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Powers of Two as Sums of Three Lucas Numbers

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Abstract

In this paper, we find all positive integer solutions of the Diophantine equation $L_k + L_l + L_t = 2^d$ in non-negative integers k, l, t, and d, where $(L_n)_{n\geq 0}$ is the Lucas sequence. The tools used to solve our main theorem are linear forms in logarithms, properties of continued fractions, and a version of the Baker-Davenport reduction method in Diophantine approximation.

1 Introduction

The Lucas sequence $(L_k)_{k\geq 0}$ is a linear recurrence given by $L_0 = 2, L_1 = 1$ and

$$L_{k+2} = L_{k+1} + L_k, \quad \text{for } k \geq 0.$$

It satisfies the same recurrence as the Fibonacci sequence $(F_k)_{k\geq 0}$ given by $F_0 = 0, F_1 = 1$ and

$$F_{k+2} = F_{k+1} + F_k, \quad \text{for } k \geq 2,$$

whose numbers are found everywhere in nature. The Fibonacci numbers are famous for possessing many wonderful and amazing properties.

In 2014, Bravo and Luca [3] studied the Diophantine equation

$$L_k + L_l = 2^t$$

in positive integers k, l and t. Similar equations involving Fibonacci and Padovan sequences are solved in [5, 7]. E. Bravo and J. Bravo [2] found also all powers of 2 which are sums of three Fibonacci numbers. Specifically, they proved the following theorems.

Theorem 1. *The only solutions (k, l, t) of the Diophantine equation $L_k + L_l = 2^t$ in positive integers k, l, t and with $k \geq l$ are*

$$(0, 0, 2); (1, 1, 1); (3, 3, 3); (2, 1, 2); (4, 1, 3); (7, 2, 5).$$

Theorem 2. *All solutions (k, l, t, d) of the Diophantine equation*

$$F_k + F_l + F_t = 2^d$$

in non-negative integers k, l, t, with $k \geq l \geq t$ and d are

$$(3, 1, 1, 2); (3, 2, 2, 2); (3, 2, 1, 2); (4, 4, 3, 3); (5, 3, 1, 3); (5, 3, 2, 3); (6, 5, 4, 4);$$

$$(7, 3, 1, 4); (7, 3, 2, 4); (8, 6, 4, 5); (10, 6, 1, 6); (10, 6, 2, 6); (11, 9, 5, 7); (13, 8, 3, 8); (16, 9, 4, 10).$$

In this paper, we prove an extension of Theorem 1 when the two Lucas numbers are replaced by three Lucas numbers and determine all the solutions of the Diophantine equation

$$L_k + L_l + L_t = 2^d$$

in non-negative integers k, l, t and d. We prove the following result.

Theorem 3. *All solutions (k, l, t, d) of the Diophantine equation*

$$L_k + L_l + L_t = 2^d \tag{1}$$

in non-negative integers $k \geq l \geq t$ and d, are

$$(1, 1, 0, 2), (2, 2, 0, 3), (3, 0, 0, 3), (3, 2, 1, 3), (4, 4, 0, 4), (5, 2, 0, 4), (5, 3, 1, 4), (6, 4, 4, 5),$$

$$(6, 5, 2, 5), (7, 1, 0, 5), (10, 2, 0, 7), (10, 3, 1, 7), (17, 13, 3, 12).$$

Our method of proof is similiar to the method described in [3, 2].

2 Preliminaries

Before proceeding further, we recall the Binet formula for the Lucas numbers $(L_k)_{k\geq 0}$, namely

$$L_k = \alpha^k + \beta^k, \quad \text{for} \quad k \geq 0,$$

where

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}$$

are the roots of the characteristic equation $x^2 - x - 1 = 0$. In particular, the inequality

$$\alpha^{k-1} \leq L_k \leq 2\alpha^k \tag{2}$$

holds for all $k \geq 0$.

To prove Theorem 3, using a result on linear forms in two logarithms., we require some notation. Let δ be an algebraic number of degree d with minimal polynomial

$$a_0 x^d + a_1 x^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d} (X - \delta^{(i)})$$

where the a_i are relatively prime integers with $a_0 > 0$ and the $\delta^{(i)}$ denotes the conjugates of δ. Then

$$h(\delta) = \frac{1}{d}(\log a_0 + \sum_{i=1}^{d} \log(\max\{|\delta^{(i)}|, 1\}))$$

is called the logarithmic height of δ. In particular, if $\delta = p/q$ is a rational number with $\gcd(p,q) = 1$ and $q > 0$, then

$$h(\delta) = \log \max\{|p|, q\}.$$

The following properties of the logarithmic height, will be used in the next section. Let δ, ν be algebraic numbers and $r \in \mathbb{Z}$. Then

- $h(\delta \pm \nu) \leq h(\delta) + h(\nu) + \log 2$,
- $h(\delta\nu^{\pm 1}) \leq h(\delta) + h(\nu)$,
- $h(\delta^r) = |r| h(\delta)$.

