Journal of Integer Sequences, Vol. 23 (2020),
Article 20.11.2

An Identity for Generalized Bernoulli Polynomials

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Abstract

Recognizing the great importance of Bernoulli numbers and Bernoulli polynomials in various branches of mathematics, the present paper develops two results dealing with these objects. The first one proposes an identity for the generalized Bernoulli polynomials, which leads to further generalizations for several relations involving classical Bernoulli numbers and Bernoulli polynomials. In particular, it generalizes a recent identity suggested by Gessel. The second result allows the deduction of similar identities for Fibonacci, Lucas, and Chebyshev polynomials, as well as for generalized Euler polynomials, Genocchi polynomials, and generalized numbers of Stirling.

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1 Introduction

Let $\mathbb{N}$ and $\mathbb{C}$ denote, respectively, the set of positive integers and the set of complex numbers. In his book, Roman [41, p. 93] defined generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ as follows: for all $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}$, we have

$$\sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!} = \left(\frac{t}{e^t - 1}\right)^{\alpha} e^{tx}. \tag{1}$$

The Bernoulli numbers B_n, classical Bernoulli polynomials $B_n(x)$, and generalized Bernoulli numbers $B_n^{(\alpha)}$ are, respectively, defined by

$$B_n = B_n(0),\ \ B_n(x) = B_n^{(1)}(x),\ \text{ and } B_n^{(\alpha)} = B_n^{(\alpha)}(0). \tag{2}$$

The Bernoulli numbers and the Bernoulli polynomials play a fundamental role in various branches of mathematics, such as combinatorics, number theory, mathematical analysis, and topology. Dilcher and Slavutskii [20] listed numerous publications on the properties of Bernoulli numbers and Bernoulli polynomials.

In this paper, we are mainly interested in evaluating the sum $S_{n,\ell,r}^{(\alpha)}(x, y, z)$ defined by

$$\begin{aligned} S_{n,\ell,r}^{(\alpha)}(x, y, z) = &\sum_{k=0}^{n+r} \binom{n+r}{k}\binom{\ell+k+r}{r} x^{n+r-k} B_{\ell+k}^{(\alpha)}(y) \\ &+ (-1)^{\ell+n+r+1} \sum_{k=0}^{\ell+r} \binom{\ell+r}{k}\binom{n+k+r}{r} x^{\ell+r-k} B_{n+k}^{(\alpha)}(z), \end{aligned} \tag{3}$$

where n, ℓ, and r are non-negative integers.

In 2011, Zekiri and Bencherif [53] proved, for r odd, that

$$S_{n,n,r}^{(1)}(1, 0, 0) = 0. \tag{4}$$

In 2012, Bencherif and Garici [10] improved Eq. (4), for all non-negative integers ℓ, and showed that

$$S_{n,\ell,r}^{(1)}(1, 0, 0) = 0. \tag{5}$$

In 2013, using umbral calculus, Gessel [8, Thm. 2, p. 6] generalized the result above by proving the following explicit formula for the sum $S_{n,\ell,r}^{(1)}(m, 0, 0)$:

$$S_{n,\ell,r}^{(1)}(m, 0, 0) = (r + 1)\sum_{k=1}^{m-1}\sum_{j=0}^{r+1}(-1)^{\ell+j-1}\binom{n+r}{j}\binom{\ell+r}{r+1-j} k^{\ell+j-1}(m-k)^{n+r-j}, \tag{6}$$

for all integers $m \geq 1$.

By considering the special case where $\ell = n$ and r is odd, from Eq. (6), Gessel [8] deduced the following formula:

$$\begin{aligned}\frac{1}{2}S^{(1)}_{n,n,r}(m,0,0) &= \sum_{k=0}^{n+r} m^{n+r-k}\binom{n+r}{k}\binom{n+k+r}{r}B_{n+k}\\ &= \frac{1}{2}(r+1)\sum_{k=1}^{m-1}\sum_{j=0}^{r+1}\binom{n+r}{j}\binom{n+r}{r+1-j}k^{j+n-1}(k-m)^{n+r-j}.\end{aligned}\tag{7}$$

The Relations (5), (6), and (7) are, in fact, the synthesis of a long journey of research discussed in the third section.

This paper briefly reviews some properties of generalized Bernoulli polynomials, as well as two related lemmas that help to prove the result presented in the second section. The main result proposes a simplified and useful expression for the sum $S^{(\alpha)}_{n,l,r}(x,y,z)$ where $x+y+z-\alpha$ is a non-negative integer. The result is stated and proven in the fourth section. Then the fifth section provides some applications related to the main theorem. In the last section, we establish similar identities for Fibonacci, Lucas polynomials and more.

2 Some properties of the generalized Bernoulli polynomials and lemmas

Let us consider the three following operators defined over any endomorphism of the vector space $\mathbb{C}[x]$. The classical derivation operator D, the identity operator I, and the finite difference operator Δ are respectively defined by

$$I(x^n) = x^n \text{ and } \Delta(x^n) = (x+1)^n - x^n,\ n \in \mathbb{N}. \tag{8}$$

Generalized Bernoulli polynomials can be expressed as a sequence of Appell polynomials [5, 11]. These polynomials satisfy the following well-known properties for which the proofs are straightforward [41]

$$B^{(\alpha)}_0(x) = 1, \tag{9}$$

$$D(B^{(\alpha)}_n(x)) = nB^{(\alpha)}_{n-1}(x),\ n \geq 1, \tag{10}$$

$$B^{(\alpha)}_n(x+y) = \sum_{k=0}^{n}\binom{n}{k}y^{n-k}B^{(\alpha)}_k(x), \tag{11}$$

$$\Delta(B^{(\alpha)}_n(x)) = D(B^{(\alpha-1)}_n(x)), \tag{12}$$

$$B^{(\alpha)}_n(\alpha - x) = (-1)^n B^{(\alpha)}_n(x). \tag{13}$$

For every $\alpha \in \mathbb{C}$, let us consider the endomorphism Ω_α of $\mathbb{C}[x]$ defined by

$$\Omega_\alpha(x^n) = B^{(\alpha)}_n(x),\ n \in \mathbb{N}. \tag{14}$$

Lemma 1. *For every non-negative integer n and for all complex numbers α and γ, we have*

$$\Omega_\alpha((x+\gamma)^n) = B_n^{(\alpha)}(x+\gamma). \tag{15}$$

Proof. Just by using Property (11) for $y=\gamma$, it follows that

$$\Omega_\alpha((x+\gamma)^n) = \Omega_\alpha\left(\sum_{k=0}^{n}\binom{n}{k}\gamma^{n-k}x^k\right) = \sum_{k=0}^{n}\binom{n}{k}\gamma^{n-k}B_k^{(\alpha)}(x) = B_n^{(\alpha)}(x+\gamma). \tag{16}$$

□

The operators Ω_α, Δ, and D satisfy some remarkable relations that are mentioned in the following lemma.

Lemma 2. *For every $\alpha \in \mathbb{C}$, we have*

$$D \circ \Omega_\alpha = \Omega_\alpha \circ D, \tag{17}$$

$$\Omega_\alpha \circ \Delta = \Omega_{\alpha-1} \circ D. \tag{18}$$

Proof.

1. It is sufficient to verify that the equality $(D\circ\Omega_\alpha)(x^n) = (\Omega_\alpha \circ D)(x^n)$ is true for all non-negative integers n. For $n=