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Harmonic Sums via Euler's Transform: Complementing the Approach of Boyadzhiev

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Abstract

We prove a new expression for binomial sums with harmonic numbers. Our derivation is based on an alternative argument for the Euler transform of these sums. The findings complement a result of Boyadzhiev. To demonstrate the usefulness of our alternative approach, several examples are discussed. We rediscover some known identities for harmonic numbers and present some new ones. In particular, we derive some new identities involving harmonic numbers, and Fibonacci and Lucas numbers.

1 Motivation

Harmonic numbers $(H_n)_{n\geq 0}$ are defined by

$$H_0 = 0 \quad \text{and for } n \geq 1: \qquad H_n = \sum_{k=1}^{n} \frac{1}{k} = H_{n-1} + \frac{1}{n}.$$

They have the following integral form

$$H_n = \int_0^1 \frac{1-x^n}{1-x} dx.$$

[1]Disclaimer: Statements and conclusions made in this article are entirely those of the author. They do not necessarily reflect the views of LBBW.

Harmonic numbers and generalized harmonic numbers have been studied recently by many mathematicians and a considerable amount of research results has been produced (see [2] - [13], to name a few articles). In 2009, Boyadzhiev [2] studied binomial sums with harmonic numbers using the Euler transform. His main result is the following identity valid for $n \geq 1$

$$\sum_{k=1}^{n} \binom{n}{k} a^k b^{n-k} H_k = (a+b)^n H_n - \Big(b(a+b)^{n-1} + \frac{b^2}{2}(a+b)^{n-2} + \cdots + \frac{b^n}{n}\Big), \tag{1}$$

where a and b are arbitrary complex numbers. His proof is based on the Euler transform for power series and the existence of a power series near zero of the form

$$\frac{\ln(1-cz)}{1-dz} = -\sum_{n=1}^{\infty} \Big(cd^{n-1} + \frac{1}{2}c^2 d^{n-2} + \cdots + \frac{1}{n}c^n\Big) z^n.$$

In this article, we show how Boyadzhiev's arguments may be modified to derive an alternative expression for the binomial sums on the left-hand side of (1). To demonstrate the usefulness of our alternative approach, several examples will be discussed. We will rediscover some known identities and present some new. In particular, we will derive some new identities involving harmonic numbers, and Fibonacci and Lucas numbers.

2 Results

Let $A(z)$ be the ordinary generating function for the harmonic numbers, i.e.,

$$A(z) = \sum_{n=0}^{\infty} H_n z^n = -\frac{\ln(1-z)}{1-z}.$$

For $a, b \in \mathbb{C}$, let further $S_n(a,b)$ be defined as

$$S_n(a,b) = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} H_k. \tag{2}$$

Then we have the following theorem.

Theorem 1. *For all $n \geq 1$, we have the identity*

$$S_n(a,b) = \Big((a+b)^n - b^n\Big) H_n - a \sum_{k=0}^{n-1} (a+b)^k b^{n-1-k} H_{n-1-k}. \tag{3}$$

Proof. Let $S(z)$ be the ordinary generating function for the sum $S_n(a,b)$. Then, by Euler's transform

$$\begin{aligned} S(z) = \sum_{n=0}^{\infty} S_n(a,b) z^n &= \frac{1}{1-bz} A\Big(\frac{az}{1-bz}\Big) \\ &= -\frac{\ln(1-(a+b)z)}{1-(a+b)z} + \frac{\ln(1-bz)}{1-(a+b)z}. \end{aligned}$$

Now, instead of searching for a power series for the second summand, we observe that

$$\frac{\ln(1-bz)}{1-(a+b)z} = \frac{az}{1-(a+b)z}\frac{\ln(1-bz)}{1-bz} + \frac{\ln(1-bz)}{1-bz}.$$

Hence,

$$\begin{aligned} S(z) &= -\frac{\ln(1-(a+b)z)}{1-(a+b)z} - \Big(-\frac{\ln(1-bz)}{1-bz}\Big) - \frac{az}{1-(a+b)z}\Big(-\frac{\ln(1-bz)}{1-bz}\Big) \\ &= \sum_{n=0}^{\infty}\Big((a+b)^n - b^n\Big)H_n z^n - a\Big(\sum_{n=0}^{\infty}(a+b)^n z^{n+1}\Big)\Big(\sum_{n=0}^{\infty} b^n H_n z^n\Big). \end{aligned}$$

