Journal of Integer Sequences, Vol. 23 (2020),
Article 20.6.7

Explicit Asymptotics for Signed Binomial Sums and Applications to the Carnevale-Voll Conjecture

Laurent Habsieger
Université de Lyon, CNRS UMR 5208
Université Claude Bernard Lyon 1
Institut Camille Jordan
43 boulevard du 11 novembre 1918
69622 Villeurbanne Cedex
France
habsieger@math.univ-lyon1.fr

Abstract

Carnevale and Voll conjectured that $\sum_j (-1)^j \binom{\lambda_1}{j}\binom{\lambda_2}{j} \neq 0$ when λ_1 and λ_2 are two distinct integers. We check the conjecture when either λ_2 or $\lambda_1 - \lambda_2$ is small. We investigate the asymptotic behaviour of the sum when the ratio $r := \lambda_1/\lambda_2$ is fixed and λ_2 goes to infinity. We find an explicit range $r \geq 5.8362$ on which the conjecture is true. We show that the conjecture is almost surely true for any fixed r. For r close to 1, we give several explicit intervals on which the conjecture is also true.

1 Introduction

Carnevale and Voll [1] studied Dirichlet series enumerating orbits of Cartesian products of maps whose orbits distributions are modelled on the distributions of finite index subgroups of free abelian groups of finite ranks. For Cartesian products of more than three maps they establish a natural boundary for meromorphic continuation. For products of two maps, they formulate two combinatorial conjectures that prove the existence of such a natural boundary. These conjectures state that some explicit polynomials have no unitary factors, i.e., polynomial factors that, for a suitable choice of variables, are univariate and have all their zeroes on the unit circle. The polynomials related to their Conjecture A [1] are given by:

$$C_{\lambda_1,\lambda_2}(x,1) = \sum_{j=0}^{\lambda_2} \binom{\lambda_1}{j}\binom{\lambda_2}{j} x^j$$

for positive integers $\lambda_1 \geq \lambda_2$, and the conjectured property is the following.

Conjecture 1. Let $\lambda_1 > \lambda_2$ be two positive integers. Then $C_{\lambda_1,\lambda_2}(-1,1) \neq 0$.

Note that

$$\begin{aligned} C_{\lambda,\lambda}(-1,1) &= \sum_{j=0}^{\lambda} (-1)^j \binom{\lambda}{j}\binom{\lambda}{\lambda-j} = \oint (1-z)^\lambda (1+z)^\lambda \frac{dz}{2i\pi z^{\lambda+1}} = \oint (1-z^2)^\lambda \frac{dz}{2i\pi z^{\lambda+1}} \\ &= \begin{cases} (-1)^{\lambda/2}\binom{\lambda}{\lambda/2}, & \text{if } \lambda \text{ is even;} \\ 0, & \text{if } \lambda \text{ is odd.} \end{cases} \end{aligned}$$

This explains why the case $\lambda_1 = \lambda_2$ is excluded.

Carnevale and Voll [1] reported that Stanton pointed out the following property: the alternating summands have increasing absolute values for $\lambda_1 > \lambda_2(\lambda_2+1) - 1$, which shows the next result.

Proposition 2. *For all* λ_2 *and* $\lambda_1 > \lambda_2(\lambda_2+1) - 1$, *we have* $C_{\lambda_1,\lambda_2}(-1,1) \neq 0$.

For a fixed λ_2, the sum $C_{\lambda_1,\lambda_2}(-1,1)$ is a polynomial in λ_1 of degree λ_2. The first values are $C_{\lambda_1,0}(-1,1) = 1$, $C_{\lambda_1,1}(-1,1) = 1 - \lambda_1$. Moreover, for $2 \leq \lambda_2 \leq 240$, we checked with `Maple` that it is an irreducible polynomial over $\mathbb{Q}$ when λ_2 is even, and that it is the product of $\lambda_1 - \lambda_2$ by an irreducible polynomial over $\mathbb{Q}$ when λ_2 is odd. The even case required much less time (238 seconds) than the odd case (63908 seconds). Since an irreducible polynomial of degree at least 2 cannot have an integer zero, we deduce that the conjecture is true for the first values of λ_2.

Proposition 3. *For all* $1 \leq \lambda_2 \leq 240$ *and* $\lambda_1 > \lambda_2$, *we have* $C_{\lambda_1,\lambda_2}(-1,1) \neq 0$.

The aim of this paper is to study the size of $C_{\lambda_1,\lambda_2}(-1,1)$ when λ_1 and λ_2 are large enough. We shall give explicit estimates in order to extend the range of validity of Conjecture 1.

We start by relating $C_{\lambda_1,\lambda_2}(x,1)$ to a complex integral formula:

$$C_{\lambda_1,\lambda_2}(x,1) = \oint (1+z)^{\lambda_1}(z+x)^{\lambda_2} \frac{dz}{2i\pi z^{\lambda_2+1}},$$

where the path is a simple one around 0. This will always be the case from now on. We thus have

$$(-1)^{\lambda_2} C_{\lambda_1,\lambda_2}(-1,1) = \oint (1+z)^{\lambda_1}(1-z)^{\lambda_2} \frac{dz}{2i\pi z^{\lambda_2+1}}.$$

Put $\lambda = \lambda_2$ and $r = \lambda_1/\lambda_2 > 1$. We get

$$(-1)^{\lambda_2} C_{\lambda_1,\lambda_2}(-1,1) = \oint \exp(\lambda f(z)) \frac{dz}{2i\pi z}\,, \tag{1}$$

with

$$f(z) = f(r,z) := r\log(1+z) + \log(1-z) - \log z\,. \tag{2}$$

We need to find the right path to be able to find the asymptotic behaviour of this kind of integral when λ goes to infinity. Let us take $z = \rho e^{i\theta}$ with $-\pi \le \theta \le \pi$. The parameter ρ will be optimal when f' vanishes on the path. Since

$$f'(z) = \frac{r}{1+z} - \frac{1}{1-z} - \frac{1}{z} = \frac{-rz^2 + (r-1)z - 1}{z(1-z^2)}\,, \tag{3}$$

and $(r-1)^2 - 4r = r^2 - 6r + 1 = (r - 3 - 2\sqrt{2})(r - 3 + 2\sqrt{2})$, we need to distinguish several cases: $1 < r < 3 + 2\sqrt{2}$, $r = 3 + 2\sqrt{2}$ and $r > 3 + 2\sqrt{2}$. We thus define

$$\rho = \rho(r) := \begin{cases} \frac{r-1-\sqrt{r^2-6r+1}}{2r}, & \text{if } r \ge 3+2\sqrt{2}; \\ \frac{1}{\sqrt{r}} = \left|\frac{r-1\pm i\sqrt{-r^2+6r-1}}{2r}\right| & \text{if } r \le 3+2\sqrt{2}. \end{cases} \tag{4}$$

By (1) we want to study the integral

$$I(\lambda) = I(r,\lambda) := \int_{-\pi}^{\pi} \exp(\lambda f(\rho e^{i\theta})) \frac{d\theta}{2\pi}\,. \tag{5}$$

In the case $r > 3 + 2\sqrt{2}$, we find $f(\rho e^{i\theta}) = f(\rho) - M\frac{\theta^2}{2} + o(\theta^3)$ for some positive real number M, when θ goes to 0. We therefore find $I(\lambda) \sim \int_{-\infty}^{\infty} \exp\left(-\lambda M \frac{\theta^2}{2}\right) \frac{d\theta}{2\pi}$. We prove an effective version of this equivalence.

Theorem 4. *For $r > 3 + 2\sqrt{2}$, put $M = M(r) := \rho^2 f''(\rho)$. Then $M > 0$ and we have*

$$\left|\frac{\sqrt{2\pi\lambda M}}{\exp(\lambda f(\rho))} I(\lambda) - 1\right| \le \frac{3(3+2\sqrt{2})\pi^5}{256\lambda M^2} + \frac{5(3+2\sqrt{2})}{24\lambda M^3} + \frac{\sqrt{2}\exp\left(-\lambda M\frac{\pi^2}{2}\right)}{\pi^{3/2}\sqrt{\lambda M}}\,.$$

This theorem shows that, for all r, we have $I(\lambda) \ne 0$ for λ large enough. We use Proposition 3 and further tools to deduce a large range of validity for Conjecture 1.

Theorem 5. *For $\lambda_1 \ge 5.8362\lambda_2 > 0$, we have $C_{\lambda_1,\lambda_2}(-1,1) \ne 0$.*

Note that the value 5.8362 is close to $3 + 2\sqrt{2} = 5.82842\ldots$, the limit of the method.

Since $r = \lambda_1/\lambda_2$ is a rational number, the case $r = 3 + 2\sqrt{2}$ cannot occur, so we shall not give it in detail. The same method would provide an effective version of the equality

$$\frac{2^{2/3}3^{1/6}\pi(\sqrt{2}+1)^{1/3}}{\Gamma(1/3)} \times \frac{\lambda^{1/3}}{2^{(2+\sqrt{2})\lambda}} I(3+2\sqrt{2},\lambda) = 1 + O\left(\frac{1}{\lambda^{1/6}}\right)\,.$$

In the case $1 < r < 3+2\sqrt{2}$, the situation is quite different. There are two conjugate points on the integrating circle where f' vanishes. Their contributions partially cancel each other, so we cannot get an exact equivalent term: for some choices of (λ_1, λ_2), the implied constant may be really small. The analog of Theorem 4 has indeed the following form.

Theorem 6. *For $1 < r < 3+2\sqrt{2}$, define $\gamma_1 = \arccos\frac{3r-1}{2\sqrt{2r}}$, $\gamma_2 = -\arccos\frac{r-3}{2\sqrt{2}}$ and $\gamma_3 = \frac{1}{2}\arcsin\frac{(r-1)^2}{4r}$. For $\lambda \geq \frac{512r^{3/2}}{(r+1)(-r^2+6r-1)^{3/2}}$, we have*

$$\left|\frac{(-r^2+6r-1)^{1/4}\sqrt{\pi\lambda}}{2^{1+\frac{(r+1)\lambda}{2}}}I(\lambda) - \cos\left((r\gamma_1+\gamma_2)\lambda+\gamma_3\right)\right| \leq \frac{16336}{\sqrt{\lambda}(-r^2+6r-1)^{11/4}}.$$

It seems quite difficult to find a lower bound for $|\cos(\lambda_1\gamma_1+\lambda_2\gamma_2+\gamma_3)|$ for every λ_1, λ_2, and thus to show that $I(\lambda) \neq 0$. However, we can upper bound the number of possible exceptions.

Theorem 7. *For $r > 1$, we have*

$$\#\{(\lambda_1,\lambda_2) \ : \ \lambda_1 = r\lambda_2\,,\ C_{\lambda_1,\lambda_2}(-1,1)\neq 0\,,\ \lambda_2\leq x\}$$
$$\leq \begin{cases} O_r(1), & \text{if } r > 3+2\sqrt{2};\\ \frac{102644}{(-r^2+6r-1)^{11/4}\log\left(\frac{1+\sqrt{5}}{2}\right)}\sqrt{x}\log x + O_r(\sqrt{x}), & \text{if } 1 < r < 3+2\sqrt{2}.\end{cases}$$

Note that this theorem implies that, for a fixed $r > 1$, we have $C_{\lambda_1,\lambda_2}(-1,1)\neq 0$ almost surely. We can get more explicit estimates when r is close to 1, following a suggestion of Dennis Stanton. This case is also of special interest since $C_{\lambda_1,\lambda_2}(-1,1)$ can be reduced to a sum with at most $\left\lfloor\frac{\lambda_1}{2}\right\rfloor - \left\lceil\frac{\lambda_2}{2}\right\rceil$ terms, using a hypergeometric transformation. This enables us to prove the following analogue of Proposition 3.

Proposition 8. *For all $1 \leq \lambda_1-\lambda_2 \leq 701$, we have $C_{\lambda_1,\lambda_2}(-1,1)\neq 0$.*

We then prove the following theorems.

Theorem 9. *For $1 < \lambda_1/\lambda_2 < 3+2\sqrt{2}$, define $\gamma_1 = \arccos\frac{3\lambda_1-\lambda_2}{2\sqrt{2}\lambda_1}$ and $\gamma_2 = -\arccos\frac{\lambda_1-3\lambda_2}{2\sqrt{2}\lambda_2}$. Assume $\lambda_1-\lambda_2 \geq 702$. We then have $\left|\frac{\sqrt{\pi\lambda_2}}{2^{\frac{\lambda_1+\lambda_2+1}{2}}}I(\lambda_2) - \cos(\lambda_1\gamma_1+\lambda_2\gamma_2)\right| \leq 0.0165$ for $\lambda_1-\lambda_2\leq\sqrt{8\pi\lambda_2}$, and*

$$\sqrt{\lambda_2}\left|\frac{\sqrt{\pi\lambda_2}}{2^{\frac{\lambda_1+\lambda_2+1}{2}}}I(\lambda_2) - \cos(\lambda_1\gamma_1+\lambda_2\gamma_2)\right| \leq \begin{cases} 1.05882, & \text{if } \log\lambda_2 \leq \lambda_1-\lambda_2\leq\sqrt{\pi\lambda_2};\\ 1.30775, & \text{if } \sqrt{\pi\lambda_2} \leq \lambda_1-\lambda_2\leq\sqrt{2\pi\lambda_2};\\ 1.50929, & \text{if } \sqrt{2\pi\lambda_2} \leq \lambda_1-\lambda_2\leq\sqrt{3\pi\lambda_2};\\ 1.68876, & \text{if } \sqrt{3\pi\lambda_2} \leq \lambda_1-\lambda_2\leq\sqrt{4\pi\lambda_2};\\ 1.85482, & \text{if } \sqrt{4\pi\lambda_2} \leq \lambda_1-\lambda_2\leq\sqrt{5\pi\lambda_2};\\ 2.01189, & \text{if } \sqrt{5\pi\lambda_2} \leq \lambda_1-\lambda_2\leq\sqrt{6\pi\lambda_2};\\ 2.1626, & \text{if } \sqrt{6\pi\lambda_2} \leq \lambda_1-\lambda_2\leq\sqrt{7\pi\lambda_2};\\ 2.30865, & \text{if } \sqrt{7\pi\lambda_2} \leq \lambda_1-\lambda_2\leq\sqrt{8\pi\lambda_2}.\end{cases}$$

Theorem 10. *We have* $C_{\lambda_1,\lambda_2}(-1,1) \neq 0$ *in the following cases:*

- $\lambda_1+\lambda_2 \equiv 0 \pmod 4$*:* $\lambda_2 < \lambda_1 \le \lambda_2+\sqrt{2\pi\lambda_2}-1.0443$ *or* $\lambda_2+\sqrt{2\pi\lambda_2}+3.1407 \le \lambda_1 \le \lambda_2+\sqrt{6\pi\lambda_2}-0.9275$*;*

- $\lambda_1+\lambda_2 \equiv 1 \pmod 4$*:* $\lambda_2 < \lambda_1 \le \lambda_2+\sqrt{3\pi\lambda_2}-0.984$ *or* $\lambda_2+\sqrt{3\pi\lambda_2}+3.8433 \le \lambda_1 \le \sqrt{7\pi\lambda_2}-0.9231$*;*

- $\lambda_1+\lambda_2 \equiv 2 \pmod 4$*,* $\lambda_2+\max\left(702, 2.0582\lambda_2^{1/4}\right) \le \lambda_1 \le \lambda_2+\sqrt{2\pi\lambda_2}-0.9535$ *or* $2\sqrt{\pi\lambda_2}+4.5938 \le \lambda_1-\lambda_2 \le 2\sqrt{2\pi\lambda_2}-0.9218$*;*

- $\lambda_1+\lambda_2 \equiv 3 \pmod 4$*:* $\lambda_2 < \lambda_1 \le \lambda_2+\sqrt{\pi\lambda_2}-1.1958$ *or* $\lambda_2+\sqrt{\pi\lambda_2}+2.5913 \le \lambda_1 \le \lambda_2+\sqrt{5\pi\lambda_2}-0.9367$.

