Powers of Two as Sums of Three Lucas Numbers

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Abstract
In this paper, we find all positive integer solutions of the Diophantine equation $L_k + L_l + L_t = 2^d$ in non-negative integers $k, l, t,$ and $d$, where $(L_n)_{n \geq 0}$ is the Lucas sequence. The tools used to solve our main theorem are linear forms in logarithms, properties of continued fractions, and a version of the Baker-Davenport reduction method in Diophantine approximation.
1 Introduction

The Lucas sequence \((L_k)_{k \geq 0}\) is a linear recurrence given by \(L_0 = 2, L_1 = 1\) and
\[
L_{k+2} = L_{k+1} + L_k, \quad \text{for } k \geq 0.
\]
It satisfies the same recurrence as the Fibonacci sequence \((F_k)_{k \geq 0}\) given by \(F_0 = 0, F_1 = 1\) and
\[
F_{k+2} = F_{k+1} + F_k, \quad \text{for } k \geq 2,
\]
whose numbers are found everywhere in nature. The Fibonacci numbers are famous for possessing many wonderful and amazing properties.

In 2014, Bravo and Luca \([3]\) studied the Diophantine equation
\[
L_k + L_l = 2^t
\]
in positive integers \(k, l\) and \(t\). Similar equations involving Fibonacci and Padovan sequences are solved in \([5, 7]\). E. Bravo and J. Bravo \([2]\) found also all powers of 2 which are sums of three Fibonacci numbers. Specifically, they proved the following theorems.

**Theorem 1.** The only solutions \((k, l, t)\) of the Diophantine equation \(L_k + L_l = 2^t\) in positive integers \(k, l, t\) and with \(k \geq l\) are
\[
(0, 0, 2); (1, 1, 1); (3, 3, 3); (2, 1, 2); (4, 1, 3); (7, 2, 5).
\]

**Theorem 2.** All solutions \((k, l, t, d)\) of the Diophantine equation
\[
F_k + F_l + F_t = 2^d
\]
in non-negative integers \(k, l, t, d\) are
\[
(3, 1, 1, 2); (3, 2, 2, 2); (3, 2, 1, 2); (4, 4, 3, 3); (5, 3, 1, 3); (5, 3, 2, 3); (6, 5, 4, 4); (7, 3, 1, 4); (7, 3, 2, 4); (8, 6, 4, 5); (10, 6, 1, 6); (10, 6, 2, 6); (11, 9, 5, 7); (13, 8, 3, 8); (16, 9, 4, 10).
\]

In this paper, we prove an extension of Theorem 1 when the two Lucas numbers are replaced by three Lucas numbers and determine all the solutions of the Diophantine equation
\[
L_k + L_l + L_t = 2^d
\]
in non-negative integers \(k, l, t\) and \(d\). We prove the following result.

**Theorem 3.** All solutions \((k, l, t, d)\) of the Diophantine equation
\[
L_k + L_l + L_t = 2^d
\]
(1)
in non-negative integers \(k \geq l \geq t\) and \(d\), are
\[
(1, 1, 0, 2), (2, 2, 0, 3), (3, 0, 0, 3), (3, 2, 1, 3), (4, 4, 0, 4), (5, 2, 0, 4), (5, 3, 1, 4), (6, 4, 4, 5), (6, 5, 2, 5), (7, 1, 0, 5), (10, 2, 0, 7), (10, 3, 1, 7), (17, 13, 3, 12).
\]
Our method of proof is similar to the method described in \([3, 2]\).
2 Preliminaries

Before proceeding further, we recall the Binet formula for the Lucas numbers $(L_k)_{k \geq 0}$, namely

$$L_k = \alpha^k + \beta^k, \text{ for } k \geq 0,$$

where

$$\alpha = \frac{1 + \sqrt{5}}{2} \text{ and } \beta = \frac{1 - \sqrt{5}}{2}$$

are the roots of the characteristic equation $x^2 - x - 1 = 0$. In particular, the inequality

$$\alpha^{k-1} \leq L_k \leq 2\alpha^k \tag{2}$$

holds for all $k \geq 0$.

