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On 3- and 9-Regular Cubic Partitions

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Abstract

Let $a_3(n)$ and $a_9(n)$ be 3- and 9-regular cubic partitions of n. In this paper, we establish several infinite families of congruences modulo powers of 3. For example, for all non-negative integers n and α we find that

$$a_3\left(3^{2\alpha}n + \frac{3^{2\alpha}-1}{4}\right) \equiv 0 \pmod{3^\alpha}$$

and

$$a_9\left(3^{\alpha+1}n + 3^{\alpha+1} - 1\right) \equiv 0 \pmod{3^{\alpha+1}}.$$

1 Introduction

A partition of a positive integer n is a non-increasing sequence of positive integers whose sum is n. Let $p(n)$ denote the number of partitions of n and the generating function is

$$\sum_{n\geq 0} p(n)q^n = \frac{1}{f_1},$$

where, here and throughout the paper, we set

$$f_k = (q^k;q^k)_\infty = \prod_{m=1}^{\infty}(1-q^{km}).$$

Chan [1] studied the cubic partition function denoted by $a(n)$, whose generating function is

$$\sum_{n\geq 0} a(n)q^n = \frac{1}{f_1 f_2}.$$

He found the generating function

$$\sum_{n\geq 0} a(3n+2)q^n = 3\frac{f_3^3 f_6^3}{f_1^4 f_2^4},$$

which readily implies that

$$a(3n+2) \equiv 0 \pmod{3}.$$

Chan [2] established infinite family of congruence modulo powers of 3 for $a(n)$. For each $n, k \geq 1$, he proved that

$$a(3^k n + c_k) \equiv 0 \pmod{3^{k+\delta(k)}}, \tag{1}$$

where c_k is the reciprocal modulo 3^k of 8 and

$$\delta(k) := \begin{cases} 1, & \text{if } k \text{ is even;} \\ 0, & k \text{ is odd.} \end{cases}$$

Zhao and Zhong [9] studied cubic partition pairs denoted by $b(n)$, the generating function satisfied by $b(n)$ is

$$\sum_{n\geq 0} b(n)q^n = \frac{1}{f_1^2 f_2^2}. \tag{2}$$

For each $n \geq 0$, they found Ramanujan's type congruences for $b(n)$, namely

$$b(5n+4) \equiv 0 \pmod{5},$$
$$b(7n+i) \equiv 0 \pmod{7},$$
$$b(9n+7) \equiv 0 \pmod{9},$$

where $i \in \{2,3,4,6\}$.

Lin [6] studied the cubic partition pairs and established the following congruences modulo 27:

$$b(27n+16) \equiv 0 \pmod{27}, \tag{3}$$
$$b(27n+25) \equiv 0 \pmod{27}, \tag{4}$$
$$b(81n+61) \equiv 0 \pmod{27}. \tag{5}$$

Also, he proposed the following conjectures:

Conjecture 1. For each $n \geq 0$,

$$b(81n+61) \equiv 0 \pmod{81}. \tag{6}$$

Conjecture 2.

$$\sum_{n\geq 0} b(81n+7)q^n \equiv 9\frac{f_2 f_3^2}{f_6} \pmod{81}, \tag{7}$$
$$\sum_{n\geq 0} b(81n+34)q^n \equiv 36\frac{f_1 f_6^2}{f_3} \pmod{81}. \tag{8}$$

The above conjectures were proved by Gireesh and Naika [4], Chern [3], and Lin et al. [7].

Motivated by the above results, in this paper, we study 3- and 9-regular cubic partitions, which are defined as follow:

- Let $a_3(n)$ denote the number of 3-regular cubic partitions of n, whose generating function is
$$\sum_{n\geq 0} a_3(n)q^n = \frac{f_3 f_6}{f_1 f_2}. \tag{9}$$

- Let $a_9(n)$ denote the number of 9-regular cubic partitions of n, whose generating function is
$$\sum_{n\geq 0} a_9(n)q^n = \frac{f_9 f_{18}}{f_1 f_2}. \tag{10}$$