In Theorem 10 we study what happens before and after the first gap. The results obtained show that we miss at most six values in the first gap, which is quite small.

In the next section we study the case $r > 3+2\sqrt{2}$. We investigate the case $1 < r < 3+2\sqrt{2}$ in Section 3, and focus on the case r close to 1 in Section 4. We end this paper with some remarks and conjectures.

Before starting our studies, let us note that

$$\begin{aligned} f''(z) &= -\frac{r}{(1+z)^2}-\frac{1}{(1-z)^2}+\frac{1}{z^2}\,, \\ f'''(z) &= \frac{2r}{(1+z)^3}-\frac{2}{(1-z)^3}-\frac{2}{z^3}\,, \end{aligned} \tag{6}$$

and let us define the even function $g(\theta)=g(r,\theta):=\Re(f(\rho e^{i\theta}))$ and the odd function $h(\theta)=h(r,\theta):=\Im(f(\rho e^{i\theta}))$.

Since $\overline{f(\rho e^{i\theta})}=f(\rho e^{-i\theta})$, we have the useful expressions

$$\int_{-\pi}^{\pi}\exp(\lambda f(\rho e^{i\theta}))\frac{d\theta}{2\pi}=\int_{-\pi}^{\pi}\exp(\lambda g(\theta))\cos\left(\lambda h(\theta)\right)\frac{d\theta}{2\pi}=\int_{0}^{\pi}\exp(\lambda g(\theta))\cos\left(\lambda h(\theta)\right)\frac{d\theta}{\pi}\,. \tag{7}$$

2 The case $r > 3+2\sqrt{2}$

2.1 General properties

A straightforward calculation shows that

$$\rho'(r)=\frac{\sqrt{1-6r+r^2}+1-3r}{2r^2\sqrt{1-6r+r^2}}<0\,, \tag{8}$$

and we get in this case

$$0<\rho<\rho(3+2\sqrt{2})=\sqrt{2}-1\,. \tag{9}$$

From (3) we have

$$r = \frac{1+\rho}{\rho(1-\rho)} \tag{10}$$

and from (6) and (9) we get that

$$M = M(r) := \rho^2 f''(\rho) = \frac{r\rho}{(1+\rho)^2} - \frac{\rho}{(1-\rho)^2} = \frac{1-2\rho-\rho^2}{(1+\rho)(1-\rho)^2} > 0\,. \tag{11}$$

Another straightforward calculation shows that

$$M'(r) = -\rho'(r)\frac{1+\rho+5\rho^2+\rho^3}{(1+\rho)^2(1-\rho)^3} > 0\,, \tag{12}$$

and we get

$$0 < M < 1\,. \tag{13}$$

Let us now state a key lemma, which shows what kind of estimates are needed and how to use them to get results for $I(\lambda)$.

Lemma 11. *Let $\delta \in [0, \pi]$. Assume that*

1. *$g(\theta) - g(0) \leq -KM\frac{\theta^2}{2}$, for some constant $0 < K \leq 1$,*
2. *$\left|g(\theta) - g(0) + M\frac{\theta^2}{2}\right| \leq C_g\theta^4$, for some positive constant C_g,*
3. *$|h(\theta)| \leq C_h|\theta|^3$, for some positive constant C_h,*

for $0 \leq |\theta| \leq \delta$. We then have

$$\left|\frac{\sqrt{2\pi\lambda M}}{\exp\left(\lambda f(\rho)\right)} I(\lambda) - 1\right| < \frac{3C_g}{\lambda K^{5/2}M^2} + \frac{15C_h^2}{2\lambda M^3} + \frac{\sqrt{2}}{\delta\sqrt{\pi\lambda M}}\exp\left(-\lambda M\frac{\delta^2}{2}\right) + \left(1 - \frac{\delta}{\pi}\right)\sqrt{2\pi\lambda M}\exp\left(-KM\frac{\delta^2}{2}\right).$$

Proof. Note that

$$|e^x\cos y - 1| \leq |(e^x-1)\cos y| + 1 - \cos y \leq |x|e^{\max(x,0)} + \frac{y^2}{2} \tag{14}$$

for all real numbers x and y.

Let us use this property with $x = \lambda\left(g(\theta) - g(0) + M\frac{\theta^2}{2}\right)$ and $y = \lambda h(\theta)$. Condition 1 provides the upper bound $\max(x, 0) \leq \lambda(1-K)M\frac{\theta^2}{2}$. Conditions 2 and 3 imply $|x| \leq \lambda C_g\theta^4$

and $|y| \le \lambda C_h |\theta|^3$, respectively. From (14) and these bounds we deduce that

$$\begin{aligned}
\exp\left(-\lambda f(\rho)\right) & \left| \int_{-\delta}^{\delta} \exp(\lambda f(\rho e^{i\theta})) \frac{d\theta}{2\pi} - \int_{-\delta}^{\delta} \exp\left(\lambda\left(f(\rho) - M\frac{\theta^2}{2}\right)\right) \frac{d\theta}{2\pi}\right| \\
&= \left| \int_{-\delta}^{\delta} \left(e^x \cos y - 1\right) \exp\left(-\lambda M \frac{\theta^2}{2}\right) \frac{d\theta}{2\pi}\right| \\
&\le \int_{-\delta}^{\delta} \lambda C_g \theta^4 \exp\left(-\lambda K M \frac{\theta^2}{2}\right) \frac{d\theta}{2\pi} + \int_{-\delta}^{\delta} \frac{\lambda^2 C_h^2 \theta^6}{2} \exp\left(-\lambda M \frac{\theta^2}{2}\right) \frac{d\theta}{2\pi} \\
&< \frac{\lambda C_g \Gamma(5/2)}{2\pi(\lambda K M/2)^{5/2}} + \frac{\lambda^2 C_h^2 \Gamma(7/2)}{4\pi(\lambda M/2)^{7/2}} = \frac{3C_g}{\sqrt{2\pi}\lambda^{3/2}(KM)^{5/2}} + \frac{15 C_h^2}{2\sqrt{2\pi}\lambda^{3/2} M^{7/2}} .
\end{aligned}$$

Since

$$\int_{\delta \le |\theta| \le +\infty} \exp\left(-\lambda M \frac{\theta^2}{2}\right) \frac{d\theta}{2\pi} \le \int_{\delta}^{+\infty} \frac{\theta}{\delta} \exp\left(-\lambda M \frac{\theta^2}{2}\right) \frac{d\theta}{\pi} = \frac{\exp\left(-\lambda M \frac{\delta^2}{2}\right)}{\delta \lambda M \pi}$$

and

$$\int_{-\infty}^{+\infty} \exp\left(-\lambda M \frac{\theta^2}{2}\right) \frac{d\theta}{2\pi} = \frac{1}{\sqrt{2\pi\lambda M}} ,$$

we obtain

$$\left| \frac{\sqrt{2\pi\lambda M}}{\exp\left(\lambda f(\rho)\right)} \int_{-\delta}^{\delta} \exp(\lambda f(\rho e^{i\theta})) \frac{d\theta}{2\pi} - 1 \right| < \frac{3C_g}{\lambda K^{5/2} M^2} + \frac{15 C_h^2}{2\lambda M^3} + \frac{\sqrt{2}}{\delta\sqrt{\pi\lambda M}} \exp\left(-\lambda M \frac{\delta^2}{2}\right) .$$

To deal with the remaining integral $\int_{\delta \le |\theta| \le \pi} \exp(\lambda f(\rho e^{i\theta})) \frac{d\theta}{2\pi}$, we first show that g is increasing on $[-\pi, 0]$ and decreasing on $[0, \pi]$. From the definition

$$g(\theta) = \Re\left(f\left(\rho e^{i\theta}\right)\right) = r \log\left|1 + \rho e^{i\theta}\right| + \log\left|1 - \rho e^{i\theta}\right| - \log \rho$$

and (10) we deduce that

$$g(\theta) = \frac{1+\rho}{2\rho(1-\rho)} \log\left(1 + 2\cos\theta\rho + \rho^2\right) + \frac{1}{2} \log\left(1 - 2\cos\theta\rho + \rho^2\right) - \log \rho$$

and

$$\begin{aligned}
g'(\theta) &= -\frac{1+\rho}{2\rho(1-\rho)} \times \frac{2\rho \sin\theta}{1 + 2\rho\cos\theta + \rho^2} + \frac{1}{2} \times \frac{2\rho\sin\theta}{1 - 2\rho\cos\theta + \rho^2} \\
&= -\frac{\sin\theta(2\rho(1 + 2\rho - \rho^2)(1 - \cos\theta) + (1-\rho^2)(1 - 2\rho - \rho^2))}{(1-\rho)((1+\rho^2)^2 - 4\rho^2\cos^2\theta)} \le 0 ,
\end{aligned}$$

which proves this intermediate result. We therefore have

$$\begin{aligned}
\frac{1}{\exp\left(\lambda f(\rho)\right)} \left| \int_{\delta \le |\theta| \le \pi} \exp(\lambda f(\rho e^{i\theta})) \frac{d\theta}{2\pi} \right| &\le \int_{\delta \le |\theta| \le \pi} \exp\left(\lambda(g(\theta) - g(0))\right) \frac{d\theta}{2\pi} \\
&\le \frac{\pi - \delta}{2\pi} \left(\exp\left(\lambda(g(\delta) - g(0))\right) + \exp\left(\lambda(g(-\delta) - g(0))\right)\right) \le \frac{\pi - \delta}{\pi} \exp\left(-KM\frac{\delta^2}{2}\right) ,
\end{aligned}$$

and the lemma follows. □

2.2 Estimates for g and h

In this subsection, we obtain explicit versions of Conditions 1-3 in Lemma 11. We shall present two kinds of inequalities: a general inequality valid for $r > 3+2\sqrt{2}$, and a more precise one only valid when r is close to $3+2\sqrt{2}$. The first ones will provide applications when r is large enough, and the second ones when r is close to $3+2\sqrt{2}$.

Lemma 12. *For $r > 3+2\sqrt{2}$ and $|\theta| \leq \pi$, we have*

$$g(\theta) - g(0) \leq -\frac{2}{\pi^2} M\theta^2 \,.$$

For $3+2\sqrt{2} < r \leq 7.686899$ and $|\theta| \leq \pi/3$, we have

$$g(\theta) - g(0) < -M\frac{\theta^2}{2} \,.$$

Proof. From the definition of g, we get

$$\begin{aligned} g(\theta) - g(0) &= \frac{r}{2}\log\left(\frac{1+2\rho\cos\theta+\rho^2}{(1+\rho)^2}\right) + \frac{1}{2}\log\left(\frac{1-2\rho\cos\theta+\rho^2}{(1-\rho)^2}\right) \\ &= \frac{r}{2}\log\left(1-\frac{2\rho(1-\cos\theta)}{(1+\rho)^2}\right) + \frac{1}{2}\log\left(1+\frac{2\rho(1-\cos\theta)}{(1-\rho)^2}\right) \,. \end{aligned} \tag{15}$$

Without loss of generality we may assume $\theta > 0$. Since $\log(1+x) \leq x$ for $x > -1$, from (15) we obtain the following upper bound:

$$g(\theta) - g(0) \leq \left(\frac{\rho}{(1-\rho)^2} - \frac{r\rho}{(1+\rho)^2}\right)(1-\cos\theta) = -2M\sin^2\left(\frac{\theta}{2}\right) \leq -2M\left(\frac{\theta}{\pi}\right)^2 \,, \tag{16}$$

which proves the first part of the lemma.

From $\log(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3}$ for $x > -1$, from (11) and (15) we deduce that

$$\begin{aligned} g(\theta) - g(0) \leq &-M(1-\cos\theta) - \left(\frac{\rho^2}{(1-\rho)^4} + \frac{r\rho^2}{(1+\rho)^4}\right)(1-\cos\theta)^2 \\ &+ \left(\frac{\rho^3}{(1-\rho)^6} - \frac{r\rho^3}{(1+\rho)^6}\right)\frac{4(1-\cos\theta)^3}{3} \,. \end{aligned} \tag{17}$$

Let us put

$$\varphi_1(\theta) := \cos\theta - 1 + \frac{\theta^2}{2} - \frac{(1-\cos\theta)^2}{6} - \frac{2(1-\cos\theta)^3}{45} \,,$$

so that $\varphi_1(0) = \varphi_1'(0) = 0$ and $\varphi_1''(\theta) = \frac{2(1-\cos\theta)^3}{5}$. By Taylor's formula and using the parity of φ_1, there exists $t_\theta \in [0, |\theta|]$ such that

$$0 \leq \varphi_1(\theta) = \frac{2(1-\cos t_\theta)^3}{5} \times \frac{\theta^2}{2} \leq \frac{\theta^2(1-\cos\theta)^3}{5} \,. \tag{18}$$

From (17), we get

$$\begin{aligned} g(\theta)-g(0) &\leq M\left(-\frac{\theta^2}{2}+\frac{(1-\cos\theta)^2}{6}+\frac{2(1-\cos\theta)^3}{45}+\frac{\theta^2(1-\cos\theta)^3}{5}\right) \\ &\quad -\left(\frac{\rho^2}{(1-\rho)^4}+\frac{r\rho^2}{(1+\rho)^4}\right)(1-\cos\theta)^2+\left(\frac{\rho^3}{(1-\rho)^6}-\frac{r\rho^3}{(1+\rho)^6}\right)\frac{4(1-\cos\theta)^3}{3} \\ &= -M\frac{\theta^2}{2}+c_1(\rho)(1-\cos\theta)^2+c_2(\rho,\theta)(1-\cos\theta)^3\,. \end{aligned}$$

with

$$\begin{aligned} c_1(\rho) &:= \frac{M}{6}-\left(\frac{\rho^2}{(1-\rho)^4}+\frac{r\rho^2}{(1+\rho)^4}\right)=\frac{1-8\rho+9\rho^2-32\rho^3-9\rho^4-8\rho^5-\rho^6}{6(1+\rho)^3(1-\rho)^4} \\ c_2(\rho,\theta) &:= M\left(\frac{2}{45}+\frac{\theta^2}{5}\right)+\frac{4}{3}\left(\frac{\rho^3}{(1-\rho)^6}-\frac{r\rho^3}{(1+\rho)^6}\right)\,. \end{aligned}$$

For $3+2\sqrt{2}<r\leq 7.686899$, we have $0.19186222\leq\rho(r)<\sqrt{2}-1$, and we check that

$$c_1(\rho)+(1-\cos\theta)c_2(\rho,\theta)\leq c_1(\rho)+\frac{c_2(\rho,\pi/3)}{2}<0$$

for $|\theta|\leq\pi/3$. The lemma follows. □

Lemma 13. *Let $c\in[-1,1]$. For $r>3+2\sqrt{2}$ and $\cos\theta\geq c$, we have*

$$\left|g(\theta)-g(0)+M\frac{\theta^2}{2}\right|\leq C_g\theta^4\,,$$

with

$$C_g:=\max\left(\frac{1-2\rho-\rho^2}{24(1+\rho)(1-\rho)^2},\frac{\rho^4+6\rho^3+2\rho^2+4\rho-1}{24(1+\rho)(1-\rho)^4}+\frac{\rho}{4(1-\rho^2)(1+\rho^2+2\rho c)}\right)\,.$$

Moreover, for $r\leq 7.494$, we have

$$C_g=\frac{\rho^4+6\rho^3+2\rho^2+4\rho-1}{24(1+\rho)(1-\rho)^4}+\frac{\rho}{4(1-\rho^2)(1+\rho^2+2\rho c)}\,.$$

Proof. It follows from (16) that $g(\theta)-g(0)\leq M(\cos\theta-1)$. The upper bound

$$g(\theta)-g(0)+M\frac{\theta^2}{2}\leq M\frac{\theta^4}{24}=\frac{1-2\rho-\rho^2}{24(1+\rho)(1-\rho)^2}\theta^4 \tag{19}$$

follows from (11) and from $\cos\theta\leq 1-\frac{\theta^2}{2}+\frac{\theta^4}{24}$.

