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A Polynomial Variant of Perfect Numbers

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Abstract

We study the binary polynomials A such that the sum of all their divisors $D \neq A$ is a perfect polynomial, and A is a power of an irreducible polynomial.

1 Introduction

We will work with *binary* polynomials, i.e., polynomials in one variable over the finite field $\mathbb{F}_2$. In order to describe our work, we require some terminology. Let $A \in \mathbb{F}_2[x]$ be a polynomial. We say that A is *even* if A has a linear factor; otherwise A is *odd*. We let $\mathbb{N}$ denote the set of positive integers. A polynomial $A \in \mathbb{F}_2[x]$ is *Mersenne* if $A = 1 + x^a(x+1)^b$ for some $a, b \in \mathbb{N}$. If A is irreducible, then we say that A is *Mersenne prime*.

We let $\omega(A)$ denote the number of distinct irreducible (or *prime*) factors of A over $\mathbb{F}_2$, we let $\sigma(A)$ denote the sum of all divisors of A, including 1 and A, (e.g., $\sigma(0) = 0$, $\sigma(1) = 1$, $\sigma(x) = x+1$, $\sigma(x^2) = x^2+x+1$, $\sigma(x^2+x) = x^2+x$, $\sigma(x^2+x+1) = 1+(x^2+x+1) = x^2+x$). We explain in more detail why $\sigma(x^2) = x^2+x+1$ and $\sigma(x^2+x+1) = x^2+x$. Indeed, since the list of all divisors of x^2 is $[1, x, x^2]$, their sum $1+x+x^2$ gives $\sigma(x^2)$ and $\sigma(x^2+x+1) = 1+(x^2+x+1)$ because $P = x^2+x+1$ is irreducible so that $\sigma(P) = 1+P$.

Observe that σ is a multiplicative function (i.e., $\sigma(XY) = \sigma(X)\sigma(Y)$, provided that $\gcd(X, Y) = 1$ in $\mathbb{F}_2[x]$). If $\sigma(A) = A$, then we say that A is *perfect*. The first examples (besides 0 and 1) of perfect polynomials are those of the form $x^{2^n-1}(x+1)^{2^n-1} = (x^2+x)^{2^n-1}$, where $n \in \mathbb{N}$. We call them *trivial perfect*. Canaday [2, Theorem 17] proved that these are the only perfect polynomials A with $\omega(A) = 2$. Gallardo and Rahavandrainy [5, 7] obtained some results about odd perfect polynomials, but we do not know (as in the integer case) whether or not there exist odd perfect polynomials. Mersenne primes play an important role for (the known) nontrivial perfect polynomials over $\mathbb{F}_2$, that we call *sporadic perfect*. Proposition 1 contains the list of all known sporadic perfect polynomials. Indeed, up to two exceptions, they all have factorizations with only Mersenne primes as odd divisors. We are unable to describe a general form of sporadic perfect polynomials, in contrary to what happens in the integer case, where any even perfect number $n = 2^{p-1}(2^p-1)$, in which both p and 2^p-1 are prime numbers, has exactly two distinct prime factors $f_1 = 2$ and $f_2 = 2^p-1$. The set of Mersenne prime numbers, as well as the set of Mersenne prime polynomials are examples of sets for which we do not know whether they are finite. Testing irreducibility for polynomials (in particular for trinomials) over a finite field, remains difficult, even if the problem has been addressed several times. Brent et al., Fredricksen and Wisniewski, Swan, and Zierler [1, 4, 12, 13] obtained several results for trinomials. A difficulty in working over $\mathbb{F}_2[x]$ is that, in contrast to the set $\mathbb{Z}$ of integers, we have no order relation in $\mathbb{F}_2[x]$, an important tool in many proofs of results for perfect numbers. In the *Online Encyclopedia of Integer Sequences* (OEIS) [10], one finds several sequences related to binary polynomials, e.g., A001037, that count for every degree n, the number of irreducible binary polynomials of degree n.