Using Cauchy's product rule for power series and comparing the coefficients of z^n gives the result. □

For $(a;b)=(-1;1)$ we get as a special case

$$S_n(-1,1) = \sum_{k=0}^{n}\binom{n}{k}(-1)^k H_k = -H_n + H_{n-1} = -\frac{1}{n}, \tag{4}$$

which is an old result and has reappeared as Identity 20 in Spivey's paper [12]. In addition, using the integral form for H_n it is not difficult to show that

$$\sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}\frac{1}{k} = H_n. \tag{5}$$

This shows that the sequences $(H_n)_{n\geq 1}$ and $(1/n)_{n\geq 1}$ are connected by the binomial transform. More information about the binomial transform can be found in the book [1].

Corollary 2. *For $n\geq 1$, the harmonic numbers allow the representation*

$$H_n = \sum_{k=1}^{n}\Big(\frac{1}{k}2^{n-k} - 2^{k-1}H_{n-k}\Big). \tag{6}$$

Proof. Setting $(a;b)=(a;a)$ in (3) yields

$$\sum_{k=0}^{n}\binom{n}{k}H_k = (2^n-1)H_n - \sum_{k=0}^{n-1}2^k H_{n-1-k}.$$

Now, compare with the well-known identity ([2, Equation (20)], or [12, Identity 14])

$$\sum_{k=0}^{n}\binom{n}{k}H_k = 2^n\Big(H_n - \sum_{k=1}^{n}\frac{1}{k2^k}\Big).$$

□

Corollary 3. *For $n \geq 1$, it is true that*

$$\sum_{k=0}^{n} \binom{n}{k} 2^k H_k = (3^n - 1) H_n - 2 \sum_{k=0}^{n-1} 3^k H_{n-1-k}. \tag{7}$$

Proof. Set $(a; b) = (2; 1)$ in (3). □

Corollary 4. *For $n \geq 1$, we have the identity*

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} 2^{k-1} H_k = \begin{cases} \sum_{k=0}^{n-1} (-1)^k H_{n-1-k}, & \text{if } n \text{ is even;} \\ H_n - \sum_{k=0}^{n-1} (-1)^k H_{n-1-k}, & \text{if } n \text{ is odd.} \end{cases} \tag{8}$$

Proof. Setting $(a; b) = (2; -1)$ in (3) yields

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} 2^k H_k = (1 - (-1)^n) H_n - 2 \sum_{k=0}^{n-1} (-1)^{n-1-k} H_{n-1-k},$$

from which the result is deduced easily. □

Theorem 1 also allows to establish some identities involving harmonic numbers and Fibonacci (Lucas) numbers. Recall, that Fibonacci numbers $(F_n)_{n\geq 0}$ and Lucas numbers $(L_n)_{n\geq}$ are defined by $F_0 = 0, F_1 = 1, L_0 = 2, L_1 = 1$, and for $n \geq 2$ we have the recurrences $F_n = F_{n-1} + F_{n-2}$ and $L_n = L_{n-1} + L_{n-2}$.

Corollary 5. *Let F_n and L_n be the Fibonacci and Lucas numbers, respectively. Then, we have the relations*

$$\sum_{k=0}^{n} \binom{n}{k} F_k H_k = F_{2n} H_n - \sum_{k=0}^{n-1} F_{2k+1} H_{n-1-k}, \tag{9}$$

and

$$\sum_{k=0}^{n} \binom{n}{k} L_k H_k = (L_{2n} - 2) H_n - \sum_{k=0}^{n-1} L_{2k+1} H_{n-1-k}. \tag{10}$$

Proof. Evaluate (3) at $(a; b) = (\alpha; 1)$ and $(a; b) = (\beta; 1)$, respectively, where $\alpha = (1 + \sqrt{5})/2$ and $\beta = -1/\alpha$. This gives

$$S_n(\alpha, 1) = (\alpha^{2n} - 1) H_n - \sum_{k=0}^{n-1} \alpha^{2k+1} H_{n-1-k}$$

and

$$S_n(\beta, 1) = (\beta^{2n} - 1) H_n - \sum_{k=0}^{n-1} \beta^{2k+1} H_{n-1-k},$$

where we have used the relations $\alpha^2 = \alpha + 1$ and $\beta^2 = \beta + 1$. Now, calculate $S_n(\alpha, 1) \pm S_n(\beta, 1)$ and use the Binet forms for F_n and L_n, respectively. □