Since $\log(1-x) \geq -x - \frac{x^2}{2(1-x)}$ and $\log(1+x) \geq x - \frac{x^2}{2}$ for $x \in (0,1)$, from (15) we deduce the lower bound

$$g(\theta) - g(0) \geq -M(1-\cos\theta) - \left(\frac{r\rho^2}{(1+\rho)^4(1 - \frac{2\rho(1-\cos\theta)}{(1+\rho)^2})} + \frac{\rho^2}{(1-\rho)^4} \right) (1-\cos\theta)^2 \,.$$

From (18) we get

$$\cos\theta - 1 \geq -\frac{\theta^2}{2} + \frac{(1-\cos\theta)^2}{6} + \frac{2(1-\cos\theta)^3}{45} \geq -\frac{\theta^2}{2} + \frac{(1-\cos\theta)^2}{6} \,,$$

and hence $g(\theta) - g(0) + M\frac{\theta^2}{2} \geq -\varphi_2(\rho, c)(1-\cos\theta)^2$, with

$$\begin{aligned} \varphi_2(\rho, c) :&= -\frac{M}{6} + \frac{r\rho^2}{(1+\rho)^4 \left(1 - \frac{2\rho(1-c)}{(1+\rho)^2}\right)} + \frac{\rho^2}{(1-\rho)^4} \\ &= \frac{\rho^4 + 6\rho^3 + 2\rho^2 + 4\rho - 1}{6(1+\rho)(1-\rho)^4} + \frac{\rho}{(1-\rho^2)(1+\rho^2+2\rho c)} \,. \end{aligned}$$

From $(1-\cos\theta)^2 \leq \theta^4/4$, we obtain $g(\theta) - g(0) + M\frac{\theta^2}{2} \geq -\max\left(\varphi_2(\rho,c), 0\right) \frac{\theta^4}{4}$ and therefore

$$\left| g(\theta) - g(0) + M\frac{\theta^2}{2} \right| \leq \max\left(\frac{\varphi_2(\rho,c)}{4}, 0, \frac{1-2\rho-\rho^2}{24(1+\rho)(1-\rho)^2} \right) \theta^4 = C_g \theta^4 \,.$$

We note that

$$\varphi_2(\rho, c) - \frac{M}{6} \geq \varphi_2(\rho, 1) - \frac{M}{6} = \frac{\rho^6 + 5\rho^5 + 3\rho^4 + 14\rho^3 - 3\rho^2 + 5\rho - 1}{2(1-\rho)^4(1+\rho)^3} \geq 0$$

for $\rho \geq 0.2002734$. The second part of the lemma follows. □

Lemma 14. *For $r > 3 + 2\sqrt{2}$ and $\cos\theta \geq c$, we have $|h(\theta)| \leq C_h |\theta|^3$ with*

$$C_h = C_h(r,c) := \max\left(\frac{(1+\rho^2)(\rho^2+4\rho-1)}{6(1-\rho)^3(1+\rho)^2}, \frac{(1+\rho^2)(1-2\rho(1+c)-\rho^2)}{6(1-\rho)((1+\rho^2)^2 - 4\rho^2c^2)} \right) \,.$$

Moreover, for $r \leq 6.537$ and $c \geq 1/2$, we have

$$C_h = \frac{(1+\rho^2)(\rho^2+4\rho-1)}{6(1-\rho)^3(1+\rho)^2} \,.$$

Proof. We have

$$h(\theta) = \frac{f\left(\rho e^{i\theta}\right) - f\left(\rho e^{-i\theta}\right)}{2i}$$

so that $h(0) = 0$ and

$$h'(\theta) = \frac{\rho e^{i\theta} f'\left(\rho e^{i\theta}\right) + \rho e^{-i\theta} f'\left(\rho e^{-i\theta}\right)}{2} = \Re\left(\rho e^{i\theta} f'\left(\rho e^{i\theta}\right)\right)$$
$$= \Re\left(\frac{r\rho e^{i\theta}}{1+\rho e^{i\theta}} - \frac{\rho e^{i\theta}}{1-\rho e^{i\theta}} - 1\right) = \frac{r\rho(\cos\theta+\rho)}{1+\rho^2+2\rho\cos\theta} - \frac{\rho(\cos\theta-\rho)}{1+\rho^2-2\rho\cos\theta} - 1\,.$$

Since $h'(0) = 0$, we find

$$\begin{aligned} h'(\theta) &= \frac{r\rho(\cos\theta+\rho)}{1+\rho^2+2\rho\cos\theta} - \frac{r\rho}{1+\rho} - \frac{\rho(\cos\theta-\rho)}{1+\rho^2-2\rho\cos\theta} + \frac{\rho}{1-\rho} \\ &= -\frac{r\rho(1-\rho)(1-\cos\theta)}{(1+\rho)(1+\rho^2+2\rho\cos\theta)} + \frac{\rho(1+\rho)(1-\cos\theta)}{(1-\rho)(1+\rho^2-2\rho\cos\theta)} \\ &= \left(\frac{\rho(1+\rho)}{(1-\rho)(1+\rho^2-2\rho\cos\theta)} - \frac{1}{1+\rho^2+2\rho\cos\theta}\right) \times (1-\cos\theta)\,. \end{aligned}$$

Since

$$\frac{\rho(1+\rho)}{(1-\rho)(1+\rho^2-2\rho\cos\theta)} - \frac{1}{1+\rho^2+2\rho\cos\theta} \le \frac{\rho(1+\rho)}{(1-\rho)^3} - \frac{1}{(1+\rho)^2} = \frac{(1+\rho^2)(\rho^2+4\rho-1)}{(1-\rho)^3(1+\rho)^2} =: \varphi_3(\rho)\,,$$

we deduce that $h'(\theta) \le \max(\varphi_3(\rho), 0)\frac{\theta^2}{2}$.

Similarly we get $h'(\theta) \ge -\varphi_4(\rho, c)(1-\cos\theta) \ge -\max(\varphi_4(\rho, c), 0)\frac{\theta^2}{2}$ with

$$\varphi_4(\rho, c) := \frac{1}{1+\rho^2+2\rho c} - \frac{\rho(1+\rho)}{(1-\rho)(1+\rho^2-2\rho c)} = \frac{(1+\rho^2)(1-2\rho(1+c)-\rho^2)}{(1-\rho)((1+\rho^2)^2-4\rho^2c^2)}\,.$$

We thus obtain

$$|h'(\theta)| \le \max\left(\varphi_3(\rho), 0, \varphi_4(\rho, c)\right)\frac{|\theta|^2}{2}\,.$$

Since $-\varphi_4 \le \varphi_3$, the first part of the lemma follows by integrating.

For $c \ge 1/2$, note that

$$\varphi_3(\rho) - \varphi_4(\rho, c) \ge \varphi_3(\rho) - \varphi_4(\rho, 1/2) = \frac{(1+\rho^2)(2\rho^6+7\rho^5-3\rho^4-2\rho^3+3\rho^2+7\rho-2)}{(1-\rho)^3(1+\rho)^2(\rho^2+\rho+1)(\rho^2-\rho+1)} \ge 0$$

for $\rho \ge 0.26101$. The second part of the lemma follows. □

2.3 Proof of Theorem 4

In view of Lemma 11, we just need to estimate K, C_g, and C_h, when $\delta = \pi$. Because of Lemma 12, we may choose $K = 4/\pi^2$. By Lemma 13, and using the notation in its proof, we may choose $C_g = \max\left(\frac{M}{24}, \frac{\varphi_2(\rho,-1)}{4}\right)$. By (13) we already know $M/24 < 1/24$. Since

$$\frac{\partial \varphi_2}{\partial \rho}(\rho,-1) = \frac{\rho^5 + 15\rho^4 + 10\rho^3 + 46\rho^2 + 17\rho + 7}{(1+\rho)^2(1-\rho)^5} > 0$$

and $\varphi_2(\sqrt{2}-1,-1) = (3+2\sqrt{2})/2$, we find $C_g \leq (3+2\sqrt{2})/8$.

With the notation in the proof of Lemma 14, we may put $C_h = \max(\varphi_3(\rho), \varphi_4(\rho,-1))/6$. Since

$$\varphi_3(\rho) - \varphi_4(\rho,-1) = \frac{2(1+\rho^2)(\rho^2+2\rho-1)}{(1-\rho)^3(1+\rho)^2} \leq 0$$

and

$$\frac{\partial \varphi_4}{\partial \rho}(\rho,-1) = \frac{1+5\rho+\rho^2+\rho^3}{6(1+\rho)^2(1-\rho)^3} \geq 0\,,$$

we find $C_h \leq \varphi_4(\sqrt{2}-1,-1)/6 = (\sqrt{2}+1)/6$.

These estimates prove the theorem.

2.4 Proof of Theorem 5

Let $\Phi_1(r,\lambda)$ denote the upper bound given in Theorem 4. It follows from (12) that Φ_1 is a decreasing function of r. The function Φ_1 is also obviously a decreasing function of λ. We thus obtain

$$\Phi_1(r,\lambda) \leq \Phi_1(5.941893, 241) < 0.9999978502\,,$$

for $r \geq 5.941893$ and $\lambda \geq 241$. Theorem 4 therefore implies that $I(r,\lambda) \neq 0$ for $r \geq 5.941893$ and $\lambda \geq 241$. Proposition 3 ensures us that $I(r,\lambda) \neq 0$ for r and $\lambda \leq 240$. We thus get $I(r,\lambda) \neq 0$ for $r \geq 5.941893$ and positive integers λ.

We can now assume $r \leq 5.941893$. Let us get a better version of Theorem 4 in this case. By Lemma 12, we may choose $K = 1$ when $\delta \leq \frac{\pi}{3}$. We also put $c = \cos\delta$ in the definitions of C_g and C_h. We then get $\left|\frac{\sqrt{2\pi\lambda M}}{\exp(\lambda f(\rho))} I(\lambda) - 1\right| \leq \Phi_2(r,\lambda,\delta)$, where

$$\Phi_2(r,\lambda,\delta) := \frac{3C_g(r,c)}{\lambda M^2} + \frac{15 C_h(r,c)^2}{2\lambda M^3} + \left(\frac{\sqrt{2}}{\delta\sqrt{\pi\lambda M}} + \left(1 - \frac{\delta}{\pi}\right)\sqrt{2\pi\lambda M}\right)\exp\left(-\lambda M \frac{\delta^2}{2}\right)\,.$$

Let us show that Φ_2 is a decreasing function of r, to get results on an interval rather than at a point. Let us study each term defining Φ_2.

We find

$$\frac{\partial C_g}{\partial \rho}(\rho,c) = \frac{P(\rho,c) + 7(1-\rho)}{6(1+\rho)^2(1-\rho)^5(1+\rho^2+2\rho c)^2}\,,$$

where P is a polynomial in c and ρ with nonnegative coefficients. We obtain $\frac{\partial C_g}{\partial \rho}(\rho,c) \geq 0$, and the first term in Φ_2 is therefore a decreasing function of r by (8) and (12).

We also find

$$\frac{\partial C_h}{\partial \rho}(\rho) = \frac{\rho^5 + 9\rho^4 + 8\rho^3 + 28\rho^2 - \rho + 3}{6(1-\rho)^4(1+\rho)^3} \geq 0\,,$$

and the second term in Φ_2 is also a decreasing function of r by (8) and (12).

The third term is easily seen to be a decreasing function of r by (12) when $\lambda M\delta^2 \geq 1$. Therefore, for $\lambda M\delta^2 \geq 1$, the function $\Phi_2(r, \lambda, \delta)$ is a decreasing function of r and λ, the monotonicity in λ being easy under the condition $\lambda M\delta^2 \geq 1$. Computations then shows $\Phi_2(5.8478, 241, 0.75) < 0.9936$ and $\Phi_2(5.8362, 980, 0.5) < 0.9999$, while $\lambda M\delta^2 > 20$ in these two cases. We thus proved $I(\lambda) \neq 0$ for $r \geq 5.8362$, except when $241 \leq \lambda \leq 979$ and $5.8362 \leq r < 5.8478$. There are a finite number of possible exceptions, corresponding to the cases $241 \leq \lambda_2 \leq 979$ and $5.8362\lambda_2 \leq \lambda_1 < 5.8478\lambda_2$. We checked these cases with `Maple` in 5470 seconds. Here the limitation comes from the size of λ_2 that must be handled by `Maple` in the computations. For our program, the limitation is $\lambda_2 \leq 998$.

3 The case $r < 3 + 2\sqrt{2}$

3.1 General properties

In this case, the derivative f' has exactly two zeroes. These zeroes are the conjugate complex numbers

$$z_r = \frac{r - 1 + i\sqrt{-r^2 + 6r - 1}}{2r} = \rho e^{i\alpha} \quad \text{and} \quad \bar{z}_r = \frac{r - 1 - i\sqrt{-r^2 + 6r - 1}}{2r} = \rho e^{-i\alpha}\,,$$

with $\rho = \frac{1}{\sqrt{r}}$ and $\alpha \in (0, \pi/2)$. Note that

$$\cos\alpha = \frac{r-1}{2\sqrt{r}} \quad \text{and} \quad \sin\alpha = \frac{\sqrt{-r^2+6r-1}}{2\sqrt{r}}\,. \tag{20}$$

Many properties from the previous section can be rewritten. We still have

$$r = \frac{1 + z_r}{z_r(1 - z_r)} \quad \text{and} \quad f''(z_r) = \frac{1 - 2z_r - z_r^2}{(1 + z_r)z_r^2(1 - z_r)^2}\,,$$

from which we find

$$f''(z_r)z_r^2 = \frac{\sqrt{-r^2+6r-1}}{2}\left(\frac{(r+1)\sqrt{-r^2+6r-1} - i(r-1)^2}{4r}\right) = \frac{\sqrt{-r^2+6r-1}}{2}e^{-i\beta} \tag{21}$$

with $\beta \in (0, \pi/2)$. Note that

$$\cos\beta = \frac{(r+1)\sqrt{-r^2+6r-1}}{4r} \quad \text{and} \quad \sin\beta = \frac{(r-1)^2}{4r}\,. \tag{22}$$

We also use (3) to compute

$$\frac{\partial}{\partial\theta}\left(f\left(\rho e^{i\theta}\right)\right) = i\frac{-r\rho^2 e^{2i\theta} + (r-1)\rho e^{i\theta} - 1}{1-\rho^2 e^{2i\theta}} = ir\frac{-e^{2i\theta} + 2e^{i\theta}\cos\alpha - 1}{r - e^{2i\theta}}.$$

We then obtain the useful expressions

$$g'(\theta) + ih'(\theta) = ir\frac{(e^{i\theta} - e^{i\alpha})(e^{-i\alpha} - e^{i\theta})}{r - e^{2i\theta}} = 2ir\frac{\cos\alpha - \cos\theta}{(r-1)\cos\theta - i(r+1)\sin\theta}. \tag{23}$$

As in the previous section, we now state our key lemma.

Lemma 15. *Let $\delta \in [0, \alpha]$. Assume that*

1. $\left|f\left(\rho e^{i\theta}\right) - f\left(\rho e^{i\alpha}\right) + \frac{\sqrt{-r^2+6r-1}}{4}e^{-i\beta}(\theta-\alpha)^2\right| \le C_f|\theta-\alpha|^3$, *for some positive constant C_f,*

2. $g(\theta) - g(\alpha) \le -\frac{(r+1)(-r^2+6r-1)}{16r}(\theta-\alpha)^2 + C_g|\theta-\alpha|^3$, *for some constant $C_g > 0$,*

for $\alpha - \delta \le \theta \le \alpha + \delta$.