Gallardo and Rahavandrainy [8] proved (under a mild condition) that all even perfect polynomials over $\mathbb{F}_2$ which are products of Mersenne primes are equal to nine of the eleven known sporadic perfect polynomials, all of which are even. More precisely, M_{11a} and M_{11b} (see below) are the unique (known) sporadic perfect polynomials that are *not* a product of Mersenne primes (since P_{4c} is not a Mersenne prime polynomial).

Cengiz et al. [3] proved the following proposition, part (b) of which is due to Canaday [2].

Proposition 1. (a) *A possible new even perfect polynomial must have a degree exceeding* 200*.*

(b) *Let* $P_2 = x^2+x+1$*,* $P_{3a} = x^3+x+1$*,* $P_{3b} = P_{3a}(x+1)$*,* $P_{4a} = x^4+x^3+1$*,* $P_{4b} = P_{4a}(x+1)$*, and* $P_{4c} = x^4+x+1$*. The list of all known sporadic perfect polynomials* $M_* \in \mathbb{F}_2[x]$*, in non-decreasing order of their degrees, and factored in terms of prime polynomials* P_**, is* $M_{5a} = x(x+1)^2P_2$*,* $M_{5b} = (x+1)x^2P_2$*,* $M_{11a} = x(x+1)^2{P_2}^2P_{4c}$*,* $M_{11b} = x^2(x+1){P_2}^2P_{4c}$*,* $M_{11c} = x^3(x+1)^4P_{4a}$*,* $M_{11d} = x^4(x+1)^3P_{4b}$*,* $M_{15a} = x^3(x+1)^6P_{3a}P_{3b}$*,* $M_{15b} = x^6(x+1)^3P_{3a}P_{3b}$*,* $M_{16} = x^4(x+1)^4P_{4a}P_{4b}$*,* $M_{20a} = x^4(x+1)^6P_{3a}P_{3b}P_{4b}$*,* $M_{20b} = x^6(x+1)^4P_{3a}P_{3b}P_{4a}$*.*

Canaday [2, Theorem 1, Theorem 2] also proved the following.

Lemma 2. *Every odd perfect polynomial over* $\mathbb{F}_2$ *is a square.*

This result is the analogue of the well known fact that any odd perfect number $n > 1$ (i.e., n equals the sum of all its proper positive divisors), can be written as $n = p^{4k+1}m^2$, where k is a non-negative integer, p is an odd prime number congruent to 1 modulo 4, and m is a positive integer.

A *special perfect* polynomial is an odd perfect polynomial that equals the square of a square-free polynomial. Gallardo and Rahavandrainy [5, Theorem 5.3, Theorem 5.5] proved the following.

Lemma 3. *Let $A \in \mathbb{F}_2[x]$ be a special perfect polynomial and P a prime divisor of A. Then $\omega(A) \geq 10$, $\deg(P) \geq 30$, $\deg(P)$ is even, and $P \equiv 1 \pmod{x^2 + x + 1}$.*

Gallardo and Rahavandrainy [6, Lemma 2.6]) also proved the following.

Lemma 4. *Let P be a Mersenne prime and m a positive integer. Then $\sigma(P^{2m})$ is square-free.*

We now have two more results.

Lemma 5. *Let P be a Mersenne polynomial (prime or not) and $n \in \mathbb{N}$ such that $S = 1 + P + \cdots + P^{2^n-1}$ is perfect. Then $P = 1 + (x^2 + x)^a$ with $a = \frac{2^m-1}{2^n-1}$ for some multiple m of n.*

Proof. Put $P = 1 + x^a(x + 1)^b$. The polynomial $S = 1 + P + \cdots + P^{2^n-1} = (1 + P)^{2^n-1} = (x^a(x + 1)^b)^{2^n-1}$ is trivial perfect. So, $(2^n - 1)a = (2^n - 1)b = 2^m - 1$ for some integer m. Thus, $a = b$ and $2^n - 1$ divides $2^m - 1$. Therefore, $2^n - 1 = \gcd(2^n - 1, 2^m - 1) = 2^{\gcd(n,m)} - 1$, i.e., n divides m. □

