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Relating Balancing Polynomials to Lucas-Balancing Polynomials via Bernoulli Numbers

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Abstract

We derive relations for balancing and Lucas-balancing polynomials. The results provide extensions of some results proved recently by Frontczak.

1 Introduction

The Fibonacci numbers F_n and Lucas numbers L_n [2, 6] are sequences satisfying the Fibonacci recursion relation

$$X_{n+1} = X_n + X_{n-1}. \tag{1}$$

The initial conditions are, respectively, $F_0 = 0,\ F_1 = 1$ and $L_0 = 2,\ L_1 = 1$. The first elements of the sequences are

$$F : 0, 1, 1, 2, 3, 5, 8, 13, 21, \cdots$$

and

$$L : 2, 1, 3, 4, 7, 11, 18, 29, 47, \cdots .$$

Fibonacci numbers $(F_n)_{n\geq 0}$ and Lucas numbers $(L_n)_{n\geq 0}$ are regarded as the some of the most important sequences in mathematics. Relationships between them have been studied in the past. Hoggatt [6] states as exercises several identities relating Fibonacci numbers with Lucas numbers.

These numbers are also related to balancing and Lucas-balancing polynomials. Balancing polynomials $B_n^*(x)$ are defined by the recurrence (see [4, 5, 7])

$$B_n^*(x) = 6xB_{n-1}^*(x) - B_{n-2}^*(x),\ n \geq 2, \tag{2}$$

with initial terms $B_0^*(x) = 0$ and $B_1^*(x) = 1$. Lucas-balancing polynomials $C_n(x)$ are defined by

$$C_n(x) = 6xC_{n-1}(x) - C_{n-2}(x),\ n \geq 2, \tag{3}$$

with initial terms $C_0(x) = 1$ and $C_1(x) = 3x$. The numbers $B_n^* = B_n^*(1)$ (or $C_n = C_n(1)$) are so-called balancing numbers (or Lucas-balancing numbers). The Binet formulas for $B_n^*(x)$ and $C_n(x)$ are

$$B_n^*(x) = \frac{\lambda^n(x) - \lambda^{-n}(x)}{\lambda(x) - \lambda^{-1}(x)} \text{ and } C_n(x) = \frac{1}{2}\left(\lambda^n(x) + \lambda^{-n}(x)\right), \tag{4}$$

where $\lambda(x) = 3x + \sqrt{9x^2-1}$ and $\lambda^{-1}(x) = 3x - \sqrt{9x^2-1}$. We refer to Frontczak [4] for more details. Accordingly, the exponential generating functions of polynomials $B_n^*(x)$ and $C_n(x)$ are given by Frontczak [3, Lemma 2, p. 3] as follows:

$$F(x,t) = \frac{e^{3xt}}{\sqrt{9x^2-1}} \sinh\left(t\sqrt{9x^2-1}\right) = \sum_{n\geq 0} B_n^*(x)\frac{t^n}{n!}, \tag{5}$$

and

$$G(x,t) = e^{3xt}\cosh\left(t\sqrt{9x^2-1}\right) = \sum_{n\geq 0} C_n(x)\frac{t^n}{n!}, \tag{6}$$

where sinh and cosh are the hyperbolic sine and cosine functions. The relations that we spoke about previously are given in the papers [3, 4] as follows:

$$B_n^*\left(\frac{L_{2m}}{6}\right) = \frac{F_{2mn}}{F_{2m}},\ C_n\left(\frac{L_{2m}}{6}\right) = \frac{L_{2m}}{2}, \tag{7}$$

and

$$B_n^*\left(\frac{i}{6}L_{2m+1}\right) = i^{n-1}\frac{F_{(2m+1)n}}{F_{2m+1}},\ C_n\left(\frac{i}{6}L_{2m+1}\right) = i^n\frac{L_{(2m+1)n}}{2}, \tag{8}$$

where m is an integer and $i = \sqrt{-1}$ is the imaginary unit.

The Cauchy product of exponential generating functions is defined by

$$\left(\sum_{n\geq 0} a_n(x)\frac{t^n}{n!}\right)\left(\sum_{n\geq 0} b_n(x)\frac{t^n}{n!}\right) = \sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k}a_k(x)b_{n-k}(x)\right)\frac{t^n}{n!}. \tag{9}$$

This product allows us to provide extensions of Theorem 3 and Corollary 3 proved by Frontczak [3] in 2019.

