j \leq n$ yields

$$\frac{(-1)^n}{n!} L_n(a) - L_0(a) = \sum_{j=1}^{n} \frac{(-1)^j}{j!} a \log^j a.$$

Note that $L_0(a) = a - 1$. Thus,

$$\frac{(-1)^n}{n!} L_n(a) = -1 + \sum_{j=0}^{n} \frac{(-1)^j}{j!} a \log^j a = -1 + \sum_{j=0}^{n} \frac{(-1)^{-j}}{j!} a \log^j a,$$

and

$$L_n(a) = (-1)^{n+1} n! + a \sum_{j=0}^{n} (-1)^{n-j} \frac{n!}{j!} \log^j a.$$

After a change of variables in the summation we get

$$L_n(a) = (-1)^{n+1} n! + a \sum_{j=0}^{n} (-1)^j \, P(n,j) \log^{n-j} a.$$

Letting $a = e^x$ for $x \geq 0$ we get (2), which concludes the proof. □

4 Proof of Theorem 2

Proof. We conclude from (4) that

$$\begin{aligned}
\sum_{n=1}^{\infty} \left(D_n - \frac{n!}{e} \right) &= \sum_{n=1}^{\infty} (-1)^n \, \frac{L_n(e)}{e} = \frac{1}{e} \lim_{N \to \infty} \sum_{n=1}^{N} (-1)^n \, L_n(e) \\
&= \frac{1}{e} \lim_{N \to \infty} \sum_{n=1}^{N} (-1)^n \int_1^e \log^n t \, dt = \frac{1}{e} \lim_{N \to \infty} \int_1^e \sum_{n=1}^{N} (-\log t)^n \, dt \\
&= \frac{1}{e} \lim_{N \to \infty} \int_1^e - \frac{\log t}{1 + \log t} \left(1 + (-\log t)^{N+1} \right) dt.
\end{aligned}$$

Now we use the bounded convergence theorem [3, Theorem 3.26] to interchange the limit and integral in the last relation. Consequently,

$$\begin{aligned}
\sum_{n=1}^{\infty} \left(D_n - \frac{n!}{e} \right) &= -\frac{1}{e} \int_1^e \lim_{N \to \infty} \frac{\log t}{1 + \log t} \left(1 + (-\log t)^{N+1} \right) dt \\
&= -\frac{1}{e} \int_1^e \frac{\log t}{1 + \log t} \left(1 + \lim_{N \to \infty} (-\log t)^{N+1} \right) dt \\
&= -\frac{1}{e} \int_1^e \frac{\log t}{1 + \log t} \, dt = -\frac{1}{e} \int_1^e \frac{\log t}{1 + \log t} \, dt.
\end{aligned}$$

To evaluate the last integral we apply the change of variable $-z = 1 + \log t$, satisfying $t = e^{-1-z}$ and $dt = -t \, dz = -\frac{1}{e} \, e^{-z} dz$. Therefore

$$\begin{aligned}
\int_1^e \frac{\log t}{1 + \log t} \, dt &= \frac{1}{e} \int_{-2}^{-1} \left(1 + \frac{1}{z} \right) e^{-z} \, dz \\
&= \frac{1}{e} \int_{-2}^{-1} e^{-z} \, dz - \frac{1}{e} \left(- \int_{-2}^{-1} \frac{e^{-z}}{z} \right) dz = \frac{e^2 - e}{e} - \frac{\mathrm{Ei}(2) - \mathrm{Ei}(1)}{e}.
\end{aligned}$$

This gives (5). To prove (6) we follow an argument due to LeVeque [10], which has been described by Aigner and Ziegler [1, Chapter 9]. In (1) we apply the change of variable $z = \log t$, satisfying $t = e^z$ and $dt = e^z dz$. Accordingly,

$$L_n(e) = \int_0^1 z^n e^z \, dz. \tag{11}$$

Repeated use of (11) shows that

$$L_n(e)^2 = \left(\int_0^1 x^n e^x \, dx\right)\left(\int_0^1 y^n e^y \, dy\right) = \int_0^1 \int_0^1 (xy)^n \, e^{x+y} \, dA_{x,y}.$$

Hence, we conclude from (4) that

$$\begin{aligned}\sum_{n=1}^{\infty}\left(D_n - \frac{n!}{e}\right)^2 &= -\frac{L_0(e)^2}{e^2} + \frac{1}{e^2}\sum_{n=0}^{\infty} L_n(e)^2 \\ &= -\frac{(e-1)^2}{e^2} + \frac{1}{e^2}\sum_{n=0}^{\infty}\int_0^1\int_0^1 (xy)^n \, e^{x+y}\, dA_{x,y}.\end{aligned}$$