Lemma 6. *Let P be a prime in $\mathbb{F}_2[x]$ and let h be a positive integer. Then $\sigma(P^{2h})$ is not a square.*

Proof. Assume, contrary to what we want to prove, that $A = \sigma(P^{2h})$ is a square in $\mathbb{F}_2[x]$. Thus $A' = 0$, where A' is the formal derivative of A relative to x. Observe that $P' \neq 0$. Since $A = (1 + P)(1 + P + \cdots + P^{h-1})^2 + P^{2h}$, one has

$$A' = P'(1 + P + \cdots + P^{h-1})^2 \neq 0, \text{ a contradiction.}$$

This proves the lemma. □

Remember that a positive integer $n \in \mathbb{N}$ is *perfect* if $\sigma(n) = 2n$ (a property which is equivalent to the definition used in the paragraph after Lemma 2).

Lescot [9] recently proved the following:

Theorem 7. *Consider the following equation in which $\sigma(n)$ is the sum of all positive divisors of n.*

$$\sigma(n) = (\sigma(n) - 2n)^2. \tag{1}$$

One has

(a) *If n satisfies (1) and if n is a prime power, then $n \in \{1, 3\}$.*

(b) *If m is a perfect number for which $2m - 1$ is a prime number, then $n = m(2m - 1)$ satisfying (1).*

(c) *If n satisfies (1) and if n is even, then there is an even perfect number m such that $n/m = 2m - 1$.*

We cannot generalize condition (1) to $\mathbb{F}_2[x]$. Indeed, by considering degrees, the equation $\sigma(A) = (\sigma(A) - 2A)^2$ in $\mathbb{F}_2[x]$ implies that $\sigma(A) \in \{0, 1\}$. So, it has only the trivial solutions $A = 0$ or $A = 1$. However, a generalization of (1) to the equation $\sigma(A) = C^2$ in $\mathbb{F}_2[x]$, is possible. There are many solutions, e.g., $A = x^4 + x$ and $C = x^2 + x$, in which C divides A, or $A = x^6 + x^4 + x^3 + x$ and $C = x^3 + x^2$, in with C does not divide A. Even adding more constraints, such as $C \mid A$ or $\omega(A) = \omega(C)$, one has (for example) a solution $A = x(x^2 + x + 1)(x + 1)^5$ and $C = x(x + 1)(x^2 + x + 1)$. The full solution of the equation might be very difficult to obtain, since a possible solution is $A = C^2$, with C^2 being an odd perfect polynomial. Since in characteristic 2, the analogue of squares are indeed the expressions of the form $C^2 + C$, a possible appropriate generalization would be the following equation in the unknown A,

$$\sigma(A) = C^2 + C. \tag{2}$$

Remark 8. There are many solutions of (2); for example, $A = x^4 + x^3 + x^2$ and $C = x^2 + x$ or $A = x^4 + x$ and $C = x^2$. One non-trivial possibility is to ask for solutions of (2) with $A = P$, a prime polynomial, so that a complete solution of (2) amounts to finding all prime polynomials $P \in \mathbb{F}_2[x]$ of the form $P = C^2 + C + 1$. We found some of them by computation, e.g., $\{(P = x^2+x+1, C = x), (P = x^4+x+1, C = x^2+x), (P = x^8+x^6+x^4+x^3+x^2+x+1, C = x^4+x^3+x\}$, but a complete theoretical description of all solutions appears to be out of reach.

As an analogue to (1) over the ring $\mathbb{F}_2[x]$, we propose, instead, the following condition (in which the polynomial A is the unknown).

Condition 9. A in $\mathbb{F}_2[x]$ satisfies

$$\sigma(A) + A \text{ is perfect.} \tag{3}$$

It might be impossible to find all polynomials A for which Condition 9 holds.