Using the above notation, we restate Laurent, Mignotte, and Nesterenko's result [6, Cor. 1].

Theorem 4. *Let δ_1, δ_2 be two non-zero algebraic numbers, and let $\log \delta_1$ and $\log \delta_2$ be any determinations of their logarithms. Set*

$$D = [\mathbb{Q}(\delta_1, \delta_2) : \mathbb{Q}]/[\mathbb{R}(\delta_1, \delta_2) : \mathbb{R}]$$

and

$$\Gamma := b_2 \log \delta_2 - b_1 \log \delta_1,$$

where b_1 *and* b_2 *are positive integers. Further, let* $A_1, A_2 > 1$ *be real numbers such that*

$$\log A_i \geq \max\{h(\delta_i), \frac{|h(\delta_i)|}{D}, \frac{1}{D}\}, \quad i = 1, 2.$$

Then, assuming that δ_1 *and* δ_2 *are multiplicatively independent, we have*

$$\log|\Gamma| > -30.9 \cdot D^4 (\max\{\log b', \frac{21}{D}, \frac{1}{2}\})^2 \log A_1 \log A_2,$$

where

$$b' = \frac{b_1}{D \log A_2} + \frac{b_2}{D \log A_1}.$$

We also need the following general lower bound for linear forms in logarithms due to Matveev [8].

Theorem 5. *Assume that* $\delta_1, \ldots, \delta_t$ *are positive real algebraic numbers in a real algebraic number field* $\mathbb{K}$ *of degree* D*. Let* $b_1, \cdots, b_n$ *be rational integers, and*

$$\Lambda := \delta_1^{b_1} \cdots \delta_t^{b_t} - 1$$

be not zero. Then

$$|\Lambda| > \exp\left(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2 (1 + \log D)(1 + \log B) A_1 \cdots A_t\right),$$

where

$$B \geq \max\{|b_1|, \ldots, |b_t|\},$$

and

$$A_i \geq \max\{Dh(\delta_i), |\log \delta_i|, 0.16\}, \quad \text{for all} \quad i = 1, \cdots, t.$$

Finally, we present a version of the reduction method based on the Baker-Davenport Lemma [1], from Dujella and Pethő [4]. This will be one of the key tools used to reduce the upper bounds on the variables of the equation (1).

Lemma 6. *Let* N *be a positive integer, let* p/q *be a convergent of the irrational number* γ *such that* $q > 6N$*, and let* A, B, μ *be real numbers with* $A > 0$ *and* $B > 1$*. Define*

$$\xi := \|\mu q\| - N\|\gamma q\|,$$

where $\|\cdot\|$ *denotes the distance to the nearest integer. If* $\xi > 0$*, then there is no solution to the inequality*

$$0 < u\gamma - v + \mu < AB^{-w},$$

in positive integers u*,* v*, and* w*, with* $u \leq N$ *and* $w \geq \frac{\log(Aq/\xi)}{\log B}$.

3 The Proof of Theorem 3

First of all, observe that if $k = l = t$, then equation (1) becomes $3L_k = 2^d$. Since $3 \nmid 2$, then equation (1) has no solution. Subsequently, we assume that either $k > l$ or $l > t$.

If $k \leq 250$, then a brute force search using `Sagemath` in the range $0 \leq t \leq l \leq k \leq 250$ produces the solutions

$$(1,1,0,2),(2,2,0,3),(3,0,0,3),(3,2,1,3),(4,4,0,4),(5,2,0,4),(5,3,1,4),(6,4,4,5),(6,5,2,5),$$
$$(7,1,0,5),(10,2,0,7),(10,3,1,7),(17,13,3,12).$$

Thus, for the remainder of the paper, we assume that $k > 250$. Let us now establish a relation between k and d.

Combining (1) with the right inequality of (2), one gets that

$$2^d \leq 2\alpha^k + 2\alpha^l + 2\alpha^t < 6\alpha^k < 6 \cdot 2^k < 2^{k+3},$$

which leads to $d \leq k + 2$.