To prove Theorem 3, using a result on linear forms in two logarithms., we require some notation. Let $\delta$ be an algebraic number of degree $d$ with minimal polynomial

$$a_0 x^d + a_1 x^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d} (X - \delta^{(i)})$$

where the $a_i$ are relatively prime integers with $a_0 > 0$ and the $\delta^{(i)}$ denotes the conjugates of $\delta$. Then

$$h(\delta) = \frac{1}{d} \left( \log a_0 + \sum_{i=1}^{d} \log(\max\{|\delta^{(i)}|, 1\}) \right)$$

is called the logarithmic height of $\delta$. In particular, if $\delta = p/q$ is a rational number with $\gcd(p, q) = 1$ and $q > 0$, then

$$h(\delta) = \log \max\{|p|, q\}.$$

The following properties of the logarithmic height, will be used in the next section. Let $\delta, \nu$ be algebraic numbers and $r \in \mathbb{Z}$. Then

- $h(\delta \pm \nu) \leq h(\delta) + h(\nu) + \log 2$,
- $h(\delta \nu^{\pm 1}) \leq h(\delta) + h(\nu)$,
- $h(\delta^r) = |r|h(\delta)$.

Using the above notation, we restate Laurent, Mignotte, and Nesterenko’s result [6, Cor. 1].

**Theorem 4.** Let $\delta_1, \delta_2$ be two non-zero algebraic numbers, and let $\log \delta_1$ and $\log \delta_2$ be any determinations of their logarithms. Set

$$D = [\mathbb{Q}(\delta_1, \delta_2) : \mathbb{Q}] / [\mathbb{R}(\delta_1, \delta_2) : \mathbb{R}]$$

3
\[ \Gamma := b_2 \log \delta_2 - b_1 \log \delta_1, \]

where \( b_1 \) and \( b_2 \) are positive integers. Further, let \( A_1, A_2 > 1 \) be real numbers such that
\[
\log A_i \geq \max \{ h(\delta_i), \left| \frac{h(\delta_i)}{D} \right|, \frac{1}{D} \}, \quad i = 1, 2.
\]

Then, assuming that \( \delta_1 \) and \( \delta_2 \) are multiplicatively independent, we have
\[
\log |\Gamma| > -30.9 \cdot D^4 (\max \{ \log b', \frac{21}{D}, \frac{1}{2} \})^2 \log A_1 \log A_2,
\]

where
\[
b' = \frac{b_1}{D \log A_2} + \frac{b_2}{D \log A_1}.
\]

We also need the following general lower bound for linear forms in logarithms due to Matveev [8].

**Theorem 5.** Assume that \( \delta_1, \ldots, \delta_t \) are positive real algebraic numbers in a real algebraic number field \( \mathbb{K} \) of degree \( D \). Let \( b_1, \ldots, b_n \) be rational integers, and
\[
\Lambda := \delta_1^{b_1} \cdots \delta_t^{b_t} - 1
\]
be not zero. Then
\[
|\Lambda| > \exp \left( -1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2 (1 + \log D)(1 + \log B)A_1 \cdots A_t \right),
\]
where
\[
B \geq \max \{ |b_1|, \ldots, |b_t| \},
\]
and
\[
A_i \geq \max \{ Dh(\delta_i), |\log \delta_i|, 0.16 \}, \quad \text{for all} \quad i = 1, \ldots, t.
\]

Finally, we present a version of the reduction method based on the Baker-Davenport Lemma [1], from Dujella and Peth\H{o} [4]. This will be one of the key tools used to reduce the upper bounds on the variables of the equation (1).

**Lemma 6.** Let \( N \) be a positive integer, let \( p/q \) be a convergent of the irrational number \( \gamma \) such that \( q > 6N \), and let \( A, B, \mu \) be real numbers with \( A > 0 \) and \( B > 1 \). Define
\[
\xi := \| \mu q \| - N \| q \|,
\]
where \( \| \cdot \| \) denotes the distance to the nearest integer. If \( \xi > 0 \), then there is no solution to the inequality
\[
0 < u \gamma - v + \mu < AB^{-w},
\]
in positive integers \( u, v, \) and \( w \), with \( u \leq N \) and \( w \geq \frac{\log (Aq/\xi)}{\log B} \).
3 The Proof of Theorem 3

First of all, observe that if $k = l = t$, then equation (1) becomes $3L_k = 2^d$. Since $3 \nmid 2$, then equation (1) has no solution. Subsequently, we assume that either $k > l$ or $l > t$.

If $k \leq 250$, then a brute force search using Sagemath in the range $0 \leq t \leq l \leq k \leq 250$ produces the solutions

$$(1, 1, 0, 2), (2, 2, 0, 3), (3, 0, 0, 3), (3, 2, 1, 3), (4, 4, 0, 4), (5, 2, 0, 4), (5, 3, 1, 4), (6, 4, 4, 5), (6, 5, 2, 5),$$

$$(7, 1, 0, 5), (10, 2, 0, 7), (10, 3, 1, 7), (17, 13, 3, 12).$$

Thus, for the remainder of the paper, we assume that $k > 250$. Let us now establish a relation between $k$ and $d$.