We then have

$$\left|\frac{(-r^2+6r-1)^{1/4}\sqrt{\pi\lambda}}{2^{1+\frac{r+1}{2}\lambda}}I(\lambda) - \cos\left((r\gamma_1 + \gamma_2)\lambda + \frac{\beta}{2}\right)\right| \le \frac{128r^2 C_f e^{C_g\lambda\delta^3}}{\sqrt{\pi\lambda}(r+1)^2(-r^2+6r-1)^{7/4}}$$

$$+\frac{8re^{-\frac{(r+1)(-r^2+6r-1)}{16r}\lambda\delta^2}}{\delta\sqrt{\pi\lambda}(r+1)(-r^2+6r-1)^{3/4}} + \frac{(\pi-2\delta)(-r^2+6r-1)^{1/4}\sqrt{\lambda}}{2\sqrt{\pi}}e^{-\frac{(r+1)(-r^2+6r-1)}{16r}\lambda\delta^2 + C_g\lambda\delta^3},$$

with $\gamma_1 = \arccos\frac{3r-1}{2\sqrt{2r}}$, and $\gamma_2 = -\arccos\frac{r-3}{2\sqrt{2}}$.

Proof. Instead of (14), we use the bound

$$|e^z - 1| \le |z|e^{\max(\Re z, 0)}, \tag{24}$$

with $z = f\left(\rho e^{i\theta}\right) - f\left(\rho e^{i\alpha}\right) + \frac{\sqrt{-r^2+6r-1}}{4}e^{-i\beta}(\theta-\alpha)^2$. Conditions 1 and 2 then imply $|z| \le C_f\lambda|\theta-\alpha|^3$ and $\Re z \le C_g\lambda|\theta-\alpha|^3$, using (22). We thus find

$$\left|\int_{\alpha-\delta}^{\alpha+\delta} e^{\lambda\left(f\left(\rho e^{i\theta}\right) - f\left(\rho e^{i\alpha}\right)\right)}\frac{d\theta}{2\pi} - \int_{\alpha-\delta}^{\alpha+\delta} e^{-\frac{\sqrt{-r^2+6r-1}}{4}e^{-i\beta}\lambda(\theta-\alpha)^2}\frac{d\theta}{2\pi}\right|$$

$$= \left|\int_{\alpha-\delta}^{\alpha+\delta}(e^z - 1)e^{-\frac{\sqrt{-r^2+6r-1}}{4}e^{-i\beta}\lambda(\theta-\alpha)^2}\frac{d\theta}{2\pi}\right|$$

$$\le \int_{\alpha-\delta}^{\alpha+\delta} C_f\lambda|\theta-\alpha|^3\exp\left(\lambda\max\left(g(\theta)-g(\alpha), -\frac{\sqrt{-r^2+6r-1}}{4}\cos\beta(\theta-\alpha)^2\right)\right)\frac{d\theta}{2\pi}$$

$$\le C_f\lambda\int_{-\delta}^{\delta}|u|^3\exp\left(-\frac{(r+1)(-r^2+6r-1)}{16r}\lambda u^2 + C_g\lambda|u|^3\right)\frac{du}{2\pi}$$

$$\le C_f\lambda e^{C_g\lambda\delta^3}\int_0^{\pi}u^3\exp\left(-\frac{(r+1)(-r^2+6r-1)}{16r}\lambda u^2\right)\frac{du}{\pi}$$

$$= C_f e^{C_g\lambda\delta^3}\frac{128r^2}{\pi(r+1)^2(-r^2+6r-1)^2\lambda}.$$

Since

$$\left| \int_{|\theta-\alpha|>\delta} e^{-\frac{\sqrt{-r^2+6r-1}}{4} e^{-i\beta}\lambda(\theta-\alpha)^2} \frac{d\theta}{2\pi} \right| \le \int_{\delta}^{+\infty} \frac{u}{\delta} e^{-\frac{\sqrt{-r^2+6r-1}}{4}\cos\beta\lambda u^2} \frac{du}{\pi}$$

$$= \frac{8re^{-\frac{(r+1)(-r^2+6r-1)}{16r}\lambda\delta^2}}{\pi\delta\lambda(r+1)(-r^2+6r-1)} .$$

and

$$\int_{-\infty}^{+\infty} e^{-\frac{\sqrt{-r^2+6r-1}}{4} e^{-i\beta}\lambda(\theta-\alpha)^2} \frac{d\theta}{2\pi} = \frac{e^{i\beta/2}}{(-r^2+6r-1)^{1/4}\sqrt{\pi\lambda}}$$

we get

$$\left| \frac{(-r^2+6r-1)^{1/4}\sqrt{\pi\lambda}}{e^{\lambda g(\alpha)}} \int_{\alpha-\delta}^{\alpha+\delta} e^{\lambda f\left(\rho e^{i\theta}\right)} \frac{d\theta}{2\pi} - e^{i\lambda h(\alpha)+i\beta/2} \right|$$

$$\le \frac{128r^2C_f e^{C_g\lambda\delta^3}}{\sqrt{\pi\lambda}(r+1)^2(-r^2+6r-1)^{7/4}} + \frac{8re^{-\frac{(r+1)(-r^2+6r-1)}{16r}\lambda\delta^2}}{\delta\sqrt{\pi\lambda}(r+1)(-r^2+6r-1)^{3/4}} .$$

By (23) we have

$$g'(\theta) = -\frac{2r(r+1)\sin\theta}{|(r-1)\cos\theta + i(r+1)\sin\theta|^2}(\cos\alpha - \cos\theta) ,$$

which shows that g is increasing on $[0,\alpha]$ and decreasing on $[\alpha,\pi]$. From Condition 2 we deduce that

$$\left| \int_0^{\alpha-\delta} e^{\lambda\left(f\left(\rho e^{i\theta}\right) - f\left(\rho e^{i\alpha}\right)\right)} \frac{d\theta}{2\pi} \right| \le \int_0^{\alpha-\delta} e^{\lambda(g(\theta)-g(\alpha))} \frac{d\theta}{2\pi} \le \frac{\alpha-\delta}{2\pi} e^{\lambda(g(\alpha-\delta)-g(\alpha))}$$

$$\le \frac{\alpha-\delta}{2\pi} e^{-\frac{(r+1)(-r^2+6r-1)}{16r}\lambda\delta^2 + C_g\lambda\delta^3}$$

and

$$\left| \int_{\alpha+\delta}^{\pi} e^{\lambda\left(f\left(\rho e^{i\theta}\right) - f\left(\rho e^{i\alpha}\right)\right)} \frac{d\theta}{2\pi} \right| \le \frac{\pi-\alpha-\delta}{2\pi} e^{-\frac{(r+1)(-r^2+6r-1)}{16r}\lambda\delta^2 + C_g\lambda\delta^3} .$$

We obtain

$$\left| \frac{(-r^2+6r-1)^{1/4}\sqrt{\pi\lambda}}{e^{\lambda g(\alpha)}} \int_0^{\pi} e^{\lambda f\left(\rho e^{i\theta}\right)} \frac{d\theta}{2\pi} - e^{i\lambda h(\alpha)+i\beta/2} \right| \le \frac{128r^2C_f e^{C_g\lambda\delta^3}}{\sqrt{\pi\lambda}(r+1)^2(-r^2+6r-1)^{7/4}}$$

$$+ \frac{8re^{-\frac{(r+1)(-r^2+6r-1)}{16r}\lambda\delta^2}}{\delta\sqrt{\pi\lambda}(r+1)(-r^2+6r-1)^{3/4}} + \frac{(\pi-2\delta)(-r^2+6r-1)^{1/4}\sqrt{\lambda}}{2\sqrt{\pi}} e^{-\frac{(r+1)(-r^2+6r-1)}{16r}\lambda\delta^2 + C_g\lambda\delta^3} .$$

By (7) we have $I(\lambda) = 2\Re\left(\int_0^\pi e^{\lambda f\left(\rho e^{i\theta}\right)} \frac{d\theta}{2\pi}\right)$. and the lemma then follows from (22) and

$$e^{i\gamma_1} = \frac{1+z_r}{|1+z_r|} = \frac{3r-1+i\sqrt{-r^2+6r-1}}{2\sqrt{2}r},$$
$$e^{i\gamma_2} = \frac{z_r^{-1}-1}{|z_r^{-1}-1|} = \frac{r-3-i\sqrt{-r^2+6r-1}}{2\sqrt{2}},$$
$$e^{\lambda g(\alpha)} = |1+z_r|^{r\lambda}|z_r^{-1}-1|^\lambda = \sqrt{2}^{(r+1)\lambda}.$$

□

3.2 Estimates for f and g

Lemma 16. *For $\alpha/2 \leq \theta \leq \pi - \alpha/2$, we have*

$$\left| f\left(\rho e^{i\theta}\right) - f\left(\rho e^{i\alpha}\right) + \frac{\sqrt{-r^2+6r-1}}{4} e^{-i\beta}(\theta-\alpha)^2 \right| \leq 0.33846 \frac{(r+1)^2}{r^2} |\theta - \alpha|^3 .$$

Proof. From (23) we know that

$$g'(\theta) + ih'(\theta) = ir\frac{(e^{i\theta}-e^{i\alpha})(e^{-i\alpha}-e^{i\theta})}{r-e^{2i\theta}}.$$

We also have

$$\frac{e^{-i\alpha}-e^{i\theta}}{r-e^{2i\theta}} = \frac{e^{-i\alpha}-e^{i\alpha}}{r-e^{2i\alpha}} + \frac{(e^{i\theta}-e^{i\alpha})(e^{i(\theta-\alpha)}+1-r-e^{i(\alpha+\theta)})}{(r-e^{2i\theta})(r-e^{2i\alpha})},$$

which gives

$$\left| g'(\theta) + ih'(\theta) - ir\frac{e^{-i\alpha}-e^{i\alpha}}{r-e^{2i\alpha}}(e^{i\theta}-e^{i\alpha}) \right| \leq \frac{\sqrt{-r^2+6r-1}+\sqrt{r}(r-1)}{2\sqrt{r^2+1+\sqrt{r}(r-1)}}(\theta-\alpha)^2$$

since

$$|e^{i(\theta-\alpha)}+1-r-e^{i(\alpha+\theta)}| \leq r-1+2\sin\alpha = r-1+\frac{\sqrt{-r^2+6r-1}}{\sqrt{r}},$$
$$|r-e^{2i\alpha}|^2 = r^2+1-2r\cos(2\alpha) = r^2+1-2r\left(2\frac{(r-1)^2}{4r}-1\right) = 4r,$$
$$|r-e^{2i\theta}|^2 = r^2+1-2r\cos(2\theta) \geq r^2+1-2r\cos(\alpha) = r^2+1-\sqrt{r}(r-1).$$

Moreover we find

$$\left| \frac{e^{-i\alpha}-e^{i\alpha}}{r-e^{2i\alpha}}(e^{i\theta}-e^{i\alpha}) - \frac{i(1-e^{2i\alpha})}{r-e^{2i\alpha}}(\theta-\alpha) \right| \leq \frac{\sin\alpha}{|r-e^{2i\alpha}|}(\theta-\alpha)^2 = \frac{\sqrt{-r^2+6r-1}}{4r}(\theta-\alpha)^2$$

and therefore

$$\left|g'(\theta) + ih'(\theta) + r\frac{1-e^{2i\alpha}}{r-e^{2i\alpha}}(\theta-\alpha)\right|$$
$$\leq \left(\frac{\sqrt{-r^2+6r-1}+\sqrt{r}(r-1)}{2\sqrt{r^2+1+\sqrt{r}(r-1)}} + \frac{\sqrt{-r^2+6r-1}}{4}\right)(\theta-\alpha)^2 \,.$$

We obtain the lemma by using (20), (21), (22), and checking the inequality

$$\frac{\sqrt{-r^2+6r-1}+\sqrt{r}(r-1)}{2\sqrt{r^2+1+\sqrt{r}(r-1)}} + \frac{\sqrt{-r^2+6r-1}}{4} \leq 1.01537\frac{(r+1)^2}{r^2} \,.$$

□

Lemma 17. *For $\alpha/2 \leq \theta \leq \pi - \alpha/2$, we have*

$$g(\theta) - g(\alpha) \leq -\frac{(r+1)(-r^2+6r-1)}{16r}(\theta-\alpha)^2 + \frac{r+1}{4}|\theta-\alpha|^3 \,.$$

Proof. We find

$$\begin{aligned}
g(\theta) - g(\alpha) &= \frac{r}{2}\log\left(\frac{1+\rho^2+2\rho\cos\theta}{1+\rho^2+2\rho\cos\alpha}\right) + \frac{1}{2}\log\left(\frac{1+\rho^2-2\rho\cos\theta}{1+\rho^2-2\rho\cos\alpha}\right) \\
&= \frac{r}{2}\log\left(1+\frac{2\rho(\cos\theta-\cos\alpha)}{1+\rho^2+2\rho\cos\alpha}\right) + \frac{1}{2}\log\left(1-\frac{2\rho(\cos\theta-\cos\alpha)}{1+\rho^2-2\rho\cos\alpha}\right) \\
&= \frac{r}{2}\log\left(1+\rho(\cos\theta-\cos\alpha)\right) + \frac{1}{2}\log\left(1-\rho^{-1}(\cos\theta-\cos\alpha)\right)
\end{aligned}$$

since $2\rho\cos\alpha = 1-\rho^2$. We deduce the upper bound

$$\begin{aligned}
g(\theta) - g(\alpha) &\leq -\frac{r+1}{4}(\cos\theta-\cos\alpha)^2 + \frac{1-r^2}{6\sqrt{r}}(\cos\theta-\cos\alpha)^3 \\
&= \frac{r+1}{4}\left(-(\cos\theta-\cos\alpha)^2 - \frac{4\cos\alpha}{3}(\cos\theta-\cos\alpha)^3\right) \,.
\end{aligned}$$

Let us define $\varphi_5(\theta) = \sin^2\alpha(\theta-\alpha)^2 - (\cos\theta-\cos\alpha)^2 - \frac{4\cos\alpha}{3}(\cos\theta-\cos\alpha)^3$, so that

$$\begin{aligned}
\varphi_5''(\theta) &= 2(\cos\theta-\cos\alpha)\left(6\cos\alpha\cos^2\theta + 2(1-\cos^2\alpha)\cos\theta - 3\cos\alpha\right) \\
&=: 2(\cos\theta-\cos\alpha)p(\cos\theta,\cos\alpha) \,.
\end{aligned}$$

We check that

$$\begin{aligned}
&p(\cos\theta,\cos\alpha) \leq p(\cos(\alpha/2),\cos\alpha) = 3 + (1-\cos^2\alpha)\left(\sqrt{2(1+\cos\alpha)}-3\right) \leq 3 \,, \\
&p(\cos\theta,\cos\alpha) \geq \min_{\substack{-1\leq x\leq 1 \\ 0\leq y\leq 1}} p(x,y) = p(0,1) = -3 \,.
\end{aligned}$$

This gives $|\varphi_5''(\theta)| \leq 6|\theta-\alpha|$ and $\varphi_5(\theta) \leq |\theta-\alpha|^3$, and the lemma follows using (20). □

3.3 Proof of Theorem 6

We use Lemmas 15, 16, 17 to get

$$\left|\frac{(-r^2+6r-1)^{1/4}\sqrt{\pi\lambda}}{2^{1+\frac{r+1}{2}\lambda}}I(\lambda)-\cos\left((r\gamma_1+\gamma_2)\lambda_2+\frac{\beta}{2}\right)\right|\le\frac{24.45e^{(r+1)\lambda\frac{\delta^3}{4}}}{\sqrt{\lambda}(-r^2+6r-1)^{7/4}}$$

$$+\frac{8re^{-\frac{(r+1)(-r^2+6r-1)}{16r}\lambda\delta^2}}{\delta\sqrt{\pi\lambda}(r+1)(-r^2+6r-1)^{3/4}}+\frac{(\pi-2\delta)(-r^2+6r-1)^{1/4}\sqrt{\lambda}}{2\sqrt{\pi}}e^{-\frac{(r+1)(-r^2+6r-1)}{16r}\lambda\delta^2+(r+1)\lambda\frac{\delta^3}{4}},$$

for $\delta\le\alpha/2$. We apply the inequalities $xe^{-x}\le 1$ and $x^3e^{-x}\le 27e^{-3}$ to the second and third term of the right hand side respectively to get

$$\left|\frac{(-r^2+6r-1)^{1/4}\sqrt{\pi\lambda}}{2^{1+\frac{r+1}{2}\lambda}}I(\lambda)-\cos\left((r\gamma_1+\gamma_2)\lambda_2+\frac{\beta}{2}\right)\right|\le\frac{24.45e^{(r+1)\lambda\frac{\delta^3}{4}}}{\sqrt{\lambda}(-r^2+6r-1)^{7/4}}$$

$$+\frac{128r^2}{\delta^3\lambda^{3/2}\sqrt{\pi}(r+1)^2(-r^2+6r-1)^{7/4}}+\frac{55296\sqrt{\pi}r^3e^{(r+1)\lambda\frac{\delta^3}{4}}}{e^3\delta^6\lambda^{5/2}(r+1)^3(-r^2+6r-1)^{11/4}}.$$

We now choose $\delta=\left(\frac{(r+1)\lambda}{8}\right)^{-1/3}$. By hypothesis, we have $\delta\le\frac{\sin\alpha}{2}\le\frac{\alpha}{2}$. We obtain

$$\left|\frac{(-r^2+6r-1)^{1/4}\sqrt{\pi\lambda}}{2^{1+\frac{r+1}{2}\lambda}}I(\lambda)-\cos\left((r\gamma_1+\gamma_2)\lambda_2+\frac{\beta}{2}\right)\right|$$
$$\le\frac{1}{\sqrt{\lambda}(-r^2+6r-1)^{11/4}}\left(24.45e^2(-r^2+6r-1)+\frac{16r^2(-r^2+6r-1)}{\sqrt{\pi}(r+1)}+\frac{864\sqrt{\pi}r^3}{e(r+1)}\right)$$
$$\le\frac{16336}{\sqrt{\lambda}(-r^2+6r-1)^{11/4}},$$

which proves the theorem.