3.1 Bounding $k - l$ and $k - t$ in terms of k

We rewrite (1) as

$$\alpha^k - 2^d = -\beta^k - L_l - L_t.$$

Now taking absolute values, we obtain

$$|\alpha^k - 2^d| \leq |\beta|^k + L_l + L_t < \frac{1}{2} + 2\alpha^l + 2\alpha^t.$$

Dividing both sides of the above expression by α^k and taking into account that $k \geq l \geq t$, we get

$$|1 - 2^d\alpha^{-k}| < \frac{1}{2}\alpha^{-k} + 2\alpha^{-k+l} + 2\alpha^{-k+t} < 5\alpha^{-k+l}.$$

Thus

$$|1 - 2^d\alpha^{-k}| < \frac{5}{\alpha^{k-l}}. \tag{3}$$

We apply Theorem 4 to

$$\Gamma := d\log\alpha - k\log 2.$$

Therefore the estimate (3) can be rewritten as

$$|1 - e^\Gamma| < \frac{5}{\alpha^{k-l}}. \tag{4}$$

The algebraic number field containing $2, \alpha$ is $\mathbb{Q}(\sqrt{5})$, so we can take $D := 2$. By using (1) and the Binet formula for the Lucas sequence, we have

$$\alpha^k = L_k - \beta^k < L_k + 1 \leq L_k + L_l + L_t = 2^d. \tag{5}$$

Consequently, $1 < 2^d\alpha^{-k}$ and so $\Gamma > 0$. Using the fact that $\log(1+x) \le x$ for all $x \in \mathbb{R}^+$, together with (4), gives

$$0 < \Gamma < \frac{5}{\alpha^{k-l}}, \tag{6}$$

Hence,

$$\log\Gamma < \log 5 - (k-l)\log\alpha. \tag{7}$$

Note further that $h(\alpha) = \log\alpha/2$ and $h(2) = \log 2$. Thus, we can choose

$$\log A_1 := \log\alpha \quad \text{and} \quad \log A_2 := \log 2.$$

Finally, recall that $d \le k+2$, and so

$$b' = \frac{k}{2\log 2} + \frac{d}{2\log\alpha} < 4k.$$

Since α and 2 are multiplicatively independent, we have, by Theorem 4, that

$$\log\Gamma \ge -30.9 \cdot 2^4 \cdot (\max\{\log(4k), 21/2, 1/2\})^2 \cdot \log\alpha \cdot \log 2.$$

Thus

$$\log\Gamma > -174 \cdot (\max\{\log(4k), 21/2, 1/2\})^2. \tag{8}$$

Combining (7) and (8), we obtain

$$(k-l)\log\alpha < 180 \cdot (\max\{\log(4k), 21/2\})^2. \tag{9}$$

Let us now establish a second linear form in logarithms. To this end, we rewrite equation (1) as follows

$$\alpha^k(1+\alpha^{(l-k)}) - 2^d = -\beta^k - \beta^l - L_t.$$

Taking absolute values in the above relation and using the fact that $\beta = (1-\sqrt{5})/2$ we get

$$|\alpha^k(1+\alpha^{(l-k)}) - 2^d| = |-\beta^k - \beta^l - L_t| < 2 + 2\alpha^t$$

for all $k > 250$ and $l \ge t \ge 0$. Dividing both sides of the above inequality by the first term of the left-hand side, we obtain

$$|1 - 2^d\alpha^{-k}(1+\alpha^{(l-k)})^{-1}| < \frac{2}{\alpha^k(1+\alpha^{(l-k)})} + \frac{2}{\alpha^{k-t}(1+\alpha^{(l-k)})} < \frac{4}{\alpha^{k-t}}. \tag{10}$$

We are now ready to apply Matveev's result given in Theorem 5. To do this, we take the parameters $n := 3$ and

$$\delta_1 := 2, \quad \delta_2 := \alpha, \quad \delta_3 := (1+\alpha^{(l-k)}).$$

We take $b_1 := d$, $b_2 := -k$ and $b_3 := -1$. As before, $\mathbb{K} := \mathbb{Q}(\sqrt{5})$ contains $\delta_1, \delta_2, \delta_3$ and has $D := [\mathbb{K} : \mathbb{Q}] = 2$. To see why the left-hand side of (10) is not zero, note that otherwise, we would get the relation

$$\alpha^k + \alpha^l = 2^d. \tag{11}$$

Conjugating the above relation in $\mathbb{Q}(\sqrt{5})$, we get

$$\beta^k + \beta^l = 2^d. \tag{12}$$

Further, combining (11) and (12), we obtain

$$\alpha^k < \alpha^k + \alpha^l = |\beta^k + \beta^l| < 2.$$

This is impossible because $k > 250$. Thus,

$$1 - 2^d \alpha^{-k}(1 + \alpha^{(l-k)})^{-1}$$

is not zero.

