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When the Large Divisors of a Natural Number Are in Arithmetic Progression

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Abstract

Iannucci considered the positive divisors of a natural number n that do not exceed the square root of n and found all numbers whose such divisors are in arithmetic progression. Continuing the work, we define *large divisors* to be divisors at least $\sqrt{n}$ and find all numbers whose large divisors are in arithmetic progression. The asymptotic formula for the count of these numbers not larger than x is observed to be $\frac{x \log\log x}{\log x}$.

1 Introduction

For a natural number n, let L_n denote the set of positive divisors of n that are at least $\sqrt{n}$ and strictly smaller than n; that is,

$$L_n \;:=\; \{d \,:\, d|n, \sqrt{n} \leq d < n\}.$$

Also, define

$$L'_n \;:=\; \{d \,:\, d|n, \sqrt{n} \leq d \leq n\}.$$

We call L'_n the set of *large divisors* of n. Clearly, we have $|L'_n| = |L_n| + 1$. In this paper, we will determine the set of all natural numbers n such that either L_n or L'_n forms an arithmetic progression. Since $L_n \subset L'_n$, if L'_n forms an arithmetic progression, then so does L_n. Hence, we will first focus our attention on L_n and find all n such that

$$L_n \;=\; \{d, d+a, d+2a, \ldots, d+(k-1)a\}$$

for some natural numbers d, a, and k. Note that L_n can be empty and in that case, L_n vacuously forms an arithmetic progression. Let $|L_n| = k \geq 0$.

Our work is a companion to a paper of Iannucci [3], who defined *small divisors* of n to be divisors not exceeding $\sqrt{n}$ and found all natural numbers whose small divisors are in arithmetic progression. For previous work on divisors in or not in arithmetic progression, see [1, 6] and on small divisors, see [2, 4].

As usual, we have the divisor-counting function

$$\tau(n) \ := \ \sum_{d|n} 1.$$

Since $\tau(n)$ is multiplicative, for the k distinct primes $p_1 < p_2 < \cdots < p_k$ and natural numbers $a_1, a_2, \ldots, a_k$, we have

$$\tau(p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}) \ = \ (a_1+1)(a_2+1)\cdots(a_k+1). \tag{1}$$

If $n = bc$ and $b \leq c$, then $b \leq \sqrt{n} \leq c$; hence

$$\tau(n) \ = \ \begin{cases} 2|L'_n|, & \text{if } n \text{ is not a square;} \\ 2|L'_n| - 1, & \text{if } n \text{ is a square} \end{cases} \ = \ \begin{cases} 2|L_n| + 2, & \text{if } n \text{ is not a square;} \\ 2|L_n| + 1, & \text{if } n \text{ is a square.} \end{cases} \tag{2}$$

Theorem 1. *Let n be a natural number. If numbers in L_n are in arithmetic progression, then one of the following holds:*

(i) $n = 1$, *and hence* $L_n = \emptyset$.

(ii) $n = p$ *for some prime* p, *and hence* $L_n = \emptyset$.

(iii) $n = p^2$ *for some prime* p, *and hence* $L_n = \{p\}$.

(iv) $n = p^3$ *for some prime* p, *and hence* $L_n = \{p^2\}$.

(v) $n = pq$ *for some primes* $p < q$, *and hence* $L_n = \{q\}$.

(vi) $n = p^4$ *for some prime* p, *and hence* $L_n = \{p^2, p^3\}$.

(vii) $n = p^5$ *for some prime* p, *and hence* $L_n = \{p^3, p^4\}$.

(viii) $n = p^2 q$ *for some primes* $p < q$, *and hence* $L_n = \{p^2, pq\}$ *or* $L_n = \{q, pq\}$.

(ix) $n = pq^2$ *for some primes* $p < q$, *and hence* $L_n = \{pq, q^2\}$.

(x) $n = pqr$ *for some primes* $p < q < r$, $pq < r$ *and* $p = \frac{1}{2}(q+1)$, *and hence* $L_n = \{r, rp, rq\}$.

(xi) $n = p^3 q$ *for some primes* $p > q$ *and* $q = \frac{1}{2}(p+1)$, *and hence* $L_n = \{p^2, p^2 q, p^3\}$.

To prove Theorem 1, we first find all forms of n when $|L_n| = k \leq 3$ by case analysis, then show that k cannot be larger than 3. To find all n such that L'_n forms an arithmetic progression, we need only to check the 11 forms in Theorem 1. It is straightforward to prove the following corollary, so we omit the proof.

Corollary 2. *Let n be a natural number. If numbers in L'_n are in arithmetic progression, then one of the following holds:*

(i) $n = 1$, *and hence* $L'_n = \{1\}$.

(ii) $n = p$, *and hence* $L'_n = \{p\}$.

(iii) $n = p^2$ *for some prime* p, *and hence* $L'_n = \{p, p^2\}$.

(iv) $n = p^3$ *for some prime* p, *and hence* $L'_n = \{p^2, p^3\}$.

(v) $n = pq$ *for some primes* $p < q$, *and hence* $L'_n = \{q, pq\}$.