3.4 Proof of Theorem 7

For $r>3+2\sqrt{2}$, Theorem 4 shows that $I(\lambda)\neq 0$ for λ large enough, the implied bound only depending on $M(r)$. The first part of the theorem follows.

For $1<r<3+2\sqrt{2}$, define $\mathcal{S}_r$ as the set of non negative integers λ such that $I(r,\lambda)=0$. As usual, let $\|x\|$ denote the distance of x to the nearest integer. For $\lambda\in\mathcal{S}_r$, we have

$$\left\|(r\gamma_1+\gamma_2)\lambda+\frac{\pi+\beta}{2}\right\|\le\frac{\pi}{2}\left|\sin\left((r\gamma_1+\gamma_2)\lambda+\frac{\pi+\beta}{2}\right)\right|\le\frac{25661}{\sqrt{\lambda}(-r^2+6r-1)^{11/4}},$$

by Theorem 6. For $\lambda,\lambda'\in\mathcal{S}_r$, $\lambda<\lambda'<\lambda+\frac{(-r^2+6r-1)^{11/4}\sqrt{\lambda}}{102644}$, we find

$$\|(r\gamma_1+\gamma_2)(\lambda'-\lambda)\|\le\frac{51322}{\sqrt{\lambda}(-r^2+6r-1)^{11/4}}\le\frac{1}{2(\lambda'-\lambda)}.$$

By Legendre's theorem [3, pp. 27–29], this implies that $\lambda' - \lambda$ is a denominator q_n in the continued fraction expansion of $r\gamma_1 + \gamma_2$. Since $q_n \geq F_{n+1} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right) \geq \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n}$, the number of such denominators less than q is upper bounded by $\frac{\log(q\sqrt{5})}{\log\left(\frac{1+\sqrt{5}}{2}\right)}$, and we get for $x \geq 1$:

$$\#\mathcal{S}_r \cap \left[x, x + \frac{(-r^2+6r-1)^{11/4}\sqrt{x}}{102644}\right]$$
$$\leq \frac{\log \frac{(-r^2+6r-1)^{11/4}\sqrt{5x}}{102644}}{\log\left(\frac{1+\sqrt{5}}{2}\right)} = \varphi_6\left(x + \frac{(-r^2+6r-1)^{11/4}\sqrt{x}}{102644}\right) - \varphi_6(x)\,,$$

with

$$\varphi_6(x) = \frac{102644}{(-r^2+6r-1)^{11/4}\log\left(\frac{1+\sqrt{5}}{2}\right)}\sqrt{x}\log x + O_r(\sqrt{x})\,.$$

The second part of the theorem follows.

4 The case r close to 1

When r goes to 1, the angles α and β go to $\pi/2$ and 0 respectively. So we shall prove specific estimates in this case. Before that, we establish Proposition 8.

4.1 Small values of $\lambda_1 - \lambda_2$

We have

$$\begin{aligned}
C_{\lambda_1,\lambda_2}(-1,1) &= \oint (z+1)^{\lambda_1}(1-z)^{\lambda_2}\frac{dz}{2i\pi z^{\lambda_1+1}} = \oint (1-z^2)^{\lambda_2}(z+1)^{\lambda_1-\lambda_2}\frac{dz}{2i\pi z^{\lambda_1+1}} \\
&= \sum_{\lambda_2 \leq 2j \leq \lambda_1} (-1)^j \binom{\lambda_2}{j}\binom{\lambda_1-\lambda_2}{\lambda_1-2j} \\
&= \frac{\lambda_2!}{\left\lfloor\frac{\lambda_1}{2}\right\rfloor!\left\lfloor\frac{\lambda_2}{2}\right\rfloor!} \sum_{\left\lceil\frac{\lambda_2}{2}\right\rceil \leq j \leq \left\lfloor\frac{\lambda_1}{2}\right\rfloor} (-1)^j \binom{\lambda_1-\lambda_2}{\lambda_1-2j}\frac{\left\lfloor\frac{\lambda_1}{2}\right\rfloor!}{j!}\frac{\left\lfloor\frac{\lambda_2}{2}\right\rfloor!}{(\lambda_2-j)!} \\
&= \frac{\lambda_2!(-1)^{\left\lceil\frac{\lambda_2}{2}\right\rceil}}{\left\lfloor\frac{\lambda_1}{2}\right\rfloor!\left\lfloor\frac{\lambda_2}{2}\right\rfloor!} \sum_{0 \leq j \leq \left\lfloor\frac{\lambda_1}{2}\right\rfloor - \left\lceil\frac{\lambda_2}{2}\right\rceil} (-1)^j \binom{\lambda_1-\lambda_2}{2j+\left\lceil\frac{\lambda_2}{2}\right\rceil-\left\lfloor\frac{\lambda_2}{2}\right\rfloor}\frac{\left\lfloor\frac{\lambda_1}{2}\right\rfloor!}{(\left\lceil\frac{\lambda_2}{2}\right\rceil+j)!}\frac{\left\lfloor\frac{\lambda_2}{2}\right\rfloor!}{(\left\lfloor\frac{\lambda_2}{2}\right\rfloor-j)!}\,.
\end{aligned}$$

When $\lambda_1 - \lambda_2$ is fixed, the last sum is a polynomial $\widetilde{C}$ in $\left\lfloor\frac{\lambda_2}{2}\right\rfloor$ of degree at most $\left\lfloor\frac{\lambda_1}{2}\right\rfloor - \left\lceil\frac{\lambda_2}{2}\right\rceil$, also depending on the parity of λ_1 and λ_2. More precisely we consider the four families of polynomials

$$\widetilde{C}_{l,0,0}(k) = \sum_{0\le j\le l} (-1)^j \binom{2l}{2j} \frac{(k+l)!}{(k+j)!} \frac{k!}{(k-j)!} \qquad \text{for } (\lambda_2,\lambda_1) = (2k, 2k+2l),$$

$$\widetilde{C}_{l,0,1}(k) = \sum_{0\le j\le l} (-1)^j \binom{2l+1}{2j} \frac{(k+l)!}{(k+j)!} \frac{k!}{(k-j)!} \qquad \text{for } (\lambda_2,\lambda_1) = (2k, 2k+2l+1),$$

$$\widetilde{C}_{l,1,0}(k) = \sum_{0\le j\le l-1} (-1)^j \binom{2l}{2j+1} \frac{(k+l)!}{(k+1+j)!} \frac{k!}{(k-j)!} \qquad \text{for } (\lambda_2,\lambda_1) = (2k+1, 2k+2l+1),$$

$$\widetilde{C}_{l,1,1}(k) = \sum_{0\le j\le l} (-1)^j \binom{2l+1}{2j+1} \frac{(k+l+1)!}{(k+1+j)!} \frac{k!}{(k-j)!} \qquad \text{for } (\lambda_2,\lambda_1) = (2k+1, 2k+2l+2).$$

The first values are given by $\widetilde{C}_{0,0,0}(k) = \widetilde{C}_{1,0,0}(k) = \widetilde{C}_{0,0,1}(k) = \widetilde{C}_{0,1,1}(k) = 1$, $\widetilde{C}_{0,1,0}(k) = 0$, $\widetilde{C}_{1,1,0}(k) = 2$, $\widetilde{C}_{2,1,0}(k) = 8$, $\widetilde{C}_{1,0,1}(k) = -2k+1$, $\widetilde{C}_{1,1,1}(k) = 2k+6$. We can compute the leading terms of each polynomial to show that the degree of $\widetilde{C}_{l,0,0}(k)$, $\widetilde{C}_{l,0,1}(k)$, $\widetilde{C}_{l,1,1}(k)$ is at least 2 for $l \ge 2$, and that the degree of $\widetilde{C}_{l,1,0}(k)$ is at least 2 for $l \ge 3$:

$$\begin{aligned}
\widetilde{C}_{l,0,0}(k) &= \sum_{0\le j\le l} (-1)^i \binom{2l}{2j} \left(k^l + \left(\binom{l+1}{2} - j^2 \right) k^{l-1} + \cdots \right) \\
&= \Re \left(\sum_{0\le m\le 2l} i^m \binom{2l}{m} \left(k^l + \left(\binom{l+1}{2} - \frac{m^2}{4} \right) k^{l-1} + \cdots \right) \right) \\
&= \Re \left((1+i)^{2l} k^l + \left(\binom{l+1}{2} - \frac{l(1+2li)}{4} \right) (1+i)^{2l} k^{l-1} + \cdots \right) \\
\widetilde{C}_{l,0,1}(k) &= \sum_{0\le j\le l} (-1)^j \binom{2l+1}{2j} \left(k^l + \cdots \right) = \Re \left(\sum_{0\le m\le 2l+1} i^m \binom{2l+1}{m} \left(k^l + \cdots \right) \right) \\
&= \Re \left((1+i)^{2l+1} k^l + \cdots \right) = \pm 2^l k^l + \cdots \\
\widetilde{C}_{l,1,0}(k) &= \sum_{0\le j\le l-1} (-1)^j \binom{2l}{2j+1} \left(k^{l-1} + \left(\binom{l+1}{2} - j^2 - j - 1 \right) k^{l-2} + \cdots \right) \\
&= \Im \left(\sum_{0\le m\le 2l} i^m \binom{2l}{m} \left(k^{l-1} + \left(\binom{l+1}{2} - \frac{m^2+3}{4} \right) k^{l-2} + \cdots \right) \right) \\
&= \Im \left((1+i)^{2l} k^{l-1} + \left(\binom{l+1}{2} - \frac{3 + l(1+2li)}{4} \right) (1+i)^{2l} k^{l-2} + \cdots \right) \\
\widetilde{C}_{l,1,1}(k) &= \sum_{0\le j\le l} (-1)^j \binom{2l+1}{2j+1} \left(k^l + \cdots \right) = \Im \left(\sum_{0\le m\le 2l+1} i^m \binom{2l+1}{m} \left(k^l + \cdots \right) \right) \\
&= \Im \left((1+i)^{2l+1} k^l + \cdots \right) = \pm 2^l k^l + \cdots .
\end{aligned}$$

For each of the four cases, we checked with `Maple` that $\widetilde{C}_l$ is an irreducible polynomial for $2 \le l \le 350$, which proves Proposition 8 since $\widetilde{C}_l$ then has no integer zero. Each case required between 95000 and 98000 seconds.

4.2 The approach

We give a more specific version of Lemma 15.

Lemma 18. *Let δ, δ_1 be two positive real numbers with $\delta < \min(2/3, 3(3-r)/4)$ and $\delta \le \delta_1 \le \alpha$. Assume that*

1. $\left| f\left(\rho e^{i\theta}\right) - f\left(\rho e^{i\alpha}\right) + \frac{(\theta-\alpha)^2}{2} \right| \le \frac{|\theta-\alpha|^3}{3} + \frac{r-1}{4}(\theta-\alpha)^2$, *for* $\alpha - \delta \le \theta \le \alpha + \delta$,
2. $g(\theta) - g(\alpha) \le -\frac{(\theta-\alpha)^2}{2} + \frac{|\theta-\alpha|^3}{2}$, *for* $\alpha - \delta_1 \le \theta \le \alpha + \delta_1$.

Define $\gamma_1 = \arccos \frac{3r-1}{2\sqrt{2r}}$ *and* $\gamma_2 = -\arccos \frac{r-3}{2\sqrt{2}}$. *We then have*

$$\left| \frac{\sqrt{\pi\lambda}}{2^{\frac{1+(r+1)\lambda}{2}}} I(\lambda) - \cos\left((r\gamma_1 + \gamma_2)\lambda\right) \right| \le \frac{1}{3\sqrt{2\pi\lambda}\left(\frac{3-r}{4} - \frac{\delta}{3}\right)^2} + \frac{r-1}{2^{7/2}\left(\frac{3-r}{4} - \frac{\delta}{3}\right)^{3/2}} + \frac{\sqrt{2}e^{-\lambda\frac{\delta^2}{2}}}{\sqrt{\pi\lambda}\delta}$$
$$+ \left(\frac{\pi - 2\delta_1}{\sqrt{2\pi}} + \frac{2^{3/2}}{\sqrt{\pi}\delta(2-3\delta)} \right) \sqrt{\lambda} e^{-\lambda\frac{\delta_1^2}{2} + \lambda\frac{\delta_1^3}{2}}.$$

Proof. We follow the proof of Lemma 15, and use Conditions 1 and 2:

$$\left| \int_{\alpha-\delta}^{\alpha+\delta} e^{\lambda\left(f\left(\rho e^{i\theta}\right) - f\left(\rho e^{i\alpha}\right)\right)} \frac{d\theta}{2\pi} - \int_{\alpha-\delta}^{\alpha+\delta} e^{-\lambda\frac{(\theta-\alpha)^2}{2}} \frac{d\theta}{2\pi} \right|$$
$$\le \int_{\alpha-\delta}^{\alpha+\delta} \lambda \left(\frac{|\theta-\alpha|^3}{3} + \frac{r-1}{4}(\theta-\alpha)^2 \right) e^{-\frac{3-r}{4}\lambda(\theta-\alpha)^2 + \lambda\frac{|\theta-\alpha|^3}{3}} \frac{d\theta}{2\pi}$$
$$\le \lambda \int_0^{\delta} \left(\frac{u^3}{3} + \frac{r-1}{4}u^2 \right) e^{-\lambda u^2\left(\frac{3-r}{4} - \frac{\delta}{3}\right)} \frac{du}{\pi} < \lambda \int_0^{\infty} \left(\frac{u^3}{3} + \frac{r-1}{4}u^2 \right) e^{-\lambda u^2\left(\frac{3-r}{4} - \frac{\delta}{3}\right)} \frac{du}{\pi},$$

that is

$$\left| \int_{\alpha-\delta}^{\alpha+\delta} e^{\lambda\left(f\left(\rho e^{i\theta}\right) - f\left(\rho e^{i\alpha}\right)\right)} \frac{d\theta}{2\pi} - \int_{\alpha-\delta}^{\alpha+\delta} e^{-\lambda\frac{(\theta-\alpha)^2}{2}} \frac{d\theta}{2\pi} \right| \le \frac{1}{6\pi\lambda\left(\frac{3-r}{4} - \frac{\delta}{3}\right)^2} + \frac{r-1}{16\sqrt{\pi\lambda}\left(\frac{3-r}{4} - \frac{\delta}{3}\right)^{3/2}}. \tag{25}$$