In this application of Theorem 5, we take $A_1 := 2\log 2$ and $A_2 := \log\alpha$. Since $t \leq k+2$, it follows that we can take $B := k+2$. Let us now estimate $h(\delta_3)$. We begin by observing that

$$\delta_3 = (1 + \alpha^{(l-k)}) < 2 \quad \text{and} \quad \delta_3^{-1} < 1.$$

So that

$$0 < \log\delta_3 < 1.$$

Next, notice that

$$h(\delta_3) \leq (k-l)\log\alpha + \log 2.$$

Hence, we can take

$$A_3 := 2 + (k-l)\log\alpha > \max\{2h(\delta_3), |\log\delta_3|, 0.16\}.$$

Now Theorem 5 implies that a lower bound on the left-hand side of (10) is

$$\log|\Lambda| > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2(1 + \log 2)(1 + \log(k+2)) \cdot 2\log 2 \cdot 2\log\alpha \cdot (2 + (k-l)\log\alpha).$$

So, inequality (10) yields

$$k - t < 2.8 \cdot 10^{12} \log(k+2) \cdot (2 + (k-l)\log\alpha), \tag{13}$$

where we used the inequality $1 + \log(k+2) < 2\log(k+2)$, which holds because $k > 250$.

Now using (9) in the right-most term of inequality (13) and performing the respective calculations, we obtain

$$k - t < 5.1 \cdot 10^{14} \log(k+2)(\max\{\log(4k), 21/2\})^2. \tag{14}$$

3.2 Bounding k

Finally, we consider a third linear form in logarithms. We now rewrite equation (1) as follows

$$\alpha^k(1+\alpha^{(l-k)}+\alpha^{(t-k)})-2^d=-\beta^k-\beta^l-\beta^t.$$

Taking absolute values in the above relation and using the fact that $\beta=(1-\sqrt{5})/2$, we get

$$|\alpha^k(1+\alpha^{(l-k)}+\alpha^{(t-k)})-2^d|=|-\beta^k-\beta^l-\beta^t|<3$$

for all $k>250$ and $l\geq t\geq 0$. Dividing both sides of the above inequality by the first term of the left-hand side, we obtain

$$|1-2^d\alpha^{-k}(1+\alpha^{(l-k)}+\alpha^{(t-k)})^{-1}|<\frac{3}{\alpha^k(1+\alpha^{(l-k)}+\alpha^{(t-k)})}<\frac{3}{\alpha^k}. \tag{15}$$

We apply Theorem 5 to

$$\Lambda=1-2^d\alpha^{-k}(1+\alpha^{(l-k)}+\alpha^{(t-k)})^{-1},$$

with the parameters $n:=3$, $\delta_1:=2$, $\delta_2:=\alpha$, $\delta_3:=(1+\alpha^{(l-k)}+\alpha^{(t-k)})$, $b_1:=d$, $b_2:=-k$ and $b_3:=-1$, $\mathbb{K}:=\mathbb{Q}(\sqrt{5})$ contains $\delta_1,\delta_2,\delta_3$, $D:=[\mathbb{K}:\mathbb{Q}]=2$. To see why the left-hand side of (15) is not zero, note that otherwise, we would get the relation

$$\alpha^k+\alpha^l+\alpha^t=2^d. \tag{16}$$

Conjugating the above relation in $\mathbb{Q}(\sqrt{5})$, we get

$$\beta^k+\beta^l+\beta^t=2^d. \tag{17}$$

Furthermore, combining (16) and (17), we obtain

$$\alpha^k<\alpha^k+\alpha^l+\alpha^t=|\beta^k+\beta^l+\beta^t|<3.$$

This is impossible because $k>250$. Thus,

$$1-2^d\alpha^{-k}(1+\alpha^{(l-k)}+\alpha^{(t-k)})^{-1}$$

is not zero. We now apply Theorem 5 with $A_1:=2\log 2$, $A_2:=\log\alpha$. Since $d\leq k+2$; it follows that we can take $B:=k+2$. Let us now estimate $h(\delta_3)$. We begin by observing that

$$\gamma_3=(1+\alpha^{(l-k)}+\alpha^{(t-k)})<3$$

and

$$0<\log\gamma_3<\log 3.$$

Next, notice that

$$h(\delta_3) \leq (k-l)\log\alpha + (k-t)\log\alpha + 2\log 2 \leq 2(k-t)\log\alpha + 2\log 2.$$