Since the function e^{x+y} is bounded on the region $[0,1]\times[0,1]$, uniform convergence of the geometric series allows us to change the order of sum and integrals. Accordingly,

$$\sum_{n=1}^{\infty}\left(D_n - \frac{n!}{e}\right)^2 = -\frac{(e-1)^2}{e^2} + \frac{1}{e^2}\, I,$$

where

$$I = \int_0^1\int_0^1 \frac{e^{x+y}}{1-xy}\, dA_{x,y}.$$

The same reasoning applies to the case of other moments. Thus, meanwhile we obtain (7). Let us compute I. For this purpose, we apply the change of coordinates. Let $u = \frac{y+x}{2}$ and $v = \frac{y-x}{2}$. We get the new domain of integration from old domain by first rotating it by $-45°$ and then shrinking it by a factor of $\sqrt{2}$. This new domain of integration and the function to be integrated are symmetric with respect to the u-axis. Also, $dA_{x,y} = 2dA_{u,v}$. Therefore,

$$\begin{aligned} I &= 4\int_0^{\frac{1}{2}}\int_0^{u} \frac{e^{2u}}{1-u^2+v^2}\, dv\, du + 4\int_{\frac{1}{2}}^{1}\int_0^{1-u} \frac{e^{2u}}{1-u^2+v^2}\, dv\, du \\ &= 4\int_0^{\frac{1}{2}} \frac{e^{2u}}{\sqrt{1-u^2}} \arctan\frac{u}{\sqrt{1-u^2}}\, du + 4\int_{\frac{1}{2}}^{1} \frac{e^{2u}}{\sqrt{1-u^2}} \arctan\frac{1-u}{\sqrt{1-u^2}}\, du.\end{aligned}$$

Substituting $u = 1 - z$ in the last integral and simplifying yields (6). This is the desired conclusion. □

5 Proof of Theorem 3

Proof. We conclude from the integral representation (11) that

$$L_n(e) = \int_0^1 z^n e^z \, dz = \int_0^1 z^n \sum_{j=0}^{\infty}\frac{z^j}{j!}\, dz = \int_0^1 \sum_{j=0}^{\infty}\frac{z^{n+j}}{j!}\, dz.$$

Since the last sum converges uniformly for $0 \le z \le 1$, we may change the order of sum and integral. Therefore,

$$L_n(e) = \sum_{j=0}^{\infty} \int_0^1 \frac{z^{n+j}}{j!}\, dz = \sum_{j=0}^{\infty} \frac{1}{j!(n+j+1)}.$$

An easy computation shows that

$$\frac{1}{n+b} = \sum_{k=1}^{r} (-1)^{k-1} \frac{b^{k-1}}{n^k} + \frac{(-1)^r}{n+b} \left(\frac{b}{n}\right)^r,$$

for $n + b \neq 0$. If we take $b = j + 1$, then

$$\begin{aligned} L_n(e) &= \sum_{j=0}^{\infty} \sum_{k=1}^{r} \frac{(-1)^{k-1}}{n^k} \frac{(j+1)^{k-1}}{j!} + (-1)^r \sum_{j=0}^{\infty} \frac{1}{j!(n+j+1)} \left(\frac{j+1}{n}\right)^r \\ &= \sum_{k=1}^{r} \frac{(-1)^{k-1}}{n^k} \sum_{j=0}^{\infty} \frac{(j+1)^{k-1}}{j!} + \frac{(-1)^r}{n^r} \sum_{j=0}^{\infty} \frac{(j+1)^r}{j!(n+j+1)}. \end{aligned}$$

Dobiński's formula [13, p. 178] states that the k-th Bell number B_k equals

$$B_k = \frac{1}{e} \sum_{j=0}^{\infty} \frac{j^k}{j!}.$$

On account of this formula, we have

$$\sum_{j=0}^{\infty} \frac{(j+1)^{k-1}}{j!} = \sum_{j=0}^{\infty} \frac{(j+1)^k}{(j+1)!} = \sum_{j=1}^{\infty} \frac{j^k}{j!} = \sum_{j=0}^{\infty} \frac{j^k}{j!} = e\, B_k,$$

and

$$\sum_{j=0}^{\infty} \frac{(j+1)^r}{j!(n+j+1)} \le \frac{1}{n} \sum_{j=0}^{\infty} \frac{(j+1)^r}{j!} = \frac{1}{n} \sum_{j=0}^{\infty} \frac{(j+1)^{r+1}}{(j+1)!} = \frac{e\, B_{r+1}}{n}.$$

Therefore, we obtain (8). This gives (9) when substituted in (4), and this is precisely the assertion of the theorem. □

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