2 Small cases of $|L_n|$

Assuming L_n is in arithmetic progression, we fully characterize n when $|L_n| \leq 3$ and prove that $|L_n| \neq 4$.

Lemma 3. *If L_n forms an arithmetic progression and $k \leq 3$, then one of the items in Theorem 1 is true.*

Proof. We consider four cases corresponding to each $0 \leq k \leq 3$.

Case 1: If $k = 0$, then by (2), we have $\tau(n) \in \{1, 2\}$. If $\tau(n) = 1$, then $n = 1$. If $\tau(n) = 2$, then $n = p$ for some prime p. Hence, $L_n = \emptyset$. This corresponds to items (i) and (ii) of the theorem.

Case 2: If $k = 1$, then by (2), we have $\tau(n) \in \{3, 4\}$.

If $\tau(n) = 3$, then by (1), we have $n = p^2$ for some prime p, and hence $L_n = \{p\}$. This corresponds to item (iii) of the theorem.

If $\tau(n) = 4$, then by (1), we have $n = p^3$ for some prime p or $n = pq$ for some primes $p < q$. For the former, we get $L_n = \{p^2\}$ and for the latter, we get $L_n = \{q\}$, corresponding to items (iv) and (v) of the theorem.

Case 3: If $k = 2$, then by (2), we have $\tau(n) \in \{5, 6\}$.

If $\tau(n) = 5$, then by (1), we have $n = p^4$ for some prime p, and hence $L_n = \{p^2, p^3\}$. This corresponds to item (vi).

If $\tau(n) = 6$, then by (1), we have $n = p^5$ for some prime p or $n = p^2q$ or pq^2 for some primes $p < q$.

If $n = p^5$, then $L_n = \{p^3, p^4\}$.

If $n = p^2 q$ for some primes $p < q < p^2$, then $L_n = \{p^2, pq\}$. If $n = p^2 q$ for some primes $p^2 < q$, we get $L_n = \{q, pq\}$.

If $n = pq^2$ for some primes $p < q$, then $L_n = \{pq.q^2\}$.

These correspond to items (vii), (viii), (ix).

Case 4: If $k = 3$, then by (2), we have $\tau(n) \in \{7, 8\}$.

If $\tau(n) = 7$, then by (1), we get $n = p^6$ for some prime p. Then $L_n = \{p^3, p^4, p^5\}$, which is impossible since $p^5 - p^4 \neq p^4 - p^3$.

If $\tau(n) = 8$, then by (1), we get $n = pqr$ for some distinct primes p, q, r or $p^3 q$ for some distinct primes p, q.

If $n = pqr$, we may assume that $p < q < r$. Two subcases are either $r > pq$ or $r < pq$.

$r > pq$: We have $L_n = \{r, pr, qr\}$ and so, $qr - pr = pr - r$, which implies that $p = \frac{1}{2}(q + 1)$. This is item (x).

$r < pq$: We have $L_n = \{pq, pr, qr\}$ and so, $qr - pr = pr - pq$, which implies that $p = \frac{qr}{2r-q}$. So, either $(2r - q)|q$ or $(2r - q)|r$. However, both are impossible since $2r - q > r > q$.

If $n = p^3 q$, two subcases are either $p < q$ or $p > q$.

$p < q$: If $p < q < p^3$, then $L_n = \{p^3, pq, p^2 q\}$. Either $p^2 q - pq = pq - p^3$ or $p^2 q - p^3 = p^3 - pq$. It is easy to see that both cases are impossible. If $q > p^3$, then $L_n = \{q, pq, p^2 q\}$. Since $p^2 q - pq = pq - q$, we get $p^2 = 2p - 1$, which implies that $p = 1$, a contradiction.

$p > q$: $L_n = \{p^2, p^2 q, p^3\}$. So, $p^3 - p^2 q = p^2 q - p^2$. Then $q = \frac{1}{2}(p + 1)$. This is item (xi).

□

Lemma 4. *Our set L_n cannot have exactly 4 elements.*

Proof. We prove this by contradiction. Suppose that $|L_n| = 4$. By (2), we have $\tau(n) \in \{9, 10\}$.

If $\tau(n) = 9$, then (1) implies that $n = p^8$ for some prime p or $n = p^2 q^2$ for some primes $p < q$.

If $n = p^8$, then $L_n = \{p^4, p^5, p^6, p^7\}$, which cannot form an arithmetic progression.

If $n = p^2 q^2$ for $p < q$, then $L_n = \{pq, p^2 q, q^2, pq^2\}$. So, $pq^2 + pq = p^2 q + q^2$, which implies that $p = q$, a contradiction.

If $\tau(n) = 10$, either $n = p^9$ for some prime p or $n = pq^4$ for distinct primes p, q.

If $n = p^9$, then $L_n = \{p^5, p^6, p^7, p^8\}$, which cannot form an arithmetic progression.

If $n = pq^4$, we have four subcases.