We shall show that

$$\sum_{n\geq 0} a_3(3n+2)q^n = 3\frac{f_3^3 f_6^3}{f_1^3 f_2^3} \tag{11}$$

and

$$\sum_{n\geq 0} a_9(3n+2)q^n = 3\frac{f_3^4 f_6^4}{f_1^4 f_2^4}. \tag{12}$$

These are analogous to Ramanujan's most beautiful identities [8, pp. 239, 243]

$$\sum_{n\geq 0} p(5n+4)q^n = 5\frac{f_5^5}{f_1^6} \tag{13}$$

and

$$\sum_{n\geq 0} p(7n+5)q^n = 7\frac{f_7^3}{f_1^4} + 49q\frac{f_7^7}{f_1^8}. \tag{14}$$

We also obtain infinite families of congruences modulo powers of 3 for $a_3(n)$ and $a_9(n)$, which are stated in the following theorems:

Theorem 3. *For each $n, \alpha \geq 0$,*

$$a_3\left(3^{2\alpha}n + \frac{3^{2\alpha}-1}{4}\right) \equiv 0 \pmod{3^{\alpha}}, \tag{15}$$

$$a_3\left(3^{2\alpha+1}n + \frac{3^{2\alpha+2}-1}{4}\right) \equiv 0 \pmod{3^{\alpha+1}}, \tag{16}$$

$$a_3\left(3^{2\alpha+2}n + \frac{7\times 3^{2\alpha+1}-1}{4}\right) \equiv 0 \pmod{3^{\alpha+2}}, \tag{17}$$

$$a_3\left(3^{2\alpha+2}n + \frac{11\times 3^{2\alpha+1}-1}{4}\right) \equiv 0 \pmod{3^{\alpha+2}}. \tag{18}$$

Theorem 4. *For each $n, \alpha \geq 0$,*

$$a_9\left(3^{\alpha+1}n + 3^{\alpha+1} - 1\right) \equiv 0 \pmod{3^{\alpha+1}}. \tag{19}$$

The results (15)–(19) are analogous to Ramanujan's congruences modulo powers of 5 [5], for $n, \alpha \geq 0$,

$$p\left(5^{2\alpha+1}n + \frac{19\times 5^{2\alpha+1}+1}{24}\right) \equiv 0 \pmod{5^{2\alpha+1}} \tag{20}$$

and

$$p\left(5^{2\alpha+2}n + \frac{23\times 5^{2\alpha+2}+1}{24}\right) \equiv 0 \pmod{5^{2\alpha+2}}. \tag{21}$$

2 Preliminaries

Define

$$\zeta = \frac{f_1 f_2}{q f_9 f_{18}}$$

and

$$T = \frac{f_3^4 f_6^4}{q^3 f_9^4 f_{18}^4}.$$

Let H be the "huffing" operator defined by

$$H\left(\sum a_n q^n\right) = \sum a_{3n} q^{3n}.$$

From Chan [2, (11)–(19)], for each $i \geq 1$, we have

$$H\left(\frac{1}{\zeta^i}\right) = \sum_{j=1}^{i} \frac{m_{i,j}}{T^j}, \tag{22}$$

where $m_{i,j}$'s are defined in the following matrix.

The $m_{i,j}$ form a matrix M, the first nine rows of which are

$$M = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 3^3 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 3^3 & 3^5 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2\cdot 3^2 & 2^2\cdot 3^4 & 3^7 & 0 & 0 & 0 & \cdots \\ 0 & 5 & 2\cdot 3^3\cdot 5 & 3^6\cdot 5 & 3^9 & 0 & 0 & \cdots \\ 0 & 1 & 2\cdot 3^2\cdot 7 & 3^6\cdot 5 & 2\cdot 3^9 & 3^{11} & 0 & \cdots \\ 0 & 0 & 2\cdot 3\cdot 7 & 2^2\cdot 3^4\cdot 7 & 3^8\cdot 7 & 3^{10}\cdot 7 & 3^{13} & \cdots \\ 0 & 0 & 2^3 & 2\cdot 3^3\cdot 19 & 2^4\cdot 3^7 & 2^2\cdot 3^9\cdot 7 & 2^3\cdot 3^{12} & \cdots \\ 0 & 0 & 1 & 2^2\cdot 3^4 & 3^9 & 3^9\cdot 5^2 & 2^2\cdot 3^{13} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{23}$$