Combining (1) with the right inequality of (2), one gets that

$$2^d \leq 2\alpha^k + 2\alpha^l + 2\alpha^t < 6\alpha^k < 6 \cdot 2^k < 2^{k+3},$$

which leads to $d \leq k + 2$.

3.1 Bounding $k - l$ and $k - t$ in terms of $k$

We rewrite (1) as

$$\alpha^k - 2^d = -\beta^k - L_l - L_t.$$

Now taking absolute values, we obtain

$$|\alpha^k - 2^d| \leq |\beta|^k + L_l + L_t < \frac{1}{2} + 2\alpha^l + 2\alpha^t.$$

Dividing both sides of the above expression by $\alpha^k$ and taking into account that $k \geq l \geq t$, we get

$$|1 - 2^d\alpha^{-k}| < \frac{1}{2}\alpha^{-k} + 2\alpha^{-k+l} + 2\alpha^{-k+t} < 5\alpha^{-k+l}.$$

Thus

$$|1 - 2^d\alpha^{-k}| < \frac{5}{\alpha^{k-l}}.\tag{3}$$

We apply Theorem 4 to

$$\Gamma := d\log \alpha - k \log 2.$$

Therefore the estimate (3) can be rewritten as

$$|1 - e^{\Gamma}| < \frac{5}{\alpha^{k-l}}.\tag{4}$$

The algebraic number field containing $2, \alpha$ is $\mathbb{Q}(\sqrt{5})$, so we can take $D := 2$. By using (1) and the Binet formula for the Lucas sequence, we have

$$\alpha^k = L_k - \beta^k < L_k + 1 \leq L_k + L_l + L_t = 2^d.\tag{5}$$
Consequently, $1 < 2^d \alpha^{-k}$ and so $\Gamma > 0$. Using the fact that $\log(1 + x) \leq x$ for all $x \in \mathbb{R}^+$, together with (4), gives

$$0 < \Gamma < \frac{5}{\alpha^{k-l}}, \quad (6)$$

Hence,

$$\log \Gamma < \log 5 - (k - l) \log \alpha. \quad (7)$$

Note further that $h(\alpha) = \log \alpha/2$ and $h(2) = \log 2$. Thus, we can choose

$$\log A_1 := \log \alpha \quad \text{and} \quad \log A_2 := \log 2.$$  

Finally, recall that $d \leq k + 2$, and so

$$b' = \frac{k}{2 \log 2} + \frac{d}{2 \log \alpha} < 4k.$$  

Since $\alpha$ and 2 are multiplicatively independent, we have, by Theorem 4, that

$$\log \Gamma \geq -30.9 \cdot 2^4 \cdot (\max\{\log (4k), 21/2, 1/2\})^2 \cdot \log \alpha \cdot \log 2.$$  

Thus

$$\log \Gamma > -174 \cdot (\max\{\log (4k), 21/2, 1/2\})^2. \quad (8)$$

Combining (7) and (8), we obtain

$$(k - l) \log \alpha < 180 \cdot (\max\{\log (4k), 21/2, 1/2\})^2. \quad (9)$$

Let us now establish a second linear form in logarithms. To this end, we rewrite equation (1) as follows

$$\alpha^k (1 + \alpha(l-k)) - 2^d = -\beta^k - \beta^l - L.$$  

Taking absolute values in the above relation and using the fact that $\beta = (1 - \sqrt{5})/2$ we get

$$|\alpha^k (1 + \alpha(l-k)) - 2^d| = | - \beta^k - \beta^l - L_t| < 2 + 2\alpha^t$$

for all $k > 250$ and $l \geq t \geq 0$. Dividing both sides of the above inequality by the first term of the left-hand side, we obtain

$$|1 - 2^d \alpha^{-k}(1 + \alpha(l-k))^{-1}| < \frac{2}{\alpha^k(1 + \alpha(l-k))} + \frac{2}{\alpha^{k-t}(1 + \alpha(l-k))} < \frac{4}{\alpha^{k-t}}. \quad (10)$$

We are now ready to apply Matveev’s result given in Theorem 5. To do this, we take the parameters $n := 3$ and

$$\delta_1 := 2, \quad \delta_2 := \alpha, \quad \delta_3 := (1 + \alpha(l-k)).$$
We take $b_1 := d$, $b_2 := -k$ and $b_3 := -1$. As before, $\mathbb{K} := \mathbb{Q}(\sqrt{5})$ contains $\delta_1, \delta_2, \delta_3$ and has $D := [\mathbb{K} : \mathbb{Q}] = 2$. To see why the left-hand side of (10) is not zero, note that otherwise, we would get the relation

$$\alpha^k + \alpha^l = 2^d. \tag{11}$$

Conjugating the above relation in $\mathbb{Q}(\sqrt{5})$, we get

$$\beta^k + \beta^l = 2^d. \tag{12}$$

Further, combining (11) and (12), we obtain

$$\alpha^k < \alpha^k + \alpha^l = |\beta^k + \beta^l| < 2.$$

This is impossible because $k > 250$. Thus,

$$1 - 2^d \alpha^{-k}(1 + \alpha^{(l-k)})^{-1}$$

is not zero.