Observe that any prime polynomial P satisfies (3), since 1 is perfect. Moreover, any perfect polynomial A also satisfies (3), since 0 is a perfect polynomial, but (warning!) it is currently impossible (to our knowledge) to give examples of perfect polynomials besides those already described. Moreover, for a given perfect polynomial $M \in \mathbb{F}_2[x] \setminus \mathbb{F}_2$, possible solutions A of the equation $\sigma(A) + A = M$ have a degree that exceeds the degree of M, since both A and $\sigma(A)$ are monic with same degree. Hence, there are potentially infinitely many candidates A to check, so that we cannot find all solutions of the equation by means of a computer. We checked the special case where $M = M_{5a} = x(x+1)^2(x^2+x+1) = x^5+x^4+x^2+x$ for all A up to degree 8. Indeed, it suffices to check the A's with $6 \leq \deg(A) \leq 8$, finding two solutions, $A = x^7+x^4+x^2+x = x(x+1)^3(x^3+x^2+1)$, and $A = x^7+x^6+x^4+x^3 = x^3(x^2+x+1)(x+1)^2$.

More generally, we obtain the following result:

Theorem 10. (i) *Let $A \in \mathbb{F}_2[x]$ be such that (3) holds.*

(a) *If $A = P^k$, with P prime and $k > 1$, then k is even.*

(b) *If $A = P^{2^n}$, with P a Mersenne prime and $n \geq 2$, then $P = 1 + x + x^2$, so that $A = (1 + x + x^2)^{2^n}$.*

(c) *If $A = P^{2^n u}$, with P a Mersenne prime, $n \geq 1$ and $u \geq 3$ odd, then $P = 1+x+x^2$, and $(1 + P + \cdots + P^{u-1})^{2^n}$ is odd perfect.*

(d) *Assume that $n = 1$ in* (c). Put $R = 1 + P + \cdots + P^{u-1}$ (with $P = 1 + x + x^2$). Then R is square-free, $S = R^2$ is odd perfect, $\omega(S) \geq 10$, $u \geq 4391$, so that $\deg(S) \geq 17560$, the degree of every prime divisor Q of R is even, and $Q \equiv 1 \pmod P$. Moreover, if Q is a prime divisor of R that has the minimal possible degree, then $\deg(Q) \geq 30$.

(e) If $A = B \cdot K$, with $B = \sigma(A) + A$ and $\gcd(B, K) = 1$, then $\sigma(K) = K + 1$.

(ii) *If $B \in \mathbb{F}_2[x]$ is a nonzero perfect polynomial, $K \in \mathbb{F}_2[x]$ is prime not dividing B, then $A = B \cdot K$ satisfies (3).*

Remark 11. Inspired by Theorem 10(c), let $P = x^2 + x + 1 \in \mathbb{F}_2[x]$, n be a non-negative integer, and $v \in \mathbb{N}$. If $A = (1 + P + \cdots + P^{2^v-1})^{2^n}$ is perfect, then $n = 0$ and A is trivial perfect. Indeed,

$$A = \left(\frac{1 + P^{2^v}}{1 + P}\right)^{2^n} = (1 + P^{2^n})^{2^v-1} = (1 + P)^{2^n \cdot (2^v-1)} = (x(x + 1))^{2^n \cdot (2^v-1)}. \tag{4}$$

The result now follows from (4) and [2, Lemma 1]. However, for completeness and clarity, we give the details. Let $a = 2^n(2^v - 1)$. Since A is perfect, we get from (4) that $x^a(x+1)^a = \sigma(x^a)\sigma((x + 1)^a)$. Thus $(x + 1)^a$ divides $\sigma(x^a)$. Comparing degrees we get

$$\sigma(x^a) = (x + 1)^a. \tag{5}$$

We claim that $n = 0$. Assume, contrary to what we want to prove, that $n \geq 1$. Then (5) is impossible since $(x + 1)^a$ is a square in $\mathbb{F}_2[x]$. But $1 + x + \cdots + x^a$ is not a square in $\mathbb{F}_2[x]$, hence $a = 2^v - 1$. This proves the result.