We still have

$$\left| \int_{|\theta-\alpha|>\delta} e^{-\lambda\frac{(\theta-\alpha)^2}{2}} \frac{d\theta}{2\pi} \right| \le \frac{e^{-\lambda\frac{\delta^2}{2}}}{\pi\lambda\delta} \tag{26}$$

and

$$\int_{-\infty}^{\infty} e^{-\lambda\frac{(\theta-\alpha)^2}{2}} \frac{d\theta}{2\pi} = \frac{1}{\sqrt{2\pi\lambda}}. \tag{27}$$

Let us now assume $0 \le \theta \le \alpha - \delta$. As in the proof of Lemma 15, we deduce the following from (23) and Condition 2:

$$\left|\int_0^{\alpha-\delta_1} e^{\lambda\left(f\left(\rho e^{i\theta}\right)-f\left(\rho e^{i\alpha}\right)\right)}\frac{d\theta}{2\pi}\right| \le \frac{\alpha-\delta_1}{2\pi}e^{-\lambda\frac{\delta_1^2}{2}+\lambda\frac{\delta_1^3}{2}}. \tag{28}$$

Put $u(\theta) = -\frac{(\theta-\alpha)^2}{2} - \frac{(\theta-\alpha)^3}{2}$, so that $g(\theta) \le u(\theta)$ for $\theta \le \alpha$. We check that $u'(\theta) = \frac{3}{2}(\alpha - \theta)(\frac{2}{3}-\alpha+\theta) \ge 0$ for $\alpha-2/3 \le \theta \le \alpha$, and $u'(\theta)-u'(\alpha-\delta) = -(\theta+\alpha+\delta)(3(\theta-\alpha-\delta)/2+1) \ge 0$ for $0 \le \theta \le \alpha-\delta$. We thus get $\frac{u'(\theta)}{u'(\alpha-\delta)} \ge 1$ for $0 \le \theta \le \alpha - \delta$. We get

$$\left|\int_{\alpha-\delta_1}^{\alpha-\delta} e^{\lambda\left(f\left(\rho e^{i\theta}\right)-f\left(\rho e^{i\alpha}\right)\right)}\frac{d\theta}{2\pi}\right| \le \int_{\alpha-\delta_1}^{\alpha-\delta}\frac{u'(\theta)}{u'(\alpha-\delta)}e^{\lambda u(\theta)}\frac{d\theta}{2\pi} = \frac{e^{-\lambda\frac{\delta_1^2}{2}+\lambda\frac{\delta_1^3}{2}} - e^{-\lambda\frac{\delta^2}{2}+\lambda\frac{\delta^3}{2}}}{3\pi\delta(2/3-\delta)}. \tag{29}$$

The estimates (28) and (29) give

$$\left|\int_0^{\alpha-\delta} e^{\lambda\left(f\left(\rho e^{i\theta}\right)-f\left(\rho e^{i\alpha}\right)\right)}\frac{d\theta}{2\pi}\right| \le \frac{\alpha-\delta_1}{2\pi}e^{-\lambda\frac{\delta_1^2}{2}+\lambda\frac{\delta_1^3}{2}} + \frac{e^{-\lambda\frac{\delta_1^2}{2}+\lambda\frac{\delta_1^3}{2}}}{\pi\delta(2-3\delta)}. \tag{30}$$

Similarly we have

$$\left|\int_{\alpha+\delta}^{\pi} e^{\lambda\left(f\left(\rho e^{i\theta}\right)-f\left(\rho e^{i\alpha}\right)\right)}\frac{d\theta}{2\pi}\right| \le \frac{\pi-\alpha-\delta_1}{2\pi}e^{-\lambda\frac{\delta_1^2}{2}+\lambda\frac{\delta_1^3}{2}} + \frac{e^{-\lambda\frac{\delta_1^2}{2}+\lambda\frac{\delta_1^3}{2}}}{\pi\delta(2-3\delta)}. \tag{31}$$

From (25), (26), (27), (30), (31) we obtain

$$\left|\int_0^{\pi} e^{\lambda\left(f\left(\rho e^{i\theta}\right)-f\left(\rho e^{i\alpha}\right)\right)}\frac{d\theta}{2\pi} - \frac{1}{\sqrt{2\pi\lambda}}\right| \le \frac{1}{6\pi\lambda\left(\frac{3-r}{4}-\frac{\delta}{3}\right)^2} + \frac{r-1}{16\sqrt{\pi\lambda}\left(\frac{3-r}{4}-\frac{\delta}{3}\right)^{3/2}} + \frac{e^{-\lambda\frac{\delta^2}{2}}}{\pi\lambda\delta}$$
$$+\frac{\pi-2\delta_1}{2\pi}e^{-\lambda\frac{\delta_1^2}{2}+\lambda\frac{\delta_1^3}{2}} + \frac{2e^{-\lambda\frac{\delta_1^2}{2}+\lambda\frac{\delta_1^3}{2}}}{\pi\delta(2-3\delta)},$$

and the lemma follows from

$$\Re\left(\int_0^{\pi} e^{\lambda\left(f\left(\rho e^{i\theta}\right)-g(\alpha)\right)}\frac{d\theta}{2\pi} - \frac{e^{i\lambda h(\alpha)}}{\sqrt{2\pi\lambda}}\right) = \frac{I(\lambda)}{2} - \frac{\cos(\lambda h(\alpha))}{\sqrt{2\pi\lambda}}.$$

□

4.3 Estimates for f and g

Lemma 19. *Assume $1 \le r \le 2.282$. For $\alpha - \sqrt{r-1} \le \theta \le \alpha + \sqrt{r-1}$, we have*

$$\left|f\left(\rho e^{i\theta}\right) - f\left(\rho e^{i\alpha}\right) + \frac{(\theta-\alpha)^2}{2}\right| \le \frac{|\theta-\alpha|^3}{3} + \frac{r-1}{4}(\theta-\alpha)^2.$$

Proof. We first notice that $\sqrt{r-1} \le \alpha = \arccos\left(\frac{r-1}{2\sqrt{r}}\right)$, for $1 \le r \le 2.282459$. This implies $0 \le \theta \le 2\alpha \le \pi$.

By (23) we have

$$\begin{aligned} g'(\theta) + ih'(\theta) &= ir\frac{(e^{i\theta} - e^{i\alpha})(e^{-i\alpha} - e^{i\theta})}{r - e^{2i\theta}} = ir\frac{\left(e^{i(\theta-\alpha)} - 1\right)\left(1 - e^{i(\theta+\alpha)}\right)}{r - e^{2i\theta}} \\ &= i\left(e^{i(\theta-\alpha)} - 1\right)\left(1 + \frac{e^{i(\theta+\alpha)}\left(e^{i(\theta-\alpha)} - r\right)}{r - e^{2i\theta}}\right), \end{aligned}$$

which gives

$$\left|g'(\theta) + ih'(\theta) - i\left(e^{i(\theta-\alpha)} - 1\right)\right| \le |\theta - \alpha|\frac{|e^{i(\theta-\alpha)} - 1| + r - 1}{|r - e^{2i\theta}|} \le \frac{(\theta-\alpha)^2 + (r-1)|\theta - \alpha|}{|r - e^{2i\theta}|}.$$

Since $2\alpha - 2\sqrt{r-1} \le 2\pi - 2\alpha - 2\sqrt{r-1}$, we also get

$$\begin{aligned} |r - e^{2i\theta}|^2 &= r^2 + 1 - 2r\cos(2\theta) \ge r^2 + 1 - 2r\cos\left(2\alpha - 2\sqrt{r-1}\right) \\ &\ge r^2 + 1 - 2r\cos(2\alpha) - 4r\sqrt{r-1}\sin(2\alpha) + 4r(r-1)\cos(2\alpha) \\ &= 4r - 2(r-1)^{3/2}\sqrt{-r^2 + 6r - 1} + 2(r-1)(r^2 - 4r + 1) \\ &= 4 + 2(r-1)^{3/2}\left(\sqrt{-r^2 + 6r - 1} + (r-3)\sqrt{r-1}\right) \ge 4, \end{aligned}$$

which leads to the upper bound

$$\left|g'(\theta) + ih'(\theta) - i\left(e^{i(\theta-\alpha)} - 1\right)\right| \le \frac{(\theta-\alpha)^2}{2} + \frac{r-1}{2}|\theta - \alpha|.$$

We deduce

$$|g'(\theta) + ih'(\theta) + (\theta - \alpha)| \le \frac{(\theta-\alpha)^2}{2} + \frac{r-1}{2}|\theta - \alpha| + \frac{(\theta-\alpha)^2}{2},$$

and the lemma follows by integrating. □

Lemma 20. *For $\alpha/2 \le \theta \le 3\alpha/2$ and $1 < r \le 2.11952$, we have*

$$g(\theta) - g(\alpha) \le -\frac{(\theta-\alpha)^2}{2} + \frac{|\theta-\alpha|^3}{2}.$$

Proof. From Lemma 17 we get

$$g(\theta) - g(\alpha) \le -\frac{(\theta-\alpha)^2}{2} + \frac{|\theta-\alpha|^3}{2} + (r-1)(\theta-\alpha)^2\left(\frac{r^2 - 4r - 1}{8r} + \frac{|\theta - \alpha|}{4}\right).$$

We check that

$$\frac{r^2 - 4r - 1}{16r} + \frac{\alpha}{8} \le 0$$

for $1 < r \le 2.119518\ldots$, and the lemma follows. □

4.4 Proof of Theorem 9

For $(r-1)\lambda \geq 702$ and $r-1 \leq \sqrt{\frac{8\pi}{\lambda}}$, we have $\lambda \geq \lceil \frac{702^2}{8\pi} \rceil = 19609$ and therefore $r \leq 1+\sqrt{\frac{8\pi}{19609}} < 1.03581$. We thus can apply Lemmas 19 and 20, and we set $\delta = \sqrt{r-1}$ and $\delta_1 = \alpha/2$. We then have $\delta \leq 0.19 < 2/3 < 3(3-r)/4$. We therefore can use Lemma 18, and we obtain

$$\left|\frac{\sqrt{\pi\lambda}}{2^{\frac{1+(r+1)\lambda}{2}}} I(\lambda) - \cos\left((r\gamma_1+\gamma_2)\lambda\right)\right| \leq \frac{1}{3\sqrt{2\pi\lambda}\left(\frac{3-r}{4} - \frac{\sqrt{r-1}}{3}\right)^2} + \frac{r-1}{2^{7/2}\left(\frac{3-r}{4} - \frac{\sqrt{r-1}}{3}\right)^{3/2}} + \frac{\sqrt{2}e^{-\frac{\lambda(r-1)}{2}}}{\sqrt{\pi\lambda(r-1)}} + \left(\frac{\pi-\alpha}{\sqrt{2\pi}} + \frac{2^{3/2}}{\sqrt{\pi}\delta(2-3\delta)}\right)\sqrt{\lambda}e^{-\lambda\frac{\alpha^2}{8}+\lambda\frac{\alpha^3}{16}}.$$

Since $\delta(2-3\delta) \geq \sqrt{\frac{702}{\lambda}}(2-3\sqrt{0.03581}) \geq \frac{37.949}{\sqrt{\lambda}}$ and $\alpha \leq \pi/2$, we find

$$\left|\frac{\sqrt{\pi\lambda}}{2^{\frac{1+(r+1)\lambda}{2}}} I(\lambda) - \cos\left((r\gamma_1+\gamma_2)\lambda\right)\right| \leq \frac{1}{3\sqrt{2\pi\lambda}\left(\frac{3-r}{4} - \frac{\sqrt{r-1}}{3}\right)^2} + \frac{r-1}{2^{7/2}\left(\frac{3-r}{4} - \frac{\sqrt{r-1}}{3}\right)^{3/2}} + \frac{\sqrt{2}e^{-\frac{\lambda(r-1)}{2}}}{\sqrt{\pi\lambda(r-1)}} + \left(0.6267 + 0.04206\sqrt{\lambda}\right)\sqrt{\lambda}e^{-\lambda\left(1-\frac{\pi}{4}\right)\frac{\alpha^2}{8}}.$$

Let $\Phi_4(r,\lambda)$ denote this last upper bound. The first two terms and the fourth term are increasing functions of $r \leq 1.03581$ and decreasing functions of $\lambda \geq 19609$, the third term is a decreasing function of $\lambda(r-1) \geq 702$: we already get this way $\Phi_4(r,\lambda) \leq 0.0165$, which proves the first inequality in the theorem.

Further assume $\log\lambda \leq (r-1)\lambda \leq 1.67007\sqrt{\lambda}$. We find $\lambda \geq \lambda_0 := \lceil \frac{702^2}{1.67007^2} \rceil = 176688$, $r \leq 1 + \frac{1.67007}{\lambda_0}$, $\alpha \geq \alpha_0 := \arccos\left(\frac{\sqrt{1.67007^2/\lambda_0}}{2(1+\sqrt{1.67007^2/\lambda_0})^{1/2}}\right)$ and

$$\sqrt{\lambda}\Phi_4(r,\lambda) \leq \frac{1}{3\sqrt{2\pi}\left(\frac{3-r}{4} - \frac{\sqrt{r-1}}{3}\right)^2} + \frac{1.67007}{2^{7/2}\left(\frac{3-r}{4} - \frac{\sqrt{r-1}}{3}\right)^{3/2}} + \frac{1}{\sqrt{351\pi}} + \left(0.6267 + 0.04206\sqrt{\lambda_0}\right)\lambda_0 e^{-\lambda_0\left(1-\frac{\pi}{4}\right)\frac{\alpha_0^2}{8}} < 1.05882,$$

which proves the second inequality in the theorem for $\log\lambda \leq (r-1)\lambda \leq 1.67007\sqrt{\lambda}$.

Assume now $c_1\sqrt{\lambda} \leq (r-1)\lambda \leq c_2\sqrt{\lambda}$, so that $\lambda \geq \lambda_0 := \lceil \frac{702^2}{c_2} \rceil$. We find here

$$\sqrt{\lambda}\Phi_4(r,\lambda) \leq \frac{1}{3\sqrt{2\pi}\left(\frac{3-r}{4} - \frac{\sqrt{r-1}}{3}\right)^2} + \frac{c_2}{2^{7/2}\left(\frac{3-r}{4} - \frac{\sqrt{r-1}}{3}\right)^{3/2}} + \frac{\sqrt{2}e^{-\frac{c_1\sqrt{\lambda_0}}{2}}}{\sqrt{\pi c_1\sqrt{\lambda_0}}} + \left(0.6267 + 0.04206\sqrt{\lambda_0}\right)\lambda_0 e^{-\lambda_0\left(1-\frac{\pi}{4}\right)\frac{\alpha_0^2}{8}},$$

which gives

$$\sqrt{\lambda}\Phi_4(r,\lambda) \le \begin{cases} 1.05882, & \text{if } 1.67007\sqrt{\lambda} \le (r-1)\lambda \le \sqrt{\pi\lambda}; \\ 1.30775, & \text{if } \sqrt{\pi\lambda} \le (r-1)\lambda \le \sqrt{2\pi\lambda}; \\ 1.50929, & \text{if } \sqrt{2\pi\lambda} \le (r-1)\lambda \le \sqrt{3\pi\lambda}; \\ 1.68876, & \text{if } \sqrt{3\pi\lambda} \le (r-1)\lambda \le \sqrt{4\pi\lambda}; \\ 1.85482, & \text{if } \sqrt{4\pi\lambda} \le (r-1)\lambda \le \sqrt{5\pi\lambda}; \\ 2.01189, & \text{if } \sqrt{5\pi\lambda} \le (r-1)\lambda \le \sqrt{6\pi\lambda}; \\ 2.1626, & \text{if } \sqrt{6\pi\lambda} \le (r-1)\lambda \le \sqrt{7\pi\lambda}; \\ 2.30865, & \text{if } \sqrt{7\pi\lambda} \le (r-1)\lambda \le \sqrt{8\pi\lambda}. \end{cases}$$

This completes the proof of the theorem.