In this application of Theorem 5, we take $A_1 := 2 \log 2$ and $A_2 := \log \alpha$. Since $t \leq k + 2$, it follows that we can take $B := k + 2$. Let us now estimate $h(\delta_3)$. We begin by observing that

$$\delta_3 = (1 + \alpha^{(l-k)}) < 2 \quad \text{and} \quad \delta_3^{-1} < 1.$$

So that

$$0 < \log \delta_3 < 1.$$

Next, notice that

$$h(\delta_3) \leq (k - l) \log \alpha + \log 2.$$

Hence, we can take

$$A_3 := 2 + (k - l) \log \alpha > \max\{2h(\delta_3), |\log \delta_3|, 0.16\}.$$

Now Theorem 5 implies that a lower bound on the left-hand side of (10) is

$$\log |\Lambda| > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2(1 + \log 2)(1 + \log(k + 2)) \cdot 2 \log 2 \cdot 2 \log \alpha \cdot (2 + (k - l) \log \alpha).$$

So, inequality (10) yields

$$k - t < 2.8 \cdot 10^{12} \log(k + 2) \cdot (2 + (k - l) \log \alpha), \tag{13}$$

where we used the inequality $1 + \log(k + 2) < 2 \log(k + 2)$, which holds because $k > 250$.

Now using (9) in the right-most term of inequality (13) and performing the respective calculations, we obtain

$$k - t < 5.1 \cdot 10^{14} \log(k + 2)(\max\{\log(4k), 21/2\})^2. \tag{14}$$
3.2 Bounding $k$

Finally, we consider a third linear form in logarithms. We now rewrite equation (1) as follows

\[ \alpha^k (1 + \alpha^{(l-k)} + \alpha^{(t-k)}) - 2^d = -\beta^k - \beta^l - \beta^t. \]

Taking absolute values in the above relation and using the fact that $\beta = (1 - \sqrt{5})/2$, we get

\[ |\alpha^k (1 + \alpha^{(l-k)} + \alpha^{(t-k)}) - 2^d| = |-\beta^k - \beta^l - \beta^t| < 3 \]

for all $k > 250$ and $l \geq t \geq 0$. Dividing both sides of the above inequality by the first term of the left-hand side, we obtain

\[ |1 - 2^d \alpha^{-k} (1 + \alpha^{(l-k)} + \alpha^{(t-k)})^{-1}| < \frac{3}{\alpha^k (1 + \alpha^{(l-k)} + \alpha^{(t-k)})} < \frac{3}{\alpha^k}. \tag{15} \]

We apply Theorem 5 to

\[ \Lambda = 1 - 2^d \alpha^{-k} (1 + \alpha^{(l-k)} + \alpha^{(t-k)})^{-1}, \]

with the parameters $n := 3$, $\delta_1 := 2$, $\delta_2 := \alpha$, $\delta_3 := (1 + \alpha^{(l-k)} + \alpha^{(t-k)})$, $b_1 := d$, $b_2 := -k$ and $b_3 := -1$, $K:= \mathbb{Q}(\sqrt{5})$ contains $\delta_1, \delta_2, \delta_3$, $D := [K : \mathbb{Q}] = 2$. To see why the left-hand side of (15) is not zero, note that otherwise, we would get the relation

\[ \alpha^k + \alpha^l + \alpha^t = 2^d. \tag{16} \]

Conjugating the above relation in $\mathbb{Q}(\sqrt{5})$, we get

\[ \beta^k + \beta^l + \beta^t = 2^d. \tag{17} \]

Furthermore, combining (16) and (17), we obtain

\[ \alpha^k < \alpha^k + \alpha^l + \alpha^t = |\beta^k + \beta^l + \beta^t| < 3. \]

This is impossible because $k > 250$. Thus,

\[ 1 - 2^d \alpha^{-k} (1 + \alpha^{(l-k)} + \alpha^{(t-k)})^{-1} \]

is not zero. We now apply Theorem 5 with $A_1 := 2 \log 2$, $A_2 := \log \alpha$. Since $d \leq k + 2$; it follows that we can take $B := k + 2$. Let us now estimate $h(\delta_3)$. We begin by observing that

\[ \gamma_3 = (1 + \alpha^{(l-k)} + \alpha^{(t-k)}) < 3 \]

and

\[ 0 < \log \gamma_3 < \log 3. \]
Next, notice that
\[ h(\delta_3) \leq (k - l) \log \alpha + (k - t) \log \alpha + 2 \log 2 \leq 2(k - t) \log \alpha + 2 \log 2. \]