2 Some computational results

1. We set $R_u^{2^n} = (1 + P + \cdots + P^{u-1})^{2^n}$ (with $P = 1 + x + x^2$) in part (c) of Theorem 10. Then, by using a straightforward `gp-PARI` program, we checked that for every odd number u with $1 < u < 4391$, one has $R_u^2 \neq \sigma(R_u^2)$. The computation took about 30 hours. As a consequence, we obtained in (d) that R_u^2 is not perfect for all odd numbers u between 3 and 4389. We also considered the cases when $n = 2$ and $n = 3$, with a reduced upper bound for u. More precisely, we obtained that R_u^4 is not perfect when u is odd and $1 < u < 4001$, and that R_u^8 is not perfect when u is odd, and $1 < u < 2001$. The whole computation, including the case $n = 1$, took about 50 hours.

2. Let $u > 1$ be an odd integer, n a positive integer, and P any odd polynomial in $\mathbb{F}_2[x]$, and $R_u = 1 + P + \cdots + P^{u-1}$. Inspired by part (c) of Theorem 10, in which a possible new odd perfect polynomial appears, one might ask the more general (and possibly unanswerable) question:

$$\text{Is } R_u^{2^n} \text{ perfect ?}$$

In the special case when R_u is prime, the answer is clearly *no*. However, consider, for instance, the case when $\omega(R_u) > 1$ and R_u is square-free (cf. Lemma 4, when P is a Mersenne prime and $\omega(R_u) > 1$). In this case, we are unable to give an answer. In the special case when $P = x^2 + x + 1$, it follows by computation, that one may conjecture that R_u is prime if and only if u is a prime number such that 2 is a primitive root modulo u (i.e., u belongs to the OEIS sequence A001122). Surely, by [11, Theorem 2.47], $1 + x + \cdots + x^{u-1}$ is prime if and only if u belongs to the sequence A001122 and $u - 1$ belongs to the sequence A071642.

3. An odd integer $2m-1$ appears when considering perfect numbers, since an even perfect number has the form $n = m(2m - 1)$, with $m = 2^{p-1}$ and a prime number $2m - 1 = 2^p - 1$, as observed in the Introduction. But in $\mathbb{F}_2[x]$ we do not know the form of a general perfect polynomial. There is no "natural" candidate to replace $2m - 1$. Moreover, it is easy to check that a positive integer p is prime if and only if $\sigma(p) = p + 1$, whereas for an $A \in \mathbb{F}_2[x]$, only the implication: A prime $\Longrightarrow \sigma(A) = A + 1$ is correct. For example, when the degree of A is at most 10, the polynomials $A \in \{x^4 + x^2, x^{10} + x^9 + x^8 + x^6 + x^5 + x^4 + 1\}$ are composite, since $x^4 + x^2 = x^2(x + 1)^2$ and $x^{10} + x^9 + x^8 + x^6 + x^5 + x^4 + 1 = (x^3 + x + 1)(x^3 + x^2 + 1)(x^4 + x + 1)$. However, one has $\sigma(A) = A + 1$.

   Furthermore, we do not know what happens when $k > 2$ in Theorem 10(a). Indeed, we do not know what happens, even in the case when $k = 2$, provided that A is not Mersenne.

   In fact, when $k = 2$, all known primes P for which $\sigma(P^2) + P^2 = P + 1$ is perfect, belong to $\{x^2+x+1, x^5+x^2+1, x^5+x^4+x^2+x+1\}$. This is proved by a computation when $P = M + 1$, and M is sporadic perfect. The proof when M is trivial perfect, i.e., when $M = (x(x + 1))^{2^n-1}$ for some positive integer n, is as follows. One can see that $P = M + 1 = (x(x + 1))^{2^n-1} + 1$ is divisible by $x^2 + x + 1$, provided that $n \geq 2$. Thus, P is prime only in the special case when $n = 1$, for which we obtain $P = x^2 + x + 1$.