Hence, we can take

$$A_3 := 4 + 2(k-t)\log\alpha > \max\{2h(\delta_3), |\log\delta_3|, 0.16\}.$$

Now, from Theorem 5 we have

$$\log|\Lambda| > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2(1+\log 2)(1+\log(k+2)) \cdot 2\log 2 \cdot 2\log\alpha \cdot (4 + 2(k-t)\log\alpha).$$

So, inequality (15) gives

$$k < 10^{13}\log(k+2) \cdot (2 + (k-t)\log\alpha), \tag{18}$$

where we used the inequality $1 + \log(k+2) < 2\log(k+2)$, which holds because $k > 250$.

Now using (13) in the rightmost term of the above inequality (18) and performing the respective calculations, we obtain

$$k < 2.6 \cdot 10^{27}(\log(k+2))^2(\max\{\log(4k), 21/2\})^2. \tag{19}$$

If $\max\{\log(4k), 21/2\} = 21/2$, it then follows from (19) that

$$k < 287 \cdot 10^{27}(\log(k+2))^2,$$

giving

$$k < 15 \cdot 10^{32}.$$

If on the other hand we have that $\max\{\log(4k), 21/2\} = \log(4k)$, then inequality (19) gives that

$$k < 2.6 \cdot 10^{27}(\log(k+2))^2(\log(4k))^2,$$

and so

$$k < 12 \cdot 10^{34}.$$

In any case, we have that

$$k < 12 \cdot 10^{34}$$

always holds. We summarize what we have so far in the following lemma.

Lemma 7. *If (k, l, t, d) is a solution in positive integers of equation (1) with $k \geq l \geq t$ and $k > 250$, then inequalities*

$$d \leq k+2 \quad \text{and} \quad k < 12 \cdot 10^{34}$$

hold.

4 The final computations

In this section, we will reduce the upper bound on k. Firstly, we determine a suitable upper bound on $k-l, k-t$, and later we use Lemma 6 to conclude that k must be smaller than 250.

Turning back to inequality (6), we obtain

$$0 < d\log 2 - k\log\alpha < \frac{5}{\alpha^{k-l}}.$$

Dividing across by $\log\alpha$, we get

$$0 < d\gamma - k < \frac{11}{\alpha^{k-l}}, \tag{20}$$

where

$$\gamma := \frac{\log 2}{\log\alpha}.$$

Let $[a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, \ldots] = [1, 2, 3, 1, 2, 3, 2, 4\ldots]$ be the continued fraction expansion of γ, and let denote p_n/q_n its nth convergent. Recall also that $d < 12\cdot 10^{34}$ by Lemma 7. A quick inspection using `Sagemath` reveals that

$$37527245802242661673724926130723830 = q_{73} < 12\cdot 10^{34} <$$
$$q_{74} = 175184858909722330004986691804684639.$$

Furthermore, $a_N := \max\{a_i; i = 0, 1, \ldots, 44\} = a_{17} = 134$. So, from the known properties of continued fractions, we obtain that

$$|d\gamma - k| > \frac{1}{(a_N+2)d}. \tag{21}$$

Comparing estimates (20) and (21), we get right away that

$$\alpha^{k-l} < 11\cdot 136\cdot d < 18\cdot 10^{37},$$

leading to $k-l < 184$.

Let us now go back to (10) and determine an improved upper bound on $k-t$. Put

$$\omega_1 := d\log 2 - k\log\alpha - \log(1+\alpha^{-(k-l)}). \tag{22}$$

Therefore, (10) implies that

$$|1 - e^{\omega_1}| < \frac{4}{\alpha^{k-t}}. \tag{23}$$

Note that $\omega_1 \neq 0$, by using (1) and the Binet formula for the Lucas sequence, we have

$$\alpha^k + \alpha^l = L_t - \beta^k - \beta^l < L_k + L_l + L_t = 2^d.$$

Therefore,

$$1 < 2^d \alpha^{-k}(1+\alpha^{(l-k)})^{-1}$$

and so $\omega_1 > 0$. Thus

$$0 < \omega_1 \le e^{\omega_1} - 1 < \frac{4}{\alpha^{k-t}}. \tag{24}$$

Replacing ω_1 in the above inequality by its formula (22) and dividing both sides of the resulting inequality by $\log\alpha$, we get

$$0 < d\left(\frac{\log 2}{\log\alpha}\right) - k - \frac{\log(1+\alpha^{-(k-l)})}{\log\alpha} < \frac{9}{\alpha^{k-t}}. \tag{25}$$