2 Connection between balancing polynomials and Lucas-balancing polynomials

As usual, we use B_n for the n^{th} *Bernoulli number* and E_n for the n^{th} *Euler number*. These classical numbers (see Apostol [1]) are defined respectively by

$$B(t) = \frac{t}{e^t - 1} = \sum_{k \geq 0} B_k \frac{t^k}{k!}, \ |t| < 2\pi \tag{10}$$

and

$$E(t) = \frac{2}{e^t + 1} = \sum_{k \geq 0} E_k \frac{t^k}{k!}, \ |t| < \pi. \tag{11}$$

The first few Bernoulli numbers are $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$. Also $B_{2n+1} = 0$ for $n \geq 1$. The first few Euler numbers are $E_0 = 1$, $E_2 = -1$, $E_4 = 5$ and $E_{2n+1} = 0$ for $n \geq 0$.

Frontczak [3, Theorem 3] proved the following identities

$$\sum_{\substack{k=0 \\ n \equiv k \pmod 2}}^{n} \binom{n}{k} \left(2\sqrt{9x^2 - 1}\right)^{n-k} B_{n-k} B_k^*(x) = n C_{n-1}(x) \tag{12}$$

and

$$\sum_{\substack{k=0 \\ n \equiv k \pmod 2}}^{n} \binom{n}{k} \left(2\sqrt{9x^2 - 1}\right)^{n-k} \left(2^{n-k} - 1\right) B_{n-k} C_k(x) = n\left(9x^2 - 1\right) B_{n-1}^*(x), \tag{13}$$

where $n \equiv k \pmod 2$ means that $n - k$ is a multiple of 2. An improvement of the identities (12) and (13) is given in the following theorem:

Theorem 1. *Let $n \geq 1$. Then*

$$\sum_{k=0}^{n} \binom{n}{k} \left(2\sqrt{9x^2 - 1}\right)^{n-k} B_{n-k} B_k^*(x) = n\left(C_{n-1}(x) - \sqrt{9x^2 - 1} B_{n-1}^*(x)\right) \tag{14}$$

and

$$\sum_{k=0}^{n} \binom{n}{k} \left(2\sqrt{9x^2 - 1}\right)^{n-k} \left(2^{n-k} - 1\right) B_{n-k} C_k(x) = n\left(9x^2 - 1\right) B_{n-1}^*(x) - n\sqrt{9x^2 - 1} C_{n-1}(x). \tag{15}$$

Proof. Let the polynomial

$$A_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}^*(x) \left(2\sqrt{9x^2 - 1}\right)^k B_k.$$

Then $A_n(x)$ is defined in means of the Cauchy product of the functions

$$\frac{e^{3xt}}{\sqrt{9x^2-1}}\sinh\left(t\sqrt{9x^2-1}\right) \text{ and } \frac{2t\sqrt{9x^2-1}}{e^{2t\sqrt{9x^2-1}}-1}.$$

But the product is

$$A(x,t)=\frac{2te^{3xt}}{e^{2t\sqrt{9x^2-1}}-1}\sinh\left(t\sqrt{9x^2-1}\right).$$

Since

$$\coth t = 1+\frac{2}{e^{2t}-1} = 1+\frac{1}{t}\sum_{n\geq 0}2^nB_n\frac{t^n}{n!}$$

then

$$A(x,t)=te^{3xt}\left(\coth\left(t\sqrt{9x^2-1}\right)-1\right)\sinh\left(t\sqrt{9x^2-1}\right).$$

Furthermore

$$A(x,t)=te^{3xt}\cosh\left(t\sqrt{9x^2-1}\right)-te^{3xt}\sinh\left(t\sqrt{9x^2-1}\right)$$

Finally

$$A(x,t)=tG(x,t)-t\sqrt{9x^2-1}F(x,t)$$

and

$$\frac{2te^{3xt}}{e^{2t\sqrt{9x^2-1}}-1}\sinh\left(t\sqrt{9x^2-1}\right)=\sum_{n\geq 0}n\left(C_{n-1}(x)-\sqrt{9x^2-1}B^*_{n-1}(x)\right)\frac{t^n}{n!}.$$

Thus $A_n(x)$ is identical with the polynomial

$$n\left(C_{n-1}(x)-\sqrt{9x^2-1}B^*_{n-1}(x)\right),$$

and the result (14) follows. Let the polynomial

$$D_n(x)=\sum_{k=0}^{n}\binom{n}{k}C_k(x)\left(2\sqrt{9x^2-1}\right)^{n-k}\left(2^{n-k}-1\right)B_{n-k}.$$