Hence, we can take
\[ A_3 := 4 + 2(k - t) \log \alpha > \max\{2h(\delta_3), |\log \delta_3|, 0.16\}. \]

Now, from Theorem 5 we have
\[ \log |\Lambda| > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2(1 + \log 2)(1 + \log(k + 2)) \cdot 2 \log 2 \cdot 2 \log \alpha \cdot (4 + 2(k - t) \log \alpha). \]

So, inequality (15) gives
\[ k < 10^{13} \log(k + 2) \cdot (2 + (k - t) \log \alpha), \tag{18} \]
where we used the inequality \(1 + \log(k + 2) < 2 \log(k + 2)\), which holds because \(k > 250\).

Now using (13) in the rightmost term of the above inequality (18) and performing the respective calculations, we obtain
\[ k < 2.6 \cdot 10^{27}(\log(k + 2))^2(\max\{\log(4k), 21/2\})^2. \tag{19} \]

If \(\max\{\log(4k), 21/2\} = 21/2\), it then follows from (19) that
\[ k < 287 \cdot 10^{27}(\log(k + 2))^2, \]
giving
\[ k < 15 \cdot 10^{32}. \]

If on the other hand we have that \(\max\{\log(4k), 21/2\} = \log(4k)\), then inequality (19) gives that
\[ k < 2.6 \cdot 10^{27}(\log(k + 2))^2(\log(4k))^2, \]
and so
\[ k < 12 \cdot 10^{34}. \]

In any case, we have that
\[ k < 12 \cdot 10^{34} \]
always holds. We summarize what we have so far in the following lemma.

**Lemma 7.** If \((k, l, t, d)\) is a solution in positive integers of equation (1) with \(k \geq l \geq t\) and \(k > 250\), then inequalities
\[ d \leq k + 2 \quad \text{and} \quad k < 12 \cdot 10^{34} \]
hold.
4 The final computations

In this section, we will reduce the upper bound on $k$. Firstly, we determine a suitable upper bound on $k - l, k - t$, and later we use Lemma 6 to conclude that $k$ must be smaller than 250.

Turning back to inequality (6), we obtain

$$0 < d \log 2 - k \log \alpha < \frac{5}{\alpha^{k-l}}.$$  

Dividing across by $\log \alpha$, we get

$$0 < d \gamma - k < \frac{11}{\alpha^{k-l}}, \tag{20}$$

where

$$\gamma := \frac{\log 2}{\log \alpha}.$$

Let $[a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, \ldots] = [1, 2, 3, 1, 2, 3, 2, 4 \ldots]$ be the continued fraction expansion of $\gamma$, and let denote $p_n/q_n$ its nth convergent. Recall also that $d < 12 \cdot 10^{34}$ by Lemma 7. A quick inspection using Sagemath reveals that

$$37527245802242661673724926130723830 < q_{73} < 12 \cdot 10^{34} < q_{74} = 17518485890972330004986691804684639.$$  

Furthermore, $a_N := \max\{a_i; i = 0, 1, \ldots, 44\} = a_{17} = 134$. So, from the known properties of continued fractions, we obtain that

$$|d \gamma - k| > \frac{1}{(a_N + 2)d}. \tag{21}$$

Comparing estimates (20) and (21), we get right away that

$$\alpha^{k-l} < 11 \cdot 136 \cdot d < 18 \cdot 10^{37},$$

leading to $k - l < 184$.

Let us now go back to (10) and determine an improved upper bound on $k - t$. Put

$$\omega_1 := d \log 2 - k \log \alpha - \log(1 + \alpha^{-(k-t)}). \tag{22}$$

Therefore, (10) implies that

$$|1 - e^{\omega_1}| < \frac{4}{\alpha^{k-l}}. \tag{23}$$

Note that $\omega_1 \neq 0$, by using (1) and the Binet formula for the Lucas sequence, we have

$$\alpha^k + \alpha^l = L_k - \beta^k - \beta^l < L_k + L_l + L_l = 2^d.$$
Therefore,

\[ 1 < 2^d \alpha^{-k} (1 + \alpha^{(l - k)})^{-1} \]

and so \( \omega_1 > 0 \). Thus

\[ 0 < \omega_1 \leq e^{\omega_1} - 1 < \frac{4}{\alpha^{k-t}}. \tag{24} \]