We now need lower bounds for $|\cos(\lambda_1\gamma_1 + \lambda_2\gamma_2)|$, and this is the aim of the next subsection.

4.5 Additional lemmas

Lemma 21. *For* $1 \le r \le \frac{9+\sqrt{73}}{4} = 4.386\ldots$, *we have*

$$0 \le r\gamma_1 + \gamma_2 - (r-3)\frac{\pi}{4} + \frac{(r-1)^2}{4} \le \frac{(r-1)^3}{8}.$$

Proof. Define $\gamma(r) = r\gamma_1 + \gamma_2$. We have $\gamma_1(1) = \frac{\pi}{4}$, $\gamma_2(1) = -\frac{3\pi}{4}$, $\gamma_1'(r) = -\frac{1}{r\sqrt{-r^2+6r-1}}$, $\gamma_2'(r) = \frac{1}{\sqrt{-r^2+6r-1}}$, and therefore $\gamma(1) = -\frac{\pi}{2}$, $\gamma' = \gamma_1$, $\gamma''(r) = -\frac{1}{r\sqrt{-r^2+6r-1}}$, $\gamma''(1) = -\frac{1}{2}$,

$$\gamma^{(3)}(r) = \frac{-2r^2+9r-1}{r^2(-r^2+6r-1)^{3/2}} \quad \text{and} \quad \gamma^{(4)}(r) = -2\frac{3r^4-27r^3+70r^2-15r+1}{r^3(-r^2+6r-1)^{5/2}}.$$

Since $3r^4 - 27r^3 + 70r^2 - 15r + 1 > 0$, the function $\gamma^{(3)}$ is decreasing on $[1, 3+2\sqrt{2})$. From $\gamma^{(3)}(1) = 3/4$ and $\gamma^{(3)}(\frac{9+\sqrt{73}}{4}) = 0$, we get $0 \le \gamma^{(3)}(r) \le 3/4$ for $1 \le r \le \frac{9+\sqrt{73}}{4}$, and the required inequality follows. □

Lemma 22. *For* $1 \le r \le 3$, *we have*

- *For* $\lambda_1 + \lambda_2 \equiv 0 \pmod 4$*:*

$$|\cos(\lambda(r\gamma_1+\gamma_2))| \ge \begin{cases} \cos\left(\frac{(r-1)^2}{4}\lambda\right), & \text{if } \lambda\frac{(r-1)^2}{4} \le \frac{\pi}{2}; \\ \min\left(\cos\left(\pi - \frac{(r-1)^2}{4}\lambda\right), \cos\left(\pi - \frac{3-r}{2}\frac{(r-1)^2}{4}\lambda\right)\right), & \text{if } \frac{\pi}{3-r} \le \lambda\frac{(r-1)^2}{4} \le \frac{3\pi}{2}. \end{cases}$$

- *For* $\lambda_1 + \lambda_2 \equiv 1 \pmod 4$*:*

$$|\cos(\lambda(r\gamma_1+\gamma_2))| \ge \begin{cases} \min\left(\frac{1}{\sqrt{2}}, \cos\left(\frac{\pi}{4} - \lambda\frac{(r-1)^2}{4}\right)\right), & \text{if } \lambda\frac{(r-1)^2}{4} \le \frac{3\pi}{4}; \\ \min\left(\cos\left(\frac{5\pi}{4} - \frac{(r-1)^2}{4}\lambda\right), \cos\left(\frac{5\pi}{4} - \frac{3-r}{2}\frac{(r-1)^2}{4}\lambda\right)\right), & \text{if } \frac{3\pi}{2(3-r)} \le \lambda\frac{(r-1)^2}{4} \le \frac{7\pi}{4}. \end{cases}$$

- *For* $\lambda_1+\lambda_2 \equiv 2 \pmod 4$*:*

$$|\cos(\lambda(r\gamma_1+\gamma_2))| \geq \begin{cases} \min\left(\cos\left(\frac{\pi}{2}-\lambda\frac{(r-1)^2}{4}\right), \cos\left(\frac{\pi}{2}-\frac{3-r}{2}\lambda\frac{(r-1)^2}{4}\right)\right), & \textit{if } \lambda\frac{(r-1)^2}{4} \leq \pi; \\ \min\left(\cos\left(\frac{3\pi}{2}-\lambda\frac{(r-1)^2}{4}\right), \cos\left(\frac{3\pi}{2}-\frac{3-r}{2}\lambda\frac{(r-1)^2}{4}\right)\right), & \textit{if } \frac{2\pi}{3-r} \leq \lambda\frac{(r-1)^2}{4} \leq 2\pi. \end{cases}$$

- *For* $\lambda_1+\lambda_2 \equiv 3 \pmod 4$*:*

$$|\cos(\lambda(r\gamma_1+\gamma_2))| \geq \begin{cases} \cos\left(\frac{\pi}{4}+\lambda\frac{(r-1)^2}{4}\right), & \textit{if } \lambda\frac{(r-1)^2}{4} \leq \frac{\pi}{4}; \\ \min\left(\cos\left(\frac{3\pi}{4}-\lambda\frac{(r-1)^2}{4}\right), \cos\left(\frac{3\pi}{4}-\frac{3-r}{2}\lambda\frac{(r-1)^2}{4}\right)\right), & \textit{if } \frac{\pi}{2(3-r)} \leq \lambda\frac{(r-1)^2}{4} \leq \frac{5\pi}{4}. \end{cases}$$

Proof. From Lemma 21 we deduce the inequalities

$$0 \leq \lambda_1\gamma_1+\lambda_2\gamma_2-(\lambda_1-3\lambda_2)\frac{\pi}{4}+\lambda_2\frac{(r-1)^2}{4} \leq \lambda_2\frac{(r-1)^3}{8}\,.$$

Let us define $\eta=\lambda_1\gamma_1+\lambda_2\gamma_2-(\lambda_1-3\lambda_2)\frac{\pi}{4}$ so that we have

$$-\frac{(r-1)^2}{4}\lambda_2 \leq \eta \leq \frac{r-3}{2}\frac{(r-1)^2}{4}\lambda_2 \leq 0\,,$$

for $1 \leq r \leq 3$. We then find

$$|\cos(\lambda_1\gamma_1+\lambda_2\gamma_2)| = \begin{cases} \cos\eta, & \text{if } \lambda_1+\lambda_2\equiv 0 \pmod 4 \text{ and } -\frac{\pi}{2}\leq\eta\leq 0; \\ \cos(\pi+\eta), & \text{if } \lambda_1+\lambda_2\equiv 0 \pmod 4 \text{ and } -\frac{3\pi}{2}\leq\eta\leq -\frac{\pi}{2}; \\ \cos\left(\frac{\pi}{4}+\eta\right), & \text{if } \lambda_1+\lambda_2\equiv 1 \pmod 4 \text{ and } -\frac{3\pi}{4}\leq\eta\leq 0; \\ \cos\left(\frac{5\pi}{4}+\eta\right), & \text{if } \lambda_1+\lambda_2\equiv 1 \pmod 4 \text{ and } -\frac{7\pi}{4}\leq\eta\leq -\frac{3\pi}{4}; \\ \cos\left(\frac{\pi}{2}+\eta\right), & \text{if } \lambda_1+\lambda_2\equiv 2 \pmod 4 \text{ and } -\pi\leq\eta\leq 0; \\ \cos\left(\frac{3\pi}{2}+\eta\right), & \text{if } \lambda_1+\lambda_2\equiv 2 \pmod 4 \text{ and } -2\pi\leq\eta\leq -\pi; \\ \cos\left(-\frac{\pi}{4}+\eta\right), & \text{if } \lambda_1+\lambda_2\equiv 3 \pmod 4 \text{ and } -\frac{\pi}{4}\leq\eta\leq 0; \\ \cos\left(\frac{3\pi}{4}+\eta\right), & \text{if } \lambda_1+\lambda_2\equiv 3 \pmod 4 \text{ and } -\frac{5\pi}{4}\leq\eta\leq -\frac{\pi}{4}. \end{cases}$$

The lemma follows. □

4.6 Proof of Theorem 10

In order to get a contradiction, assume $I(\lambda_2)=0$ with $\lambda_1-\lambda_2 \geq 702$, by Proposition 8. Theorem 9 then gives an upper bound for $|\cos(\lambda_1\gamma_1+\lambda_2\gamma_2)|$ that may be smaller than the lower bound given in Lemma 22. Because of the form of Theorem 9 and Lemma 22, we need to distinguish several cases, according to the residue class of $\lambda_1+\lambda_2$ modulo 4 and to the size of $\frac{(\lambda_1-\lambda_2)^2}{4\lambda_2}$. When needed, we shall use the upper bound $r \leq 1.03581$ and the estimate $\arccos x \geq \frac{\pi}{2}-\frac{x}{\sqrt{1-x^2}}$ for $0<x<1$.

4.6.1 The case $\lambda_1 + \lambda_2 \equiv 0 \pmod 4$

- For $702 \le \lambda_1 - \lambda_2 \le \log \lambda_2$, we obtain the bounds $\lambda_2 \ge e^{702}$ and $\frac{(\lambda_1-\lambda_2)^2}{4\lambda_2} \le \frac{702^2}{4e^{702}} < \frac{\pi}{2}$. From Theorem 9 and Lemma 22 we get the following contradiction:

$$0.9999 \le \cos\left(\frac{702^2}{4e^{702}}\right) \le \cos\left(\frac{(\lambda_1-\lambda_2)^2}{4\lambda_2}\right) \le |\cos(\lambda(r\gamma_1+\gamma_2))| \le 0.0165\,.$$

- For $\max(702, \log\lambda_2) \le \lambda_1 - \lambda_2 \le \sqrt{2\pi\lambda_2} - 1.0443$, we obtain the lower bound $\lambda_2 \ge \left\lceil \frac{703.0443^2}{2\pi} \right\rceil = 78666$. From Theorem 9 and Lemma 22 we deduce the inequality $\cos\left(\frac{(\lambda_1-\lambda_2)^2}{4\lambda_2}\right) \le \frac{1.30775}{\sqrt{\lambda_2}}$, and we get the contradiction

$$\frac{(\lambda_1-\lambda_2)^2}{4\lambda_2} \ge \arccos\frac{1.30775}{\sqrt{\lambda_2}} \ge \frac{\pi}{2} - \frac{1.30775}{\sqrt{\lambda_2 - 1.30775^2}} > \left(\sqrt{\frac{\pi}{2}} - \frac{0.52215}{\sqrt{\lambda_2}}\right)^2\,.$$

- For $\max(702, \sqrt{2\pi\lambda_2} + 3.1407) \le \lambda_1 - \lambda_2 \le \sqrt{3\pi\lambda_2}$, we obtain the lower bound $\lambda_2 \ge \left\lceil \frac{702^2}{3\pi} \right\rceil = 52289$. From Theorem 9 and Lemma 22 we deduce the inequality $\cos\left(\pi - \frac{3-r}{2}\frac{(r-1)^2}{4}\lambda_2\right) \le \frac{1.50929}{\sqrt{\lambda_2}}$, and we get the contradiction

$$\frac{\pi}{2} - \frac{1.50929}{\sqrt{\lambda_2 - 1.50929^2}} \le \pi - \frac{3-r}{2}\frac{(r-1)^2}{4}\lambda_2 \le \pi - \frac{\pi}{2}\left(1 - \frac{\sqrt{3\pi}}{2\sqrt{\lambda_2}}\right)\left(1 + \frac{3.1407}{\sqrt{2\pi\lambda_2}}\right)^2\,.$$

- For $\max(702, \sqrt{3\pi\lambda_2}) \le \lambda_1 - \lambda_2 \le \sqrt{\frac{16}{5-r}\pi\lambda_2} < \sqrt{12.68\lambda_2}$, we obtain the lower bound $\lambda_2 \ge \left\lceil \frac{702^2}{12.68} \right\rceil = 38865$. From Theorem 9 and Lemma 22 we deduce the inequalities $\cos\left(\pi - \frac{3-r}{2}\frac{(r-1)^2}{4}\lambda_2\right) \le \frac{1.85482}{\sqrt{\lambda_2}} < 0.00941$, and we get the contradiction

$$1.5613 < \arccos(0.00941) < \pi - \frac{3-r}{2}\frac{(r-1)^2}{4}\lambda_2 \le \pi - \frac{3-r}{8}3\pi \le 0.8276\,.$$

- For $\max\left(702, \sqrt{\frac{16}{5-r}\pi\lambda_2}\right) \le \lambda_1 - \lambda_2 \le \sqrt{6\pi\lambda_2} - 0.9275$, we obtain the lower bound $\lambda_2 \ge \left\lceil \frac{702.9275^2}{6\pi} \right\rceil = 26214$. From Theorem 9 and Lemma 22 we deduce the inequality $\cos\left(\pi - \frac{(r-1)^2}{4}\lambda_2\right) \le \frac{2.01189}{\sqrt{\lambda_2}}$, and we get the contradiction

$$\frac{\pi}{2} - \frac{2.01189}{\sqrt{\lambda_2 - 2.01189^2}} \le \lambda_2\frac{(r-1)^2}{4} - \pi \le \frac{3\pi}{2}\left(1 - \frac{0.9275}{\sqrt{6\pi\lambda_2}}\right)^2 - \pi\,.$$

4.6.2 The case $\lambda_1 + \lambda_2 \equiv 1 \pmod 4$

- For $703 \le \lambda_1 - \lambda_2 \le \sqrt{2\pi\lambda_2}$, we obtain the bound $\lambda_2 \ge \left\lceil \frac{703^2}{2\pi} \right\rceil = 78656$. From Theorem 9 and Lemma 22 we obtain the following contradiction:

$$\frac{1}{\sqrt{2}} \le |\cos(\lambda(r\gamma_1 + \gamma_2))| \le \min\left(0.0165, \frac{1.30775}{\sqrt{\lambda_2}}\right).$$

- For $\max(703, \sqrt{2\pi\lambda_2}) \le \lambda_1 - \lambda_2 \le \sqrt{3\pi\lambda_2} - 0.984$, we obtain the bound $\lambda_2 \ge \left\lceil \frac{703.984^2}{3\pi} \right\rceil = 52585$. The inequality $\cos\left(\frac{\pi}{4} - \frac{(r-1)^2}{4}\lambda_2\right) \le \frac{1.50929}{\sqrt{\lambda_2}}$ follows from Theorem 9 and Lemma 22, and we get the contradiction

$$\frac{(\lambda_1 - \lambda_2)^2}{4\lambda_2} \ge \frac{\pi}{4} + \arccos\left(\frac{1.50929}{\sqrt{\lambda_2}}\right) \ge \frac{3\pi}{4} - \frac{1.50929}{\sqrt{\lambda_2 - 1.50929^2}} > \left(\sqrt{\frac{3\pi}{4}} - \frac{0.492}{\sqrt{\lambda_2}}\right)^2.$$