We now put

$$\gamma := \frac{\log 2}{\log\alpha}, \quad \mu := -\frac{\log(1+\alpha^{-(k-l)})}{\log\alpha}, \quad A := 9 \quad \text{and} \quad B := \alpha.$$

Clearly γ is an irrational number. We also put $N := 12\cdot 10^{34}$, which is an upper bound on d by Lemma 7. We therefore apply Lemma 6 to inequality (25) for all choices $k-l \in \{1,\ldots,184\}$ except when $k-l = 1,3$ and get that

$$k - t < \frac{\log(Aq/\xi)}{\log B},$$

where $q > 6N$ is a denominator of a convergent of the continued fraction of γ such that $\xi = \|\mu q\| - N\|\gamma q\| > 0$. Indeed, using `Sagemath`, we have that

$$q = q_{75} = 1439006117080021301713618460568200942.$$

We find that if (k,l,t,d) is a possible solution of the equation (1) with $\omega_1 > 0$ and $k-l \in \{1,\ldots,184\}$ except when $k-l=1,3$, then

$$k - t < 178.$$

Let us now treat the cases where $k-l = 1$ and 3. The discussion of these cases will be different from the previous ones, because when applying Lemma 6 to the expression (25), the corresponding parameter μ appearing in Lemma 6 is

$$\frac{\log(1+\alpha^{-(k-l)})}{\log\alpha} = \begin{cases} -1, & \text{if } k-l=1; \\ 1 - \frac{\log 2}{\log\alpha}, & \text{if } k-l=3. \end{cases}$$

In both cases, the parameters γ and μ are linearly dependent, which yields that the corresponding value of ξ from Lemma 6 is always negative and therefore the reduction method is not useful for reducing the bound on $k-t$ in these instances. One can see that if $k-l = 1,3$, then the resulting inequality from (25) has the shape

$$0 < |a\gamma - b| < \frac{9}{\alpha^{k-t}},$$

with γ being an irrational number and $a, b \in \mathbb{Z}$. So, one can appeal to the known properties of the convergents of the continued fractions to obtain a nontrivial lower bound for

$$|a\gamma - b|.$$

When $k - l = 1$, from (25), we get that

$$0 < d\gamma - (k+1) < \frac{9}{\alpha^{k-t}}. \tag{26}$$

Let $[a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, ...] = [1, 2, 3, 1, 2, 3, 2, 4 \ldots]$ be the continued fraction expansion of γ, and let denote p_n/q_n its nth convergent. Recall also that $d < 12 \cdot 10^{34}$ by Lemma 4. Furthermore, $a_N := \max\{a_i : i = 0, 1, \ldots, 44\} = a_{17} = 134$. So, from the known properties of continued fractions, we obtain that

$$|d\gamma - (k+1)| > \frac{1}{(a_N + 2)d}. \tag{27}$$

Comparing estimates (26) and (27), we get right away that

$$\alpha^{k-t} < 9 \cdot 136 \cdot d < 15 \cdot 10^{37},$$

leading to $k - t < 184$.

By the same argument as the one we did before, we get that $k - t < 184$ in the case when $k - l = 3$. This completes the analysis of the cases when $k - l = 1, 3$. Consequently, $k - t < 184$ always holds.

Finally, we shall use (15) to reduce the upper bound on k. Put

$$\omega_2 = d \log 2 - k \log \alpha - \log \varphi(u, v) \tag{28}$$

where φ is the function given by the formula $\varphi(u, v) = 1 + \alpha^{-u} + \alpha^{-v}$, where $u = k - l, v = k - t$. Note that $\omega_2 \neq 0$. Thus, we distinguish the following cases. If $\omega_2 > 0$ then, from (15), we obtain

$$0 < \omega_2 \leq e^{\omega_2} - 1 < \frac{3}{\alpha^k}.$$

Replacing ω_2 in the above inequality by its formula (28) and dividing both sides of the resulting inequality by $\log \alpha$, we get

$$0 < d\left(\frac{\log 2}{\log \alpha}\right) - k - \frac{\log(1 + \alpha^{-u} + \alpha^{-v})}{\log \alpha} < \frac{7}{\alpha^k}. \tag{29}$$