Replacing \( \omega_1 \) in the above inequality by its formula (22) and dividing both sides of the resulting inequality by \( \log \alpha \), we get

\[ 0 < d \left( \frac{\log 2}{\log \alpha} \right) - k - \frac{\log(1 + \alpha^{-(k-l)})}{\log \alpha} < \frac{9}{\alpha^{k-t}}. \tag{25} \]

We now put

\[ \gamma := \frac{\log 2}{\log \alpha}, \quad \mu := -\frac{\log(1 + \alpha^{-(k-l)})}{\log \alpha}, \quad A := 9 \quad \text{and} \quad B := \alpha. \]

Clearly \( \gamma \) is an irrational number. We also put \( N := 12 \cdot 10^{34} \), which is an upper bound on \( d \) by Lemma 7. We therefore apply Lemma 6 to inequality (25) for all choices \( k - l \in \{1, \ldots, 184\} \) except when \( k - l = 1, 3 \) and get that

\[ k - t < \frac{\log(Aq/\xi)}{\log B}, \]

where \( q > 6N \) is a denominator of a convergent of the continued fraction of \( \gamma \) such that \( \xi = \|\mu q\| - N\|\gamma q\| > 0 \). Indeed, using Sagemath, we have that

\[ q = q_{75} = 1439006117080021301713618460568200942. \]

We find that if \( (k, l, t, d) \) is a possible solution of the equation (1) with \( \omega_1 > 0 \) and \( k - l \in \{1, \ldots, 184\} \) except when \( k - l = 1, 3 \), then

\[ k - t < 178. \]

Let us now treat the cases where \( k - l = 1 \) and 3. The discussion of these cases will be different from the previous ones, because when applying Lemma 6 to the expression (25), the corresponding parameter \( \mu \) appearing in Lemma 6 is

\[ \frac{\log(1 + \alpha^{-(k-l)})}{\log \alpha} = \begin{cases} -1, & \text{if } k - l = 1; \\ 1 - \frac{\log 2}{\log \alpha}, & \text{if } k - l = 3. \end{cases} \]

In both cases, the parameters \( \gamma \) and \( \mu \) are linearly dependent, which yields that the corresponding value of \( \xi \) from Lemma 6 is always negative and therefore the reduction method is not useful for reducing the bound on \( k - t \) in these instances. One can see that if \( k - l = 1, 3 \), then the resulting inequality from (25) has the shape

\[ 0 < |a\gamma - b| < \frac{9}{\alpha^{k-t}}, \]
with \( \gamma \) being an irrational number and \( a, b \in \mathbb{Z} \). So, one can appeal to the known properties of the convergents of the continued fractions to obtain a nontrivial lower bound for

\[ |a\gamma - b|. \]

When \( k - l = 1 \), from (25), we get that

\[ 0 < d\gamma - (k + 1) < \frac{9}{\alpha^{k-t}}. \]  \hspace{1cm} (26)

Let \([a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, \ldots] = [1, 2, 3, 1, 2, 3, 2, 4 \ldots] \) be the continued fraction expansion of \( \gamma \), and let denote \( p_n/q_n \) its \( n \)th convergent. Recall also that \( d < 12 \cdot 10^{34} \) by Lemma 4. Furthermore, \( a_N := \max\{a_i : i = 0, 1, \ldots, 44\} = a_{17} = 134 \). So, from the known properties of continued fractions, we obtain that

\[ |d\gamma - (k + 1)| > \frac{1}{(a_N + 2)d}. \]  \hspace{1cm} (27)

Comparing estimates (26) and (27), we get right away that

\[ \alpha^{k-t} < 9 \cdot 136 \cdot d < 15 \cdot 10^{37}, \]

leading to \( k - t < 184 \).

By the same argument as the one we did before, we get that \( k - t < 184 \) in the case when \( k - l = 3 \). This completes the analysis of the cases when \( k - l = 1, 3 \). Consequently, \( k - t < 184 \) always holds.

Finally, we shall use (15) to reduce the upper bound on \( k \). Put

\[ \omega_2 = d \log 2 - k \log \alpha - \log \varphi(u, v) \]  \hspace{1cm} (28)

where \( \varphi \) is the function given by the formula \( \varphi(u, v) = 1 + \alpha^{-u} + \alpha^{-v} \), where \( u = k - l, v = k - t \). Note that \( \omega_2 \neq 0 \). Thus, we distinguish the following cases. If \( \omega_2 > 0 \) then, from (15), we obtain

\[ 0 < \omega_2 \leq e^{\omega_2} - 1 < \frac{3}{\alpha^k}. \]