and for $i \geq 4$, $m_{i,1} = 0$, and for $j \geq 2$,

$$m_{i,j} = 9m_{i-1,j-1} + 3m_{i-2,j-1} + m_{i-3,j-1}. \tag{24}$$

In fact $m_{4i-3,j} = 0$ for $j \leq i-1$, so we can write

$$H\left(\frac{1}{\zeta^{4i-3}}\right) = \sum_{j=i}^{4i-3} \frac{m_{4i-3,j}}{T^j} = \sum_{j=1}^{3i-2} \frac{m_{4i-3,i+j-1}}{T^{i+j-1}} = \sum_{j=1}^{3i-2} \frac{a_{i,j}}{T^{i+j-1}}, \tag{25}$$

where

$$a_{i,j} = m_{4i-3,i+j-1}. \tag{26}$$

Similarly, $m_{4i-1,j} = 0$ if $j \leq i-1$, so we can write

$$H\left(\frac{1}{\zeta^{4i-1}}\right) = \sum_{j=i}^{4i-1} \frac{m_{4i-1,j}}{T^j} = \sum_{j=1}^{3i} \frac{m_{4i-1,i+j-1}}{T^{i+j-1}} = \sum_{j=1}^{3i} \frac{b_{i,j}}{T^{i+j-1}}, \tag{27}$$

where

$$b_{i,j} = m_{4i-1,i+j-1}. \tag{28}$$

And $m_{4i,j} = 0$ if $j \le i$, so we can write

$$H\left(\frac{1}{\zeta^{4i}}\right) = \sum_{j=1+i}^{4i} \frac{m_{4i,j}}{T^j} = \sum_{j=1}^{3i} \frac{m_{4i,i+j}}{T^{i+j}} = \sum_{j=1}^{3i} \frac{c_{i,j}}{T^{i+j}}, \tag{29}$$

where

$$c_{i,j} = m_{4i,i+j}. \tag{30}$$

We can write (25) as

$$H\left(\left(q\frac{f_9 f_{18}}{f_1 f_2}\right)^{4i-3}\right) = \sum_{j=1}^{3i-2} a_{i,j} \left(q^3 \frac{f_9^4 f_{18}^4}{f_3^4 f_6^4}\right)^{i+j-1}, \tag{31}$$

which can be rearranged to

$$H\left(q^{i-3}\left(\frac{f_3 f_6}{f_1 f_2}\right)^{4i-3}\right) = \sum_{j=1}^{3i-2} a_{i,j} q^{3j-3} \left(\frac{f_9 f_{18}}{f_3 f_6}\right)^{4j-1}. \tag{32}$$

The equation (27) can be written as

$$H\left(\left(q\frac{f_9 f_{18}}{f_1 f_2}\right)^{4i-1}\right) = \sum_{j=1}^{3i} b_{i,j} \left(q^3 \frac{f_9^4 f_{18}^4}{f_3^4 f_6^4}\right)^{i+j-1}, \tag{33}$$

which implies that

$$H\left(q^{i-1}\left(\frac{f_3 f_6}{f_1 f_2}\right)^{4i-1}\right) = \sum_{j=1}^{3i} b_{i,j} q^{3j-3} \left(\frac{f_9 f_{18}}{f_3 f_6}\right)^{4j-3}. \tag{34}$$

Similarly (29) is

$$H\left(\left(q\frac{f_9 f_{18}}{f_1 f_2}\right)^{4i}\right) = \sum_{j=1}^{3i} c_{i,j} \left(q^3 \frac{f_9^4 f_{18}^4}{f_3^4 f_6^4}\right)^{i+j}, \tag{35}$$

and this can be rearranged to

$$H\left(q^{i}\left(\frac{f_3 f_6}{f_1 f_2}\right)^{4i}\right) = \sum_{j=1}^{3i} c_{i,j} q^{3j} \left(\frac{f_9 f_{18}}{f_3 f_6}\right)^{4j}. \tag{36}$$

3 Generating functions

In this section, we deduce generating functions that are useful in proving our main results.