We now put

$$\gamma := \frac{\log 2}{\log \alpha}, \quad \mu := -\frac{\log(1 + \alpha^{-u} + \alpha^{-v})}{\log \alpha}, \quad A := 7 \quad \text{and} \quad B := \alpha.$$

Clearly γ is an irrational number. We also put $N := 12 \cdot 10^{34}$, which is an upper bound on d by Lemma 7. We therefore apply Lemma 6 to inequality (29) for all choices of $u \in \{1, \ldots, 184\}, v \in \{1, \ldots, 184\}$ except when

$$
\begin{aligned}
(u, v) \in \varpi_1 = \{&(4,2),(13,1),(17,4),(17,5),(18,7),(19,4),(19,5),(22,2),(23,1),(23,2),(24,1),\\
&(24,5),(25,1),(28,2),(30,4),(31,1),(32,7),(33,1),(33,4),(33,5),(36,1),(36,2),\\
&(37,2),(38,2),(39,1),(43,4),(44,1),(45,4),(48,1),(48,5),(50,1),(51,9),(52,1),\\
&(52,2),(53,4),(54,2),(55,2),(56,2),(57,2),(59,2),(61,1),(62,4),(64,1),(66,1),\\
&(67,1),(69,2),(70,5),(75,1),(75,2),(83,5),(87,1),(87,7),(89,2),(90,2),(95,1),\\
&(95,7),(97,1),(98,1),(99,5),(100,1),(102,2),(107,2),(110,4),(111,2),(112,1),\\
&(112,2),(113,1),(113,5),(113,6),(114,1),(118,1),(122,1),(122,4),(122,6),\\
&(128,2),(129,1),(129,2),(130,1),(130,4),(130,5),(132,2),(133,2),(136,4),\\
&(138,2),(139,1),(139,2),(141,1),(141,2),(142,4),(147,1),(148,1),(158,7)\}
\end{aligned}
$$

and get that

$$
k < \frac{\log(Aq/\xi)}{\log B},
$$

where $q > 6N$ is a denominator of a convergent of the continued fraction of γ such that $\xi = \|\mu q\| - N\|\gamma q\| > 0$. Indeed, using `Sagemath`, we have that

$$
q = q_{82} = 120541184448257862602122545163201065832 49.
$$

We find that if (k, l, t, d) is a possible solution of the equation (1) with $\omega_2 > 0$ and $(u, v) \notin \varpi_1$, then $k < 196$. This is false because our assumption is that $k > 250$.

Let us now work with the cases when $(u, v) \in \varpi_1$. We cannot study these cases as before because when applying Lemma 6 to the expression (29), the corresponding quantity $\|\mu q\|$ appearing in Lemma 6 is zero. In these cases, the parameters γ and μ are linearly dependent, which yields that the corresponding value of ξ from Lemma 6 is always negative and therefore the reduction method is not useful for reducing the bound on k in these instances. However, one can see that if $(u, v) = (2, 4)$, then the resulting inequality from (29) has the shape

$$
0 < |x\gamma - y| < \frac{7}{\alpha^k},
$$

with γ being an irrational number and $x, y \in \mathbb{Z}$. So, one can use to the known properties of the convergents of the continued fractions to obtain a nontrivial lower bound for

$$
|x\gamma - y|.
$$

This clearly gives us an upper bound for k. For example, when $(u, v) = (2, 4)$,

$$
-\frac{\log(1 + \alpha^{-u} + \alpha^{-v})}{\log \alpha} = 2 - 2\frac{\log 2}{\log \alpha}
$$

and from (29), we get that

$$0 < (d-2)\gamma - (k-2) < \frac{7}{\alpha^k}. \tag{30}$$

Let $[a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, \ldots] = [1, 2, 3, 1, 2, 3, 2, 4 \ldots]$ be the continued fraction expansion of γ, and let denote p_n/q_n its nth convergent. Recall also that $d < 12 \cdot 10^{34}$ by Lemma 7.