Replacing \( \omega_2 \) in the above inequality by its formula (28) and dividing both sides of the resulting inequality by \log \alpha, we get

\[ 0 < d \left( \frac{\log 2}{\log \alpha} \right) - k - \frac{\log(1 + \alpha^{-u} + \alpha^{-v})}{\log \alpha} < \frac{7}{\alpha^k}. \]  \hspace{1cm} (29)

We now put

\[ \gamma := \frac{\log 2}{\log \alpha}, \quad \mu := -\frac{\log(1 + \alpha^{-u} + \alpha^{-v})}{\log \alpha}, \quad A := 7 \quad \text{and} \quad B := \alpha. \]
Clearly $\gamma$ is an irrational number. We also put $N := 12 \cdot 10^{34}$, which is an upper bound on $d$ by Lemma 7. We therefore apply Lemma 6 to inequality (29) for all choices of $u \in \{1, \ldots, 184\}, v \in \{1, \ldots, 184\}$ except when

$$(u, v) \in \varpi_1 = \{(4, 2), (13, 1), (17, 4), (17, 5), (18, 7), (19, 4), (19, 5), (22, 2), (23, 1), (23, 2), (24, 1), (24, 5), (25, 1), (28, 2), (30, 4), (31, 1), (32, 7), (33, 1), (33, 4), (33, 5), (36, 1), (36, 2), (37, 2), (38, 2), (39, 1), (43, 4), (44, 1), (45, 4), (48, 1), (48, 5), (50, 1), (51, 9), (52, 1), (52, 2), (53, 4), (54, 2), (56, 2), (57, 2), (59, 2), (61, 1), (62, 4), (64, 1), (66, 1), (67, 1), (69, 2), (70, 5), (75, 1), (75, 2), (83, 5), (87, 1), (87, 7), (89, 2), (90, 2), (95, 1), (95, 7), (97, 1), (98, 1), (99, 5), (100, 1), (102, 2), (107, 2), (110, 4), (111, 2), (112, 1), (112, 2), (113, 1), (113, 5), (113, 6), (114, 1), (118, 1), (122, 1), (122, 4), (122, 6), (128, 2), (129, 1), (129, 2), (130, 1), (130, 4), (130, 5), (132, 2), (133, 2), (136, 4), (138, 2), (139, 1), (139, 2), (141, 1), (141, 2), (142, 4), (147, 1), (148, 1), (158, 7)\}

and get that

$$k < \frac{\log(Aq/\xi)}{\log B},$$

where $q > 6N$ is a denominator of a convergent of the continued fraction of $\gamma$ such that $\xi = \|\mu q\| - N\|\gamma q\| > 0$. Indeed, using Sagemath, we have that

$$q = q_{s_2} = 12054118444825786260212254516320106583249.$$

We find that if $(k, l, t, d)$ is a possible solution of the equation (1) with $\omega_2 > 0$ and $(u, v) \notin \varpi_1$, then $k < 196$. This is false because our assumption is that $k > 250$.

Let us now work with the cases when $(u, v) \in \varpi_1$. We cannot study these cases as before because when applying Lemma 6 to the expression (29), the corresponding quantity $\|\mu q\|$ appearing in Lemma 6 is zero. In these cases, the parameters $\gamma$ and $\mu$ are linearly dependent, which yields that the corresponding value of $\xi$ from Lemma 6 is always negative and therefore the reduction method is not useful for reducing the bound on $k$ in these instances. However, one can see that if $(u, v) = (2, 4)$, then the resulting inequality from (29) has the shape

$$0 < |x\gamma - y| < \frac{7}{\alpha^k},$$

with $\gamma$ being an irrational number and $x, y \in \mathbb{Z}$. So, one can use to the known properties of the convergents of the continued fractions to obtain a nontrivial lower bound for

$$|x\gamma - y|.$$ 

This clearly gives us an upper bound for $k$. For example, when $(u, v) = (2, 4)$,

$$-\frac{\log(1 + \alpha^{-u} + \alpha^{-v})}{\log \alpha} = 2 - 2\frac{\log 2}{\log \alpha}.$$
and from (29), we get that

\[ 0 < (d - 2)\gamma - (k - 2) < \frac{\gamma}{\alpha^k}. \]  

(30)

Let \([a_0, a_1, a_2, a_3, a_4, a_5, a_6, \ldots] = [1, 2, 3, 1, 2, 3, 2, 4, \ldots]\) be the continued fraction expansion of \(\gamma\), and let denote \(p_n/q_n\) its \(n\)th convergent. Recall also that \(d < 12 \cdot 10^{34}\) by Lemma 7.