Theorem 5. *For each* $\alpha \geq 0$,

$$\sum_{n\geq 0} a_3\left(3^{2\alpha}n+\frac{3^{2\alpha}-1}{4}\right)q^n=\sum_{i\geq 1} x_{2\alpha,i}q^{i-1}\left(\frac{f_3f_6}{f_1f_2}\right)^{4i-3} \tag{37}$$

and

$$\sum_{n\geq 0} a_3\left(3^{2\alpha+1}n+\frac{3^{2\alpha+2}-1}{4}\right)q^n=\sum_{i\geq 1} x_{2\alpha+1,i}q^{i-1}\left(\frac{f_3f_6}{f_1f_2}\right)^{4i-1}, \tag{38}$$

where the coefficient vectors $\mathbf{x}_\alpha=(x_{\alpha,1},x_{\alpha,2},\ \ldots\ )$ *are given by*

$$\mathbf{x}_0=(x_{0,1},x_{0,2},x_{0,3},\ \ldots\ )=(1,0,0,\ \ldots\ ), \tag{39}$$

and

$$\mathbf{x}_{\alpha+1}=\mathbf{x}_\alpha A \quad \textit{if } \alpha \textit{ is even}, \tag{40}$$

$$\mathbf{x}_{\alpha+1}=\mathbf{x}_\alpha B \quad \textit{if } \alpha \textit{ is odd}, \tag{41}$$

where $A=(a_{i,j})_{i,j\geq 1}$ *and* $B=(b_{i,j})_{i,j\geq 1}$.

Proof. The identity (9) is the $\alpha=0$ case of (37).

Suppose (37) holds for some $\alpha\geq 0$. Then

$$\sum_{n\geq 0} a_3\left(3^{2\alpha}n+\frac{3^{2\alpha}-1}{4}\right)q^n=\sum_{i\geq 1} x_{2\alpha,i}q^{i-1}\left(\frac{f_3f_6}{f_1f_2}\right)^{4i-3}, \tag{42}$$

which is equivalent to

$$\sum_{n\geq 0} a_3\left(3^{2\alpha}n+\frac{3^{2\alpha}-1}{4}\right)q^{n-2}=\sum_{i\geq 1} x_{2\alpha,i}q^{i-3}\left(\frac{f_3f_6}{f_1f_2}\right)^{4i-3}. \tag{43}$$

Applying the operator H to (43), we find that

$$\begin{aligned}\sum_{n\geq 0} a_3\left(3^{2\alpha}(3n+2)+\frac{3^{2\alpha}-1}{4}\right)q^{3n}&=\sum_{i\geq 1} x_{2\alpha,i}H\left(q^{i-3}\left(\frac{f_3f_6}{f_1f_2}\right)^{4i-3}\right)\\&=\sum_{i\geq 1} x_{2\alpha,i}\sum_{j=1}^{3i-2} a_{i,j}q^{3j-3}\left(\frac{f_9f_{18}}{f_3f_6}\right)^{4j-1}\\&=\sum_{j\geq 1}\left(\sum_{i\geq 1} x_{2\alpha,i}a_{i,j}\right)q^{3j-3}\left(\frac{f_9f_{18}}{f_3f_6}\right)^{4j-1}\\&=\sum_{j\geq 1} x_{2\alpha+1,j}q^{3j-3}\left(\frac{f_9f_{18}}{f_3f_6}\right)^{4j-1},\end{aligned}$$

which implies equation (38).