- For $\max(703, \sqrt{3\pi\lambda_2} + 3.8433) \le \lambda_1 - \lambda_2 \le \sqrt{4\pi\lambda_2}$, we obtain the lower bound $\lambda_2 \ge \left\lceil \frac{702^2}{4\pi} \right\rceil = 39217$. The inequality $\cos\left(\frac{5\pi}{4} - \frac{3-r}{2}\frac{(r-1)^2}{4}\lambda_2\right) \le \frac{1.68876}{\sqrt{\lambda_2}}$ follows from Theorem 9 and Lemma 22 , and we get the contradiction

$$\frac{\pi}{2} - \frac{1.68876}{\sqrt{\lambda_2 - 1.68876^2}} \le \frac{5\pi}{4} - \frac{3-r}{2}\frac{(r-1)^2}{4}\lambda_2 \le \frac{5\pi}{4} - \frac{3\pi}{4}\left(1 - \frac{\sqrt{\pi}}{\sqrt{\lambda_2}}\right)\left(1 + \frac{3.8433}{\sqrt{3\pi\lambda_2}}\right)^2.$$

- For $\max(703, \sqrt{4\pi\lambda_2}) \le \lambda_1 - \lambda_2 \le \sqrt{\frac{20}{5-r}\pi\lambda_2} < \sqrt{15.85\lambda_2}$, we obtain the lower bound $\lambda_2 \ge \left\lceil \frac{703^2}{15.85} \right\rceil = 31181$. From Theorem 9 and Lemma 22 we deduce the inequalities $\cos\left(\frac{5\pi}{4} - \frac{3-r}{2}\frac{(r-1)^2}{4}\lambda_2\right) \le \frac{2.01189}{\sqrt{\lambda_2}} < 0.0114$, and we get the contradiction

$$1.5593 < \arccos(0.0114) < \frac{5\pi}{4} - \frac{3-r}{2}\frac{(r-1)^2}{4}\lambda_2 \le \frac{5\pi}{4} - \frac{3-r}{8}4\pi < 0.8417.$$

- For $\max\left(703, \sqrt{\frac{20}{5-r}\pi\lambda_2}\right) \le \lambda_1 - \lambda_2 \le \sqrt{7\pi\lambda_2} - 0.9231$, we obtain the lower bound $\lambda_2 \ge \left\lceil \frac{703.9231^2}{7\pi} \right\rceil = 22533$. From Theorem 9 and Lemma 22 we deduce the inequality $\cos\left(\frac{5\pi}{4} - \frac{(r-1)^2}{4}\lambda_2\right) \le \frac{2.1626}{\sqrt{\lambda_2}}$, and we get the contradiction

$$\frac{\pi}{2} - \frac{2.1626}{\sqrt{\lambda_2 - 2.1626^2}} \le \frac{(r-1)^2}{4}\lambda_2 - \frac{5\pi}{4} \le \frac{7\pi}{4}\left(1 - \frac{0.9231}{\sqrt{7\pi\lambda_2}}\right)^2 - \frac{5\pi}{4}.$$

4.6.3 The case $\lambda_1 + \lambda_2 \equiv 2 \pmod 4$

- For $\max\left(702, 2.0582\lambda_2^{1/4}\right) \le \lambda_1 - \lambda_2 \le \sqrt{\pi\lambda_2}$, we obtain the lower bound $\lambda_2 \ge \left\lceil \frac{702^2}{\pi} \right\rceil = 156865$. The inequality $\cos\left(\frac{\pi}{2} - \frac{3-r}{2}\frac{(r-1)^2}{4}\lambda_2\right) \le \frac{1.05582}{\sqrt{\lambda_2}}$ follows from Theorem 9 and Lemma 22 , and we get the contradiction

$$\frac{\pi}{2} - \frac{1.05882}{\sqrt{\lambda_2 - 1.05882^2}} \le \frac{\pi}{2} - \frac{3-r}{2}\frac{(r-1)^2}{4}\lambda_2 \le \frac{\pi}{2} - \left(1 - \frac{2.0582}{2\lambda_2^{3/4}}\right)\frac{2.0582^2}{4\sqrt{\lambda_2}}\,.$$

- For $\max\left(702, \sqrt{\pi\lambda_2}\right) \le \lambda_1 - \lambda_2 \le \sqrt{\frac{8}{5-r}\pi\lambda_2} < \sqrt{6.34\lambda_2}$, we obtain the lower bound $\lambda_2 \ge \left\lceil \frac{702^2}{6.34} \right\rceil = 77730$. From Theorem 9 and Lemma 22 we deduce the inequalities $\cos\left(\frac{\pi}{2} - \frac{3-r}{2}\frac{(r-1)^2}{4}\lambda_2\right) \le \frac{1.50929}{\sqrt{\lambda_2}} < 0.00542$, and we get the contradiction

$$1.5653 < \arccos(0.00542) < \frac{\pi}{2} - \frac{3-r}{2}\frac{(r-1)^2}{4}\lambda_2 \le \frac{\pi}{2} - \frac{3-r}{8}\pi < 0.7995\,.$$

- For $\max\left(702, \sqrt{\frac{8}{5-r}\pi\lambda_2}\right) \le \lambda_1 - \lambda_2 \le 2\sqrt{\pi\lambda_2} - 0.9535$, we obtain the lower bound $\lambda_2 \ge \left\lceil \frac{702.9535^2}{4\pi} \right\rceil = 39323$. From Theorem 9 and Lemma 22 we deduce the inequality $\cos\left(\frac{\pi}{2} - \frac{(r-1)^2}{4}\lambda_2\right) \le \frac{1.68876}{\sqrt{\lambda_2}}$, and we get the contradiction

$$\frac{\pi}{2} - \frac{1.68876}{\sqrt{\lambda_2 - 1.68876^2}} \le \frac{(r-1)^2}{4}\lambda_2 - \frac{\pi}{2} \le \pi\left(1 - \frac{0.9535}{2\sqrt{\pi\lambda_2}}\right)^2 - \frac{\pi}{2}\,.$$

- For $\max\left(702, 2\sqrt{\pi\lambda_2} + 4.5938\right) \le \lambda_1 - \lambda_2 \le \sqrt{5\pi\lambda_2}$, we obtain the lower bound $\lambda_2 \ge \left\lceil \frac{702^2}{5\pi} \right\rceil = 31373$. From Theorem 9 and Lemma 22 we deduce the inequality $\cos\left(\frac{3\pi}{2} - \frac{3-r}{2}\frac{(r-1)^2}{4}\lambda_2\right) \le \frac{1.85482}{\sqrt{\lambda_2}}$, and we get the contradiction

$$\frac{\pi}{2} - \frac{1.85482}{\sqrt{\lambda_2 - 1.85482^2}} \le \frac{3\pi}{2} - \frac{3-r}{2}\frac{(r-1)^2}{4}\lambda_2 \le \frac{3\pi}{2} - \pi\left(1 - \frac{\sqrt{5\pi}}{2\sqrt{\lambda_2}}\right)\left(1 + \frac{4.5938}{2\sqrt{\pi\lambda_2}}\right)^2\,.$$

- For $\sqrt{5\pi\lambda_2} \le \lambda_1 - \lambda_2 \le \sqrt{\frac{24}{5-r}\pi\lambda_2} < \sqrt{19.02\lambda_2}$, we obtain the lower bound $\lambda_2 \ge \left\lceil \frac{702^2}{19.02} \right\rceil = 25910$. The inequalities $\cos\left(\frac{3\pi}{2} - \frac{3-r}{2}\frac{(r-1)^2}{4}\lambda_2\right) \le \frac{2.1626}{\sqrt{\lambda_2}} < 0.01344$ follow from Theorem 9 and Lemma 22, and we get the contradiction

$$1.5573 < \arccos(0.01344) < \frac{3\pi}{2} - \frac{3-r}{2}\frac{(r-1)^2}{4}\lambda_2 \le \frac{3\pi}{2} - \frac{3-r}{8}5\pi < 0.8558\,.$$

- For $\max\left(702, \sqrt{\frac{24}{5-r}\pi\lambda_2}\right) \le \lambda_1 - \lambda_2 \le 2\sqrt{2\pi\lambda_2} - 0.9218$, we obtain the lower bound $\lambda_2 \ge \left\lceil \frac{702.9218^2}{8\pi} \right\rceil = 19660$. From Theorem 9 and Lemma 22 we deduce the inequality $\cos\left(\frac{3\pi}{2} - \frac{(r-1)^2}{4}\lambda_2\right) \le \frac{2.30865}{\sqrt{\lambda_2}}$, and we get the contradiction

$$\frac{\pi}{2} - \frac{2.30865}{\sqrt{\lambda_2 - 2.30865^2}} \le \frac{(r-1)^2}{4}\lambda_2 - \frac{3\pi}{2} \le 2\pi\left(1 - \frac{0.9218}{2\sqrt{2\pi\lambda_2}}\right)^2 - \frac{3\pi}{2}.$$

4.6.4 The case $\lambda_1 + \lambda_2 \equiv 3 \pmod 4$

- For $703 \le \lambda_1 - \lambda_2 \le \log\lambda_2$, we still find $\frac{(\lambda_1-\lambda_2)^2}{4\lambda_2} \le \frac{703^2}{4e^{703}} < \frac{\pi}{2}$ and we get the contradiction $\cos\left(\frac{\pi}{4} + \frac{703^2}{4e^{703}}\right) \le 0.0165$.

- For $\max(703, \log\lambda_2) \le \lambda_1 - \lambda_2 \le \sqrt{\pi\lambda_2} - 1.1958$, we obtain the bound $\lambda_2 \ge \left\lceil \frac{704.1958^2}{\pi} \right\rceil = 157848$. The inequality $\cos\left(\frac{\pi}{4} + \frac{(r-1)^2}{4}\lambda_2\right) \le \frac{1.05882}{\sqrt{\lambda_2}}$ follows from Theorem 9 and Lemma 22, and we get the contradiction

$$\frac{(\lambda_1 - \lambda_2)^2}{4\lambda_2} \ge \arccos\left(\frac{1.05882}{\sqrt{\lambda_2}}\right) - \frac{\pi}{4} \ge \frac{\pi}{4} - \frac{1.05882}{\sqrt{\lambda_2 - 1.05882^2}} > \left(\sqrt{\frac{\pi}{4}} - \frac{0.5979}{\sqrt{\lambda_2}}\right)^2.$$

- For $\max(703, \sqrt{\pi\lambda_2} + 2.5913) \le \lambda_1 - \lambda_2 \le \sqrt{2\pi\lambda_2}$, we obtain the bound $\lambda_2 \ge \left\lceil \frac{703^2}{2\pi} \right\rceil = 78656$. The inequality $\cos\left(\frac{3\pi}{4} - \frac{3-r}{2}\frac{(r-1)^2}{4}\lambda_2\right) \le \frac{1.30775}{\sqrt{\lambda_2}}$ follows from Theorem 9 and Lemma 22 , and we get the contradiction

$$\frac{\pi}{2} - \frac{1.30775}{\sqrt{\lambda_2 - 1.30775^2}} \le \frac{3\pi}{4} - \frac{3-r}{2}\frac{(r-1)^2}{4}\lambda_2 \le \frac{3\pi}{4} - \frac{\pi}{4}\left(1 - \frac{\sqrt{\pi}}{\sqrt{2\lambda_2}}\right)\left(1 + \frac{2.5913}{\sqrt{\pi\lambda_2}}\right)^2.$$

- For $\max(703, \sqrt{2\pi\lambda_2}) \le \lambda_1 - \lambda_2 \le \sqrt{\frac{12}{5-r}\pi\lambda_2} < \sqrt{9.51\lambda_2}$, we obtain the lower bound $\lambda_2 \ge \left\lceil \frac{703^2}{9.51} \right\rceil = 51968$. From Theorem 9 and Lemma 22 we deduce the inequalities $\cos\left(\frac{3\pi}{4} - \frac{3-r}{2}\frac{(r-1)^2}{4}\lambda_2\right) \le \frac{1.68876}{\sqrt{\lambda_2}} < 0.00741$, and we get the contradiction

$$1.5633 < \arccos(0.00741) < \frac{3\pi}{4} - \frac{3-r}{2}\frac{(r-1)^2}{4}\lambda_2 \le \frac{3\pi}{4} - \frac{3-r}{8}2\pi < 0.8136.$$

- For $\max\left(703, \sqrt{\frac{12}{5-r}\pi\lambda_2}\right) \le \lambda_1 - \lambda_2 \le \sqrt{5\pi\lambda_2} - 0.9367$, we obtain the bound $\lambda_2 \ge \left\lceil \frac{703.9367^2}{5\pi} \right\rceil = 31547$. From Theorem 9 and Lemma 22 we deduce the inequality $\cos\left(\frac{3\pi}{4} - \frac{(r-1)^2}{4}\lambda_2\right) \le \frac{1.85482}{\sqrt{\lambda_2}}$, and we get the contradiction

$$\frac{\pi}{2} - \frac{1.85482}{\sqrt{\lambda_2 - 1.85482^2}} \le \frac{(r-1)^2}{4}\lambda_2 - \frac{3\pi}{4} \le \frac{5\pi}{4}\left(1 - \frac{0.9367}{\sqrt{5\pi\lambda_2}}\right)^2 - \frac{3\pi}{4}.$$

5 Concluding remarks

In the introduction, we discussed the irreducibility of C_{X,λ_2}, and noticed how different are the cases λ_2 even and λ_2 odd. We checked both cases for $\lambda_2 \leq 240$, but we can go further in the even case. Using `Maple` during 35101 seconds, we showed that C_{X,λ_2} is irreducible over $\mathbb{Q}$ when $\lambda_2 \leq 600$ is even. This motivates the following conjecture.

Conjecture 23. For $\lambda_2 \geq 2$ even, the polynomial C_{X,λ_2} is irreducible over $\mathbb{Q}$. For $\lambda_2 \geq 3$ odd, the polynomial C_{X,λ_2} is the product of $X - \lambda_2$ by an irreducible polynomial over $\mathbb{Q}$.

We used the same technics, together with hypergeometric transformations, to study the case $\lambda_1 - \lambda_2$ small and to prove Proposition 8. We introduced the four families of polynomials and checked their irreducibility over $\mathbb{Q}$ for $l \leq 350$. It is quite likely that this property holds for all larger values of l.

Conjecture 24. For $l \geq 3$, the polynomials $\widetilde{C}_{l,0,0}(X)$, $\widetilde{C}_{l,0,1}(X)$, $\widetilde{C}_{l,1,0}(X)$ and $\widetilde{C}_{l,1,1}(X)$ are irreducible over $\mathbb{Q}$.

Let us now discuss the results obtained in Theorem 10. The first thing we noticed is that the case $\lambda_1 + \lambda_2 \equiv 2 \pmod 4$ differs from the other cases. It would be nice to be nice to deal with the interval $702 \leq \lambda_1 - \lambda_2 \leq 2.0582\lambda_2^{1/4}$ for any λ_2, to fill the initial gap. Secondly we chose to reach the second explicit intervals with no solutions. How far could we go with this method? It would be nice to get improvements that enable to break the $\sqrt{\lambda_2}$ barrier and to go up to $\lambda_2^{1/2+\epsilon}$ for some $\epsilon > 0$.

6 Acknowledgments

The author wishes to thank the anonymous referee for substantially helping to improve the presentation of this paper in many ways.

References

[1] A. Carnevale and C. Voll, Orbit Dirichlet series and multiset permutations, *Monatsh. Math.* **186** (2018), 215–233.

[2] J. Dieudonné, *Calcul Infinitésimal*, Hermann, 1968.

[3] A. M. Legendre, *Essai Sur la Théorie des Nombres*, Duprat, Paris, 1798.

2010 *Mathematics Subject Classification*: Primary 11B65, Secondary 05A10 11B83.
Keywords: binomial sum, asymptotics

Received July 3 2019; revised versions received February 5 2020; February 9 2020; May 27 2020. Published in *Journal of Integer Sequences*, June 10 2020.

Return to Journal of Integer Sequences home page.