Furthermore, \(a_N := \max\{a_i; i = 0, 1, \ldots, 44\} = a_{17} = 134\). So, from the known properties of continued fractions, we obtain that

\[ |(d - 2)\gamma - (k - 2)| > \frac{1}{(a_N + 2)d}. \]  

(31)

Comparing estimates (30) and (31), we get that

\[ \alpha^k < 7 \cdot 136 \cdot d < 12 \cdot 10^{37}, \]  

(32)

leading to \(k < 184\). Using the above argument, we obtain \(k < 184\) in the case when \((u, v) \in \mathcal{W}_1\) except \((2, 4)\). We omit the details in order to avoid unnecessary repetitions. This completes the analysis of the cases when \((u, v) \in \mathcal{W}_1\). Consequently, \(k < 196\) always holds.

Suppose now that \(\omega_2 < 0\). First, note that \(\frac{7}{\alpha^2} < \frac{1}{2}\) since \(k > 250\). Then, from (15), we have that

\[ |1 - e^{\omega_2}| < \frac{1}{2}, \]

thus

\[ \frac{1}{2} < e^{\omega_2} < \frac{3}{2}. \]

Therefore

\[ e^{\omega_2} < 2. \]

Since \(\omega_2 < 0\), we have

\[ 0 < |\omega_2| = e^{\omega_2} - 1 \leq e^{\omega_2} - 1 = e^{\omega_2} - 1 = e^{\omega_2} - 1 < \frac{6}{\alpha^k}. \]

Then we obtain

\[ 0 < -d\log 2 + k\log \alpha + \log(1 + \alpha^{-u} + \alpha^{-v}) < \frac{6}{\alpha^k}. \]

By the same arguments used for proving (15), we obtain

\[ 0 < k \left(\frac{\log \alpha}{\log 2}\right) - d + \frac{\log(1 + \alpha^{-u} + \alpha^{-v})}{\log 2} < \frac{9}{\alpha^k}. \]  

(33)

We now put

\[ \gamma := \frac{\log \alpha}{\log 2}, \quad \mu := \frac{\log(1 + \alpha^{-u} + \alpha^{-v})}{\log 2}, \quad A := 9 \quad \text{and} \quad B := \alpha. \]
Clearly $\gamma$ is an irrational number. We also put $N := 12 \cdot 10^{34}$, which is an upper bound on $d$ by Lemma 7. We therefore apply Lemma 7 to inequality (33) for all choices of $u \in \{1, \ldots, 184\}, v \in \{1, \ldots, 184\}$ except when

$$(u, v) \in \omega_2 = \{(2, 1), (3, 3), (4, 2), (5, 2), (7, 7), (8, 2), (9, 5), (9, 6), (10, 2), (11, 4), (12, 1), (13, 1), (14, 2), (17, 4), (19, 5), (19, 6), (20, 9), (22, 1), (23, 1), (24, 1), (24, 5), (25, 1), (25, 2), (28, 2), (30, 1), (31, 4), (32, 7), (33, 1), (36, 3), (37, 2), (38, 1), (40, 2), (44, 1), (47, 2), (48, 1), (48, 6), (50, 1), (50, 2), (51, 9), (53, 1), (54, 2), (56, 2), (57, 2), (58, 5), (58, 6), (58, 8), (62, 1), (62, 4), (64, 1), (66, 1), (69, 2), (70, 6), (75, 1), (75, 2), (76, 1), (76, 4), (77, 2), (77, 3), (78, 2), (79, 2), (83, 6), (87, 1), (87, 7), (89, 1), (89, 2), (91, 1), (91, 2), (92, 1), (94, 1), (95, 1), (95, 7), (96, 2), (97, 1), (98, 1), (99, 5), (99, 6), (102, 2), (102, 3), (112, 2), (113, 1), (113, 5), (113, 6), (120, 1), (122, 1), (122, 5), (122, 6), (123, 2), (129, 2), (129, 3), (130, 4), (130, 5), (134, 7), (135, 2), (138, 1), (139, 1), (141, 1), (141, 2), (142, 4), (143, 1), (147, 1), (148, 1), (149, 2)\},
$$

and get that

$$k < \frac{\log(Aq/\xi)}{\log B},$$

where $q > 6N$ is a denominator of a convergent of the continued fraction of $\gamma$ such that $\xi = \|\mu q\| - N\|\gamma q\| > 0$. Indeed, using Sagemath, we have that

$$q = q_{s_2} = 123416550491119365182055719085668171489.$$ 

We find that if $(k, l, t, d)$ is a possible solution of the equation (1) with $\omega_2 < 0$ and $(u, v) \notin \omega_2$, then $k < 192$. This is false because our assumption is that $k > 250$. With the same arguments as in the case $\omega_2 > 0$, when $(u, v) \in \omega_2$, we obtain that $k < 192$. Thus, Theorem 3 is proven.

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## References


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