   Thus, for $k = 2$, the non-trivial case remains unknown: P is a prime such that, with $A = P^2$, one has that $M = \sigma(A) + A = 1 + P$ is an even perfect polynomial of degree $d \geq 201$ (see Proposition 1).

3 Proof of Theorem 10

(i) (a) Assume, contrary to what we want to prove, that $S = \sigma(P^k) + P^k = \sigma(P^{k-1})$ is perfect, but $k > 1$ is odd. By Lemma 6 we obtain that S is not a square. Thus, S is not a square and S is odd. This contradicts Lemma 2.

(b) In this case, $A = P^k$ with $k = 2^n$, and P is Mersenne prime. Put $P = x^a(x+1)^b+1$ for some positive integers a, b. One has that $S = \sigma(P^k) + P^k = \sigma(P^{2^n-1}) = (1 + P)^{2^n-1} = (x^a(x + 1)^b)^{2^n-1}$. Hence, S is trivial perfect and $a = b$ by Lemma 5. But P prime implies that $a = b = 1$. So, $A = (x^2 + x + 1)^{2^n}$.

(c) Put $P = 1+x^a(x+1)^b$ and $Q(u) = \sigma(P^{u-1})$. Observe that $\sigma(A)+A = \sigma(P^{k-1}) = K \cdot L$ where $K = (1 + P)^{2^n-1}$ and $L = Q(u)^{2^n}$. Observe also that u odd implies that K, L are coprime. Since $K \cdot L$ is perfect and K, L are coprime, one has

$$K \cdot L = \sigma(K) \cdot \sigma(L). \tag{6}$$

We claim that K and $\sigma(L)$ are coprime. In order to prove the claim, it suffices to prove that $D = \gcd(1 + P, \sigma(L))$ equals 1. Since $1 + P = x^a(x + 1)^b$, in order to prove that $D = 1$, it suffices to prove that $\sigma(L)$ is odd. By Lemma 4, $Q(u) = \prod_{j=1}^{r} Q_j$ for some positive integer r and for some pairwise distinct, odd prime polynomials Q_j. Then one has

$$\sigma(L) = \prod_{j=1}^{r} \sigma(Q_j^{2^n}). \tag{7}$$

Since $n \geq 1$, 2^n is even. Thus, by substituting $x = 0$ and $x = 1$ in any $\sigma(Q_j^{2^n})$ we obtain 1. In other words, $\sigma(Q_j^{2^n})$ is odd for any j. It follows then from (7) that $\sigma(L)$ is odd, thereby proving the claim. Since $\gcd(K, \sigma(L)) = 1$, it follows from (6) that K divides $\sigma(K)$, and even that $K = \sigma(K)$ since K and $\sigma(K)$ have the same degree. Again, (6) implies that $L = \sigma(L)$. It follows that L is odd and perfect. Finally, by part (b), $K = \sigma(K)$ implies that $P = 1 + x + x^2$. This proves the result.

(d) By Lemma 4, R is square-free, since P is Mersenne prime. Putting $n = 1$ in part (c), we obtain that S is odd perfect and that $P = 1 + x + x^2$. Since S is a special perfect polynomial, all the other statements, besides the numerical lower bound of u, follow from Lemma 3. A simple computation in `gp-PARI` proves that for any odd number $1 < u \leq 4389$ one has $S \neq \sigma(S)$, so that $u \geq 4391$ and $\deg(S) = 2\deg(R) = 4(u - 1) \geq 17560$, thereby proving the result.

(e) One has $\sigma(A) = \sigma(B)\sigma(K) = B\sigma(K)$ because B is perfect. On the other side, $\sigma(A) = A + B = BK + B = B(K + 1)$. Therefore, $\sigma(K) = K + 1$, since B is nonzero.

Let us now compute $S = A+\sigma(A)$. One has $S = BK+\sigma(B)\sigma(K) = BK+B(K+1) = B$, since $\sigma(B) = B$ and $\sigma(K) = K + 1$. Thus $S = B$ is perfect.

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