Now suppose (38) holds for some $\alpha \geq 0$. Then

$$\sum_{n\geq 0} a_3\left(3^{2\alpha+1}n+\frac{3^{2\alpha+2}-1}{4}\right)q^n=\sum_{i\geq 1} x_{2\alpha+1,i}q^{i-1}\left(\frac{f_3f_6}{f_1f_2}\right)^{4i-1}. \tag{44}$$

Applying the operator H to (44), we find that

$$\begin{aligned}\sum_{n\geq 0} a_3\left(3^{2\alpha+1}(3n)+\frac{3^{2\alpha+2}-1}{4}\right)q^{3n} &= \sum_{i\geq 1} x_{2\alpha+1,i}H\left(q^{i-1}\left(\frac{f_3f_6}{f_1f_2}\right)^{4i-1}\right)\\ &= \sum_{i\geq 1} x_{2\alpha+1,i}\sum_{j=1}^{3i} b_{i,j}q^{3j-3}\left(\frac{f_9f_{18}}{f_3f_6}\right)^{4j-3}\\ &= \sum_{j\geq 1}\left(\sum_{i\geq 1} x_{2\alpha+1,i}b_{i,j}\right)q^{3j-3}\left(\frac{f_9f_{18}}{f_3f_6}\right)^{4j-3}\\ &= \sum_{j\geq 1} x_{2\alpha+2,j}q^{3j-3}\left(\frac{f_9f_{18}}{f_3f_6}\right)^{4j-3}.\end{aligned}$$

After simplification, we obtain (37) with $\alpha+1$ in place of α. This completes the proof of (37) and (38) by induction. □

Theorem 6. *For each $\alpha \geq 0$,*

$$\sum_{n\geq 0} a_9\left(3^{\alpha+1}n+3^{\alpha+1}-1\right)q^n=\sum_{i\geq 1} y_{\alpha,i}q^{i-1}\left(\frac{f_3f_6}{f_1f_2}\right)^{4i} \tag{45}$$

where the coefficient vectors $\mathbf{Y}_\alpha=(y_{\alpha,1},y_{\alpha,2},\ \ldots\ )$ are given by

$$\mathbf{Y}_0=(y_{0,1},y_{0,2},y_{0,3},\ \ldots\ )=(3,0,0,\ \ldots\ ), \tag{46}$$

and

$$\mathbf{Y}_{\alpha+1}=\mathbf{Y}_\alpha C, \tag{47}$$

where $C=(c_{i,j})_{i,j\geq 1}$.

Proof. We prove this by induction on α. The identity (12) is the $\alpha=0$ case of (45).

Suppose (45) holds for some $\alpha \geq 0$. Then

$$\sum_{n\geq 0} a_9\left(3^{\alpha+1}n+3^{\alpha+1}-1\right)q^n=\sum_{i\geq 1} y_{\alpha,i}q^{i-1}\left(\frac{f_3f_6}{f_1f_2}\right)^{4i}, \tag{48}$$

which is equivalent to

$$\sum_{n\geq 0} a_9\left(3^{\alpha+1}n+3^{\alpha+1}-1\right)q^{n-2} = q^{-3}\sum_{i\geq 1} y_{\alpha,i}q^i\left(\frac{f_3f_6}{f_1f_2}\right)^{4i}. \tag{49}$$

Applying the operator H to (49), we find that

$$\begin{aligned}\sum_{n\geq 0} a_9\left(3^{\alpha+1}(3n+2)+3^{\alpha+1}-1\right)q^{3n} &= q^{-3}\sum_{i\geq 1} y_{\alpha,i}H\left(q^i\left(\frac{f_3f_6}{f_1f_2}\right)^{4i}\right)\\ &= \sum_{i\geq 1} y_{\alpha,i}\sum_{j=1}^{3i} c_{i,j}q^{3j-3}\left(\frac{f_9f_{18}}{f_3f_6}\right)^{4j}\\ &= \sum_{j\geq 1}\left(\sum_{i\geq 1} y_{\alpha,i}c_{i,j}\right)q^{3j-3}\left(\frac{f_9f_{18}}{f_3f_6}\right)^{4j}\\ &= \sum_{j\geq 1} y_{\alpha+1,j}q^{3j-3}\left(\frac{f_9f_{18}}{f_3f_6}\right)^{4j},\end{aligned}$$

which implies that

$$\sum_{n\geq 0} a_9\left(3^{\alpha+2}n+3^{\alpha+2}-1\right)q^{n} = \sum_{j\geq 1} y_{\alpha+1,j}q^{j-1}\left(\frac{f_3f_6}{f_1f_2}\right)^{4j}, \tag{50}$$

which is (45) with $\alpha+1$ for α . □

4 Congruences

Let $\nu(N)$ be the largest power of 3 that divides N. Note that $\nu(0)=+\infty$.