Furthermore, $a_N := \max\{a_i; i = 0, 1, \ldots, 44\} = a_{17} = 134$. So, from the known properties of continued fractions, we obtain that

$$|(d-2)\gamma - (k-2)| > \frac{1}{(a_N + 2)d}. \tag{31}$$

Comparing estimates (30) and (31), we get that

$$\alpha^k < 7 \cdot 136 \cdot d < 12 \cdot 10^{37}, \tag{32}$$

leading to $k < 184$. Using the above argument, we obtain $k < 184$ in the case when $(u, v) \in \varpi_1$ except $(2, 4)$. We omit the details in order to avoid unnecessary repetitions. This completes the analysis of the cases when $(u, v) \in \varpi_1$. Consequently, $k < 196$ always holds.

Suppose now that $\omega_2 < 0$. First, note that $\frac{7}{\alpha^k} < \frac{1}{2}$ since $k > 250$. Then, from (15), we have that

$$|1 - e^{\omega_2}| < \frac{1}{2},$$

thus

$$\frac{1}{2} < e^{\omega_2} < \frac{3}{2}.$$

Therefore

$$e^{|\omega_2|} < 2.$$

Since $\omega_2 < 0$, we have

$$0 < |\omega_2| \leq e^{|\omega_2|} - 1 = e^{|\omega_2|}|e^{-|\omega_2|} - 1| = e^{|\omega_2|}|e^{\omega_2} - 1| < \frac{6}{\alpha^k}.$$

Then we obtain

$$0 < -d \log 2 + k \log \alpha + \log(1 + \alpha^{-u} + \alpha^{-v}) < \frac{6}{\alpha^k}.$$

By the same arguments used for proving (15), we obtain

$$0 < k\left(\frac{\log \alpha}{\log 2}\right) - d + \frac{\log(1 + \alpha^{-u} + \alpha^{-v})}{\log 2} < \frac{9}{\alpha^k}. \tag{33}$$

We now put

$$\gamma := \frac{\log \alpha}{\log 2}, \quad \mu := \frac{\log(1 + \alpha^{-u} + \alpha^{-v})}{\log 2}, \quad A := 9 \quad \text{and} \quad B := \alpha.$$

Clearly γ is an irrational number. We also put $N := 12 \cdot 10^{34}$, which is an upper bound on d by Lemma 7. We therefore apply Lemma 7 to inequality (33) for all choices of $u \in \{1, \ldots, 184\}, v \in \{1, \ldots, 184\}$ except when

$$\begin{aligned}(u, v) \in \varpi_2 = \{&(2,1),(3,3),(4,2),(5,2),(7,7),(8,2),(9,5),(9,6),(10,2),(11,4),(12,1),(13,1),\\&(14,2),(17,4),(19,5),(19,6),(20,9),(22,1),(23,1),(24,1),(24,5),(25,1),(25,2),\\&(28,2),(30,1),(31,4),(32,7),(33,1),(36,3),(37,2),(38,1),(40,2),(44,1),(47,2),\\&(48,1),(48,6),(50,1),(50,2),(51,9),(53,1),(54,2),(56,2),(57,2),(58,5),(58,6),\\&(58,8),(62,1),(62,4),(64,1),(66,1),(69,2),(70,6),(75,1),(75,2),(76,1),\\&(76,4),(77,2),(77,3),(78,2),(79,2),(83,6),(87,1),(87,7),(89,1),(89,2),(91,1),\\&(91,2),(92,1),(94,1),(95,1),(95,7),(96,2),(97,1),(98,1),(99,5),(99,6),(102,2),\\&(102,3),(112,2),(113,1),(113,5),(113,6),(120,1),(122,1),(122,5),(122,6),\\&(123,2),(129,2),(129,3),(130,4),(130,5),(134,7),(135,2),(138,1),(139,1),\\&(141,1),(141,2),(142,4),(143,1),(147,1),(148,1),(149,2)\},\end{aligned}$$

and get that

$$k < \frac{\log(Aq/\xi)}{\log B},$$

where $q > 6N$ is a denominator of a convergent of the continued fraction of γ such that $\xi = \|\mu q\| - N\|\gamma q\| > 0$. Indeed, using `Sagemath`, we have hat

$$q = q_{82} = 12341655049111936518205571908556668171489.$$

We find that if (k, l, t, d) is a possible solution of the equation (1) with $\omega_2 < 0$ and $(u, v) \notin \varpi_2$, then $k < 192$. This is false because our assumption is that $k > 250$. With the same arguments as in the case $\omega_2 > 0$, when $(u, v) \in \varpi_2$, we obtain that $k < 192$. Thus, Theorem 3 is proven.

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