When the Large Divisors of a Natural Number Are in Arithmetic Progression

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Abstract
Iannucci considered the positive divisors of a natural number \( n \) that do not exceed the square root of \( n \) and found all numbers whose such divisors are in arithmetic progression. Continuing the work, we define large divisors to be divisors at least \( \sqrt{n} \) and find all numbers whose large divisors are in arithmetic progression. The asymptotic formula for the count of these numbers not larger than \( x \) is observed to be \( \frac{x \log \log x}{\log x} \).

1 Introduction
For a natural number \( n \), let \( L_n \) denote the set of positive divisors of \( n \) that are at least \( \sqrt{n} \) and strictly smaller than \( n \); that is,

\[
L_n := \{ d : d \mid n, \sqrt{n} \leq d < n \}.
\]

Also, define

\[
L'_n := \{ d : d \mid n, \sqrt{n} \leq d \leq n \}.
\]

We call \( L'_n \) the set of large divisors of \( n \). Clearly, we have \( |L'_n| = |L_n| + 1 \). In this paper, we will determine the set of all natural numbers \( n \) such that either \( L_n \) or \( L'_n \) forms an arithmetic progression. Since \( L_n \subset L'_n \), if \( L'_n \) forms an arithmetic progression, then so does \( L_n \). Hence, we will first focus our attention on \( L_n \) and find all \( n \) such that

\[
L_n = \{ d, d + a, d + 2a, \ldots, d + (k - 1)a \}
\]
for some natural numbers $d, a$, and $k$. Note that $L_n$ can be empty and in that case, $L_n$ vacuously forms an arithmetic progression. Let $|L_n| = k \geq 0$.

Our work is a companion to a paper of Iannucci [3], who defined small divisors of $n$ to be divisors not exceeding $\sqrt{n}$ and found all natural numbers whose small divisors are in arithmetic progression. For previous work on divisors in or not in arithmetic progression, see [1, 6] and on small divisors, see [2, 4].

As usual, we have the divisor-counting function

$$\tau(n) := \sum_{d \mid n} 1.$$

Since $\tau(n)$ is multiplicative, for the $k$ distinct primes $p_1 < p_2 < \cdots < p_k$ and natural numbers $a_1, a_2, \ldots, a_k$, we have

$$\tau(p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}) = (a_1 + 1)(a_2 + 1) \cdots (a_k + 1).$$

(1)

If $n = bc$ and $b \leq c$, then $b \leq \sqrt{n} \leq c$; hence

$$\tau(n) = \begin{cases} 2|L_n|, & \text{if } n \text{ is not a square;} \\ 2|L_n| - 1, & \text{if } n \text{ is a square} \end{cases} = \begin{cases} 2|L_n| + 2, & \text{if } n \text{ is not a square;} \\ 2|L_n| + 1, & \text{if } n \text{ is a square.} \end{cases}$$

(2)

**Theorem 1.** Let $n$ be a natural number. If numbers in $L_n$ are in arithmetic progression, then one of the following holds:

(i) $n = 1$, and hence $L_n = \emptyset$.

(ii) $n = p$ for some prime $p$, and hence $L_n = \{p\}$.

(iii) $n = p^2$ for some prime $p$, and hence $L_n = \{p\}$.

(iv) $n = p^3$ for some prime $p$, and hence $L_n = \{p^2\}$.

(v) $n = pq$ for some primes $p < q$, and hence $L_n = \{q\}$.

(vi) $n = p^4$ for some prime $p$, and hence $L_n = \{p^2, p^3\}$.

(vii) $n = p^5$ for some prime $p$, and hence $L_n = \{p^3, p^4\}$.

(viii) $n = p^2q$ for some primes $p < q$, and hence $L_n = \{p^2, pq\}$ or $L_n = \{q, pq\}$.

(ix) $n = pq^2$ for some primes $p < q$, and hence $L_n = \{pq, q^2\}$.