Proof of Theorem 3. It follows from (23) and (24) that

$$\nu(m_{i.j})\geq 3j-i-1, \tag{51}$$

and from (26), (28) and (51),

$$\nu(a_{i.j})\geq 3(i+j-1)-(4i-3)-1=3j-i-1 \tag{52}$$

and

$$\nu(b_{i.j})\geq 3(i+j-1)-(4i-1)-1=3j-i-3. \tag{53}$$

It is not hard to show that

$$\nu(x_{2\alpha,j})\geq \alpha+3j-4 \tag{54}$$

and

$$\nu(x_{2\alpha+1,j}) \geq \alpha + 1 + 3(j-1). \tag{55}$$

The identity (54) is true for $\alpha = 0$, by (39).

Suppose (54) is true for some $\alpha \geq 0$. Then

$$\begin{aligned}\nu(x_{2\alpha+1,j}) &\geq \min_{i\geq 1}\left(\nu(x_{2\alpha,i}) + \nu(a_{i,j})\right)\\ &= \nu(x_{2\alpha,1}) + \nu(a_{1,j})\\ &\geq \alpha + 3j - 2\\ &\geq \alpha + 1 + 3(j-1),\end{aligned}$$

which is (55).

Now suppose (55) is true for all $\alpha \geq 0$. Then

$$\begin{aligned}\nu(x_{2\alpha+2,j}) &\geq \min_{i\geq 1}\left(\nu(x_{2\alpha+1,i}) + \nu(b_{i,j})\right)\\ &= \nu(x_{2\alpha+1,1}) + \nu(b_{1,j})\\ &\geq \alpha + 1 + 3j - 4,\end{aligned}$$

which is (54) with $\alpha+1$ in place of α. This completes the proof of (54) and (55) by induction.

The congruence (15) follows from (37) together with (54), and the congruence (16) follows from (38) together with (55).

From (38) and (55), we have

$$\sum_{n\geq 0} a_3\left(3^{2\alpha+1}n + \frac{3^{2\alpha+2}-1}{4}\right)q^n \equiv 3^{\alpha+1}\frac{f_3^3 f_6^3}{f_1^3 f_2^3} \pmod{3^{\alpha+4}}. \tag{56}$$

By the binomial theorem, it is easy to see that

$$f_1^3 \equiv f_3 \pmod{3}. \tag{57}$$

In view of (57), the congruence (56) can be expressed as

$$\sum_{n\geq 0} a_3\left(3^{2\alpha+1}n + \frac{3^{2\alpha+2}-1}{4}\right)q^n \equiv 3^{\alpha+1}\frac{f_9 f_{18}}{f_3 f_6} \pmod{3^{\alpha+2}}. \tag{58}$$

Equating the coefficients of q^{3n+1} and q^{3n+2} in (58), we obtain (17) and (18), respectively. □

Proof of Theorem 4. It follows from (30) and (51) that

$$\nu(c_{i.j}) \geq 3(i+j) - 4i - 1 = 3j - i - 1. \tag{59}$$

It is not hard to show that

$$\nu(y_{\alpha,j}) \geq \alpha + 1 + 3(j-1). \tag{60}$$

The identity (60) is true for $\alpha = 0$, by (46).

Suppose (60) is true for some $\alpha \geq 0$. Then

$$\begin{aligned}\nu(y_{\alpha+1,j}) &\geq \min_{i\geq 1}\left(\nu(y_{\alpha,i}) + \nu(c_{i,j})\right)\\ &= \nu(y_{\alpha,1}) + \nu(c_{1,j})\\ &\geq \alpha + 1 + 3j - 2\\ &\geq \alpha + 2 + 3(j-1),\end{aligned}$$

which is (60) with $\alpha + 1$ for α.

The congruence (19) follows from (45) together with (60). □

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