$p < q$: $L_n = \{pq^2, q^3, pq^3, q^4\}$, so $pq^2 + q^4 = q^3 + pq^3$, which implies that $p = q$, a contradiction.

$q < p < q^2$: $L_n = \{q^3, pq^2, q^4, pq^3\}$, so $q^3 + pq^3 = pq^2 + q^4$, which implies that $p = q$, a contradiction.

$q^2 < p < q^4$: $L_n = \{pq, q^4, pq^2, pq^3\}$. Either $pq^3 + pq = pq^2 + q^4$ or $pq^3 + q^4 = pq + pq^2$. The former gives $q = 1$, while the latter gives $p = -\frac{q^3}{q^2-q-1}$. Both pose a contradiction.

$q^4 < p$: $L_n = \{p, pq, pq^2, pq^3\}$, so $p + pq^3 = pq + pq^2$, which implies that $q = 1$, a contradiction.

Therefore, $|L_n| \neq 4$. □

3 Proof of Theorem 1

By Lemmas 3 and 4, to prove Theorem 1, it suffices to prove that $|L_n| \leq 4$.

Proof of Theorem 1. We prove this by contradiction. Suppose that $k = |L_n| \geq 5$. Recall that

$$L_n = \{d, d+a, d+2a, \ldots, d+(k-1)a\}$$

for some natural numbers d and a. Let $\gcd(d, a) = \ell$. Write $d = \ell k_1$ and $a = \ell k_2$. Clearly, $\gcd(k_1, k_2) = 1$, so there exist integers s, t such that $sk_1 + tk_2 = 1$.

Let

$$M = \operatorname{lcm}(d, d+a, d+2a, \ldots, d+(k-1)a);$$

that is, M denotes the least common multiple of all numbers in L_n. Write

$$\begin{aligned} M &= \operatorname{lcm}(\ell k_1, \ell k_1 + \ell k_2, \ell k_1 + 2\ell k_2, \ldots, \ell k_1 + (k-1)\ell k_2) \\ &= \ell \cdot \operatorname{lcm}(k_1, k_1 + k_2, k_1 + 2k_2, \ldots, k_1 + (k-1)k_2). \end{aligned}$$

We claim that $\gcd(k_1 + (k-2)k_2, k_1 + (k-1)k_2) = 1$. Indeed, let

$$\begin{aligned} x &= k_1 + (k-1)k_2 \\ y &= k_1 + (k-2)k_2. \end{aligned}$$

We have

$$\begin{aligned} k_2 &= x - y \\ k_1 &= x - (k-1)k_2 = x - (k-1)(x-y). \end{aligned}$$

Because $sk_1 + tk_2 = 1$, we have

$$s(x - (k-1)(x-y)) + t(x-y) \; = \; 1.$$

So,

$$(t + s - s(k-1))x + ((k-1)s - t)y \; = \; 1,$$

which implies that $\gcd(x, y) = 1$. Hence,

$$N \; = \; \ell(k_1 + (k-2)k_2)(k_1 + (k-1)k_2) \; = \; \ell \cdot \mathrm{lcm}\,(k_1 + (k-2)k_2, k_1 + (k-1)k_2) \text{ divides } M.$$

Because N divides M and M divides n, we know that N divides n. Clearly, $N > \ell(k_1 + (k-1)k_2) = d + (k-1)a \geq \sqrt{n}$. Because $N \notin L_n$, we get $N = n$. So, $\ell(k_1 + (k-3)k_2)$ divides N. Hence,

$$k_1 + (k-3)k_2 \text{ divides } (k_1 + (k-2)k_2)(k_1 + (k-1)k_2).$$

Using the same argument as above, we know that $\gcd(k_1 + (k-3)k_2, k_1 + (k-2)k_2) = 1$. So,

$$k_1 + (k-3)k_2 \text{ divides } k_1 + (k-1)k_2.$$

Write $k_1 + (k-1)k_2 = u(k_1 + (k-3)k_2)$ for some integer $u \geq 2$. Simplifying the equation, we get

$$\frac{3u-1}{u-1} \; = \; \frac{k_1 + kk_2}{k_2} \; = \; \frac{k_1}{k_2} + k \; > \; 5.$$

So, $u < 2$. This contradicts that $u \geq 2$. Therefore, it must be that $|L_n| < 5$, as desired. □

Remark 5. We can estimate how often a natural number n not larger than $x > 0$ has its large divisors form an arithmetic progression. Let $f(x)$ be the function counting such numbers not larger than x.

The number of n not larger than x that is either of form p, p^2, or p^3 for a prime p is asymptotic to

$$\sum_{i=1}^{3} \pi(x^{1/i}) \; \sim \; \sum_{i=1}^{3} \frac{ix^{1/i}}{\log x}.$$

By a result of Landau [5, §56], the number of $n \leq x$ of the form pq for primes $p < q$ is asymptotic to

$$\frac{x \log\log x}{\log x}.$$

Combined with Corollary 2, we know that

$$f(x) \; \sim \; \frac{x \log\log x}{\log x},$$

which is similar to the asymptotic formula for the case of small divisors [3].

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