(x) $n = pqr$ for some primes $p < q < r$, $pq < r$ and $p = \frac{1}{2}(q + 1)$, and hence $L_n = \{r, rp, rq\}$.

(xi) $n = p^3q$ for some primes $p > q$ and $q = \frac{1}{2}(p + 1)$, and hence $L_n = \{p^2, p^2q, p^3\}$. 2
To prove Theorem 1, we first find all forms of $n$ when $|L_n| = k \leq 3$ by case analysis, then show that $k$ cannot be larger than 3. To find all $n$ such that $L'_n$ forms an arithmetic progression, we need only to check the 11 forms in Theorem 1. It is straightforward to prove the following corollary, so we omit the proof.

**Corollary 2.** Let $n$ be a natural number. If numbers in $L'_n$ are in arithmetic progression, then one of the following holds:

(i) $n = 1$, and hence $L'_n = \{1\}$.

(ii) $n = p$, and hence $L'_n = \{p\}$.

(iii) $n = p^2$ for some prime $p$, and hence $L'_n = \{p, p^2\}$.

(iv) $n = p^3$ for some prime $p$, and hence $L'_n = \{p^2, p^3\}$.

(v) $n = pq$ for some primes $p < q$, and hence $L'_n = \{q, pq\}$.

2 Small cases of $|L_n|$

Assuming $L_n$ is in arithmetic progression, we fully characterize $n$ when $|L_n| \leq 3$ and prove that $|L_n| \neq 4$.

**Lemma 3.** If $L_n$ forms an arithmetic progression and $k \leq 3$, then one of the items in Theorem 1 is true.

**Proof.** We consider four cases corresponding to each $0 \leq k \leq 3$.

Case 1: If $k = 0$, then by (2), we have $\tau(n) \in \{1, 2\}$. If $\tau(n) = 1$, then $n = 1$. If $\tau(n) = 2$, then $n = p$ for some prime $p$. Hence, $L_n = \emptyset$. This corresponds to items (i) and (ii) of the theorem.

Case 2: If $k = 1$, then by (2), we have $\tau(n) \in \{3, 4\}$.

If $\tau(n) = 3$, then by (1), we have $n = p^2$ for some prime $p$, and hence $L_n = \{p\}$. This corresponds to item (iii) of the theorem.

If $\tau(n) = 4$, then by (1), we have $n = p^3$ for some prime $p$ or $n = pq$ for some primes $p < q$. For the former, we get $L_n = \{p^3\}$ and for the latter, we get $L_n = \{q\}$, corresponding to items (iv) and (v) of the theorem.

Case 3: If $k = 2$, then by (2), we have $\tau(n) \in \{5, 6\}$.

If $\tau(n) = 5$, then by (1), we have $n = p^4$ for some prime $p$, and hence $L_n = \{p^2, p^3\}$. This corresponds to item (vi).

If $\tau(n) = 6$, then by (1), we have $n = p^5$ for some prime $p$ or $n = p^2q$ or $pq^2$ for some primes $p < q$. 

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If \( n = p^5 \), then \( L_n = \{p^3, p^4\} \).

If \( n = p^2q \) for some primes \( p < q < p^2 \), then \( L_n = \{p^2, pq\} \). If \( n = p^2q \) for some primes \( p^2 < q \), we get \( L_n = \{q, pq\} \).

If \( n = pq^2 \) for some primes \( p < q \), then \( L_n = \{pq, q^2\} \).

These correspond to items (vii), (viii), (ix).

Case 4: If \( k = 3 \), then by (2), we have \( \tau(n) \in \{7, 8\} \).

If \( \tau(n) = 7 \), then by (1), we get \( n = p^6 \) for some prime \( p \). Then \( L_n = \{p^3, p^4, p^5\} \), which is impossible since \( p^5 - p^4 \neq p^4 - p^3 \).

If \( \tau(n) = 8 \), then by (1), we get \( n = pqr \) for some distinct primes \( p, q, r \) or \( p^3q \) for some distinct primes \( p, q \).

If \( n = pqr \), we may assume that \( p < q < r \). Two subcases are either \( r > pq \) or \( r < pq \).

- \( r > pq \): We have \( L_n = \{r, pr, qr\} \) and so, \( qr - pr = pr - r \), which implies that \( p = \frac{1}{2}(q + 1) \). This is item (x).
- \( r < pq \): We have \( L_n = \{pq, pr, qr\} \) and so, \( qr - pr = pr - pq \), which implies that \( p = \frac{qr}{2r-q} \). So, either \((2r-q)|q \) or \((2r-q)|r \). However, both are impossible since \( 2r - q > r > q \).

If \( n = p^3q \), two subcases are either \( p < q \) or \( p > q \).

- \( p < q \): If \( p < q < p^3 \), then \( L_n = \{p^3, pq, p^2q\} \). Either \( p^2q - pq = pq - p^3 \) or \( p^2q - p^3 = p^3 - pq \). It is easy to see that both cases are impossible. If \( q > p^3 \), then \( L_n = \{q, pq, p^2q\} \). Since \( p^2q - pq = pq - q \), we get \( p^2 = 2p - 1 \), which implies that \( p = 1 \), a contradiction.
- \( p > q \): \( L_n = \{p^2, p^2q, p^3\} \). So, \( p^3 - p^2q = p^2q - p^2 \). Then \( q = \frac{1}{2}(p + 1) \). This is item (xi).

**Lemma 4.** Our set \( L_n \) cannot have exactly 4 elements.

**Proof.** We prove this by contradiction. Suppose that \(|L_n| = 4\). By (2), we have \( \tau(n) \in \{9, 10\} \).

If \( \tau(n) = 9 \), then (1) implies that \( n = p^8 \) for some prime \( p \) or \( n = p^2q^2 \) for some primes \( p < q \).

- If \( n = p^8 \), then \( L_n = \{p^4, p^5, p^6, p^7\} \), which cannot form an arithmetic progression.
- If \( n = p^2q^2 \) for \( p < q \), then \( L_n = \{pq, p^2q, q^2, pq^2\} \). So, \( pq^2 + pq = p^2q + q^2 \), which implies that \( p = q \), a contradiction.
If \( \tau(n) = 10 \), either \( n = p^9 \) for some prime \( p \) or \( n = pq^4 \) for distinct primes \( p, q \).

If \( n = p^9 \), then \( L_n = \{p^5, p^6, p^7, p^8\} \), which cannot form an arithmetic progression.

If \( n = pq^4 \), we have four subcases.

\( p < q \): \( L_n = \{pq^2, q^3, pq^3, q^4\} \), so \( pq^2 + q^4 = q^3 + pq^3 \), which implies that \( p = q \), a contradiction.

\( q < p < q^2 \): \( L_n = \{q^3, pq^2, q^4, pq^3\} \), so \( q^3 + pq^3 = pq^2 + q^4 \), which implies that \( p = q \), a contradiction.

\( q^2 < p < q^4 \): \( L_n = \{pq, q^4, pq^2, pq^3\} \). Either \( pq^3 + pq = pq^2 + q^4 \) or \( pq^3 + q^4 = pq + pq^2 \). The former gives \( q = 1 \), while the latter gives \( p = -\frac{q^3}{q^2 - q - 1} \). Both pose a contradiction.

\( q^4 < p \): \( L_n = \{p, pq, pq^2, pq^3\} \), so \( p + pq^3 = pq + pq^2 \), which implies that \( q = 1 \), a contradiction.

Therefore, \( |L_n| \neq 4 \). \( \square \)

### 3 Proof of Theorem 1

By Lemmas 3 and 4, to prove Theorem 1, it suffices to prove that \( |L_n| \leq 4 \).

**Proof of Theorem 1.** We prove this by contradiction. Suppose that \( k = |L_n| \geq 5 \). Recall that

\[
L_n = \{d, d + a, d + 2a, \ldots, d + (k - 1)a\}
\]

for some natural numbers \( d \) and \( a \). Let \( \gcd(d, a) = \ell \). Write \( d = \ell k_1 \) and \( a = \ell k_2 \). Clearly, \( \gcd(k_1, k_2) = 1 \), so there exist integers \( s, t \) such that \( sk_1 + tk_2 = 1 \).

Let

\[
M = \text{lcm}(d, d + a, d + 2a, \ldots, d + (k - 1)a);
\]

that is, \( M \) denotes the least common multiple of all numbers in \( L_n \). Write

\[
M = \text{lcm}(\ell k_1, \ell k_1 + \ell k_2, \ell k_1 + 2\ell k_2, \ldots, \ell k_1 + (k - 1)\ell k_2) = \ell \cdot \text{lcm}(k_1, k_1 + k_2, k_1 + 2k_2, \ldots, k_1 + (k - 1)k_2).
\]

We claim that \( \gcd(k_1 + (k - 2)k_2, k_1 + (k - 1)k_2) = 1 \). Indeed, let

\[
x = k_1 + (k - 1)k_2 \\
y = k_1 + (k - 2)k_2.
\]

We have

\[
k_2 = x - y \\
k_1 = x - (k - 1)k_2 = x - (k - 1)(x - y).
\]
Because $sk_1 + tk_2 = 1$, we have
\[
s(x - (k - 1)(x - y)) + t(x - y) = 1.
\]
So,
\[
(t + s - s(k - 1))x + ((k - 1)s - t)y = 1,
\]
which implies that $\gcd(x, y) = 1$. Hence,
\[
N = \ell(k_1 + (k - 2)k_2)(k_1 + (k - 1)k_2) = \ell \cdot \text{lcm}(k_1 + (k - 2)k_2, k_1 + (k - 1)k_2)
\]
divides $M$. Because $N$ divides $M$ and $M$ divides $n$, we know that $N$ divides $n$. Clearly, $N > \ell(k_1 + (k - 1)k_2) = d + (k - 1)a \geq \sqrt{n}$. Because $N \notin L_n$, we get $N = n$. So, $\ell(k_1 + (k - 3)k_2)$ divides $N$. Hence,
\[
k_1 + (k - 3)k_2 \text{ divides } (k_1 + (k - 2)k_2)(k_1 + (k - 1)k_2).
\]
Using the same argument as above, we know that $\gcd(k_1 + (k - 3)k_2, k_1 + (k - 2)k_2) = 1$. So,
\[
k_1 + (k - 3)k_2 \text{ divides } k_1 + (k - 1)k_2.
\]
Write $k_1 + (k - 1)k_2 = u(k_1 + (k - 3)k_2)$ for some integer $u \geq 2$. Simplifying the equation, we get
\[
\frac{3u - 1}{u - 1} = \frac{k_1 + k k_2}{k_2} = \frac{k_1}{k_2} + k > 5.
\]
So, $u < 2$. This contradicts that $u \geq 2$. Therefore, it must be that $|L_n| < 5$, as desired. \qed

**Remark 5.** We can estimate how often a natural number $n$ not larger than $x > 0$ has its large divisors form an arithmetic progression. Let $f(x)$ be the function counting such numbers not larger than $x$.

The number of $n$ not larger than $x$ that is either of form $p, p^2$, or $p^3$ for a prime $p$ is asymptotic to
\[
\sum_{i=1}^{3} \pi(x^{1/i}) \sim \sum_{i=1}^{3} \frac{ix^{1/i}}{\log x}.
\]
By a result of Landau [5, §56], the number of $n \leq x$ of the form $pq$ for primes $p < q$ is asymptotic to
\[
\frac{x \log \log x}{\log x}.
\]
Combined with Corollary 2, we know that
\[
f(x) \sim \frac{x \log \log x}{\log x},
\]
which is similar to the asymptotic formula for the case of small divisors [3].
References


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