Then $D_n(x)=J_n(x)-K_n(x)$ with

$$J_n(x)=\sum_{k=0}^{n}\binom{n}{k}C_k(x)\left(4\sqrt{9x^2-1}\right)^{n-k}B_{n-k}$$

and

$$K_n(x)=\sum_{k=0}^{n}\binom{n}{k}C_k(x)\left(2\sqrt{9x^2-1}\right)^{n-k}B_{n-k}.$$

But the polynomial $J_n(x)$ is generated by the function

$$\frac{4t\sqrt{9x^2-1}}{e^{4t\sqrt{9x^2-1}}-1}e^{3xt}\cosh\left(t\sqrt{9x^2-1}\right)$$

and the polynomial $K_n(x)$ is generated by the function

$$\frac{2t\sqrt{9x^2-1}}{e^{2t\sqrt{9x^2-1}}-1}e^{3xt}\cosh\left(t\sqrt{9x^2-1}\right).$$

Furthermore the generating function of the polynomial $D_n(x)$ is

$$D(x,t)=\frac{4t\sqrt{9x^2-1}}{e^{4t\sqrt{9x^2-1}}-1}e^{3xt}\cosh\left(t\sqrt{9x^2-1}\right)-\frac{2t\sqrt{9x^2-1}}{e^{2t\sqrt{9x^2-1}}-1}e^{3xt}\cosh\left(t\sqrt{9x^2-1}\right).$$

Then

$$D(x,t)=\frac{2t\sqrt{9x^2-1}}{e^{2t\sqrt{9x^2-1}}-1}\left(\frac{2}{e^{2t\sqrt{9x^2-1}}+1}-1\right)e^{3xt}\cosh\left(t\sqrt{9x^2-1}\right)$$

and

$$D(x,t)=-\frac{2t\sqrt{9x^2-1}}{e^{2t\sqrt{9x^2-1}}+1}e^{3xt}\cosh\left(t\sqrt{9x^2-1}\right)$$

Furthermore

$$D(x,t)=-\frac{t\sqrt{9x^2-1}}{e^{t\sqrt{9x^2-1}}}e^{3xt}$$

but

$$\frac{1}{e^{t\sqrt{9x^2-1}}}=e^{t\sqrt{9x^2-1}}-2\sinh\left(t\sqrt{9x^2-1}\right)$$

then

$$D(x,t)=t\sqrt{9x^2-1}\left(2\sinh\left(t\sqrt{9x^2-1}\right)-e^{t\sqrt{9x^2-1}}\right)e^{3xt}$$

and

$$D(x,t)=t\sqrt{9x^2-1}\left(\sinh\left(t\sqrt{9x^2-1}\right)-\cosh\left(t\sqrt{9x^2-1}\right)\right)e^{3xt}.$$

Which means that

$$D(x,t)=\left(9x^2-1\right)\sum_{n\geq 0}nB^*_{n-1}(x)\frac{t^n}{n!}-\sqrt{9x^2-1}\sum_{n\geq 0}nC_{n-1}(x)\frac{t^n}{n!}.$$

Finally we obtain

$$D_n(x)=n\left(9x^2-1\right)B^*_{n-1}(x)-n\sqrt{9x^2-1}C_{n-1}(x)$$

and the identity (15) follows. □

Frontczak [3, Corollary 4] proved the following relations between balancing numbers and Bernoulli numbers

$$\sum_{\substack{k=0\\ n\equiv k \pmod 2}}^{n}\binom{n}{k}32^{\frac{n-k}{2}}B^*_kB_{n-k}=nC_{n-1}$$

and

$$\sum_{\substack{k=0\\ n\equiv k \pmod 2}}^{n} \binom{n}{k} 32^{\frac{n-k}{2}} \left(2^{n-k}-1\right) C_k B_{n-k} = 8nB^*_{n-1}.$$

An improvement of these identities is given in the following corollary.

Corollary 2. *Let $n \geq 1$. Then*

$$\sum_{k=0}^{n} \binom{n}{k} 32^{\frac{n-k}{2}} B^*_k B_{n-k} = n\left(C_{n-1} - 2\sqrt{2}B^*_{n-1}\right) \tag{16}$$

and

$$\sum_{k=0}^{n} \binom{n}{k} 32^{\frac{n-k}{2}} \left(2^{n-k}-1\right) B_{n-k} C_k = 8nB^*_{n-1} - 2n\sqrt{2}C_{n-1}. \tag{17}$$

Proof. Evaluate (14) and (15) at the point $x = 1$ to get the desired results. □

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