Corollary 6. *Let F_n and L_n be the Fibonacci and Lucas numbers, respectively. Then, the following identities hold*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}F_{2k}H_k = F_nH_n - \sum_{k=0}^{n-1}(-1)^{n-1-k}F_{k+2}H_{n-1-k}, \tag{11}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}L_{2k}H_k = (L_n - 2(-1)^n)H_n - \sum_{k=0}^{n-1}(-1)^{n-1-k}L_{k+2}H_{n-1-k}. \tag{12}$$

Proof. Evaluate (3) at $(a;b) = (\alpha^2;-1)$ and $(a;b) = (\beta^2;-1)$, respectively. Combine the results as in the previous proof. □

Corollary 7. *Let F_n and L_n be the Fibonacci and Lucas numbers, respectively. Then*

$$\sum_{k=0}^{n}\binom{n}{k}2^k(-1)^{n-k}F_{n-k}H_k = -(F_{2n} + (-1)^nF_n)H_n + \sum_{k=0}^{n-1}F_{3k+1-n}H_{n-1-k}, \tag{13}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}2^k(-1)^{n-k}L_{n-k}H_k = (L_{2n} - (-1)^nL_n)H_n - \sum_{k=0}^{n-1}L_{3k+1-n}H_{n-1-k}. \tag{14}$$

Proof. Evaluate (3) at $(a;b) = (2;-\alpha)$ and $(a;b) = (2;-\beta)$, respectively. Simplify using $2-\alpha = \alpha^{-2}$ and $2-\beta = \alpha^2$. Finally, calculate $S_n(2,-\alpha) \pm S_n(2,-\beta)$ and keep in mind that $F_{-n} = (-1)^{n+1}F_n$ and $L_{-n} = (-1)^nL_n$, respectively. □

3 Harmonic sums with integer powers

Boyadzhiev [2, Proposition 10] obtained a representation for the combinatorial sum

$$S_n(a,1,m) = S_n(a,m) = \sum_{k=0}^{n}\binom{n}{k}k^ma^kH_k, \quad m \geq 1, \tag{15}$$

in terms of Stirling numbers of the second kind $S(m,k)$. The theorem below contains an alternative expression for the sum.

Theorem 8. *For all $n \geq 1$ we have*

$$S_n(a,m) = \Big(\sum_{k=0}^{n}\binom{n}{k}k^ma^k\Big)H_n - \sum_{j=0}^{n-1}\sum_{k=j}^{n-1}\binom{k}{j}(j+1)^ma^{j+1}H_{n-1-k}. \tag{16}$$

Proof. Apply the differential operator $(a\frac{d}{da})^m$ to both sides of (3) with $S_n(a,b) = S_n(a,1)$. Note that

$$\left(a\frac{d}{da}\right)^m (a+1)^n = \sum_{k=0}^{n} \binom{n}{k} k^m a^k.$$

□

Remark 9. The Stirling numbers of the second kind $S(n,k)$ are defined by

$$x^n = \sum_{k=0}^{n} S(n,k)(x)_k,$$

with $x_0 = 1$ and $(x)_n = x(x-1)\cdots(x-n+1), n \geq 1$, being the falling factorial. From the equation

$$(x+1)_n = (x)_n + n(x)_{n-1},$$

it becomes clear, that we can restate the above result using Stirling numbers.

Corollary 10. *For $n \geq 1$, we have the following expression*

$$\sum_{k=0}^{n} \binom{n}{k} kH_k = n2^{n-1}H_n - \sum_{k=0}^{n-1} (k+2)2^{k-1}H_{n-1-k}. \tag{17}$$

Proof. Use $(a;m) = (1;1)$ in (16) as well as the obvious identities

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \qquad \text{and} \qquad \sum_{k=0}^{n} \binom{n}{k} k = n2^{n-1}.$$

□

Corollary 11. *For $n \geq 1$, we have the identity*

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k kH_k = \begin{cases} -1, & \text{if } n = 1; \\ \frac{1}{n-1}, & \text{if } n \geq 2. \end{cases} \tag{18}$$

Proof. Evaluate (16) at $(a;m) = (-1;1)$ and simplify. □

Corollary 12. *For $n \geq 1$, we have the identity*

$$\sum_{k=j}^{n-1} \binom{k}{j} H_{n-1-k} = \binom{n}{j+1} (H_n - H_{j+1}). \tag{19}$$

Proof. We have

$$\begin{aligned} S_n(a,m) &= \Big(\sum_{k=0}^{n}\binom{n}{k}k^m a^k\Big)H_n - \sum_{j=0}^{n}\binom{n}{j}j^m a^j(H_n - H_j) \\ &= \Big(\sum_{k=0}^{n}\binom{n}{k}k^m a^k\Big)H_n - \sum_{j=1}^{n}\binom{n}{j}j^m a^j(H_n - H_j) \\ &= \Big(\sum_{k=0}^{n}\binom{n}{k}k^m a^k\Big)H_n - \sum_{j=0}^{n-1}\binom{n}{j+1}(j+1)^m a^{j+1}(H_n - H_{j+1}). \end{aligned}$$

Comparing with (16) gives the result. □

Corollary 13. *Let F_n and L_n be the Fibonacci and Lucas numbers, respectively. Then*

$$\sum_{k=0}^{n}\binom{n}{k}kF_kH_k = nF_{2n-1}H_n - \sum_{k=0}^{n-1}(kF_{2k} + F_{2k+1})H_{n-1-k}, \tag{20}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}kL_kH_k = nL_{2n-1}H_n - \sum_{k=0}^{n-1}(kL_{2k} + L_{2k+1})H_{n-1-k}. \tag{21}$$

Proof. Evaluate (16) at $(a;m) = (\alpha;1)$ and $(a;m) = (\beta;1)$, respectively, using

$$\sum_{k=1}^{n}\binom{n}{k}kx^k = nx(1+x)^{n-1},$$

in combination with

$$\sum_{k=0}^{n}\binom{n}{k}\alpha^k = \alpha^{2n}, \quad \text{and} \quad \sum_{k=0}^{n}\binom{n}{k}\beta^k = \beta^{2n}.$$

Combine the sums as in the previous proofs. □

Using similar elementary arguments we can prove the following identities for $m = 2$:

Corollary 14. *For $n \geq 1$, it is true that*

$$\sum_{k=0}^{n}\binom{n}{k}k^2H_k = n(n+1)2^{n-2}H_n - \sum_{k=0}^{n-1}(k+1)(k+4)2^{k-2}H_{n-1-k}. \tag{22}$$

Corollary 15. *For $n \geq 1$, we have the identity*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k k^2 H_k = \begin{cases} -1, & \text{if } n = 1; \\ 4, & \text{if } n = 2; \\ -\frac{n}{(n-2)(n-1)}, & \text{if } n \geq 3. \end{cases} \tag{23}$$

Corollary 16. *For $n \geq 1$, the following relations hold:*

$$\sum_{k=0}^{n} \binom{n}{k} k^2 F_k H_k = (nF_{2n-3} + n^2 F_{2n-2}) H_n - \sum_{k=0}^{n-1} \Big(kF_{2k-2} + k^2 F_{2k-1} + 2kF_{2k} + F_{2k+1} \Big) H_{n-1-k}, \qquad (24)$$

and

$$\sum_{k=0}^{n} \binom{n}{k} k^2 L_k H_k = (nL_{2n-3} + n^2 L_{2n-2}) H_n - \sum_{k=0}^{n-1} \Big(kL_{2k-2} + k^2 L_{2k-1} + 2kL_{2k} + L_{2k+1} \Big) H_{n-1-k}. \qquad (25)$$

4 Concluding comments

Using the main results of this paper, it is possible to derive more identities involving harmonic numbers and Fibonacci (Lucas) numbers. In addition, we mention that from Theorems 1 and 8 relations between harmonic numbers and other important number sequences, such as Mersenne numbers or Pell numbers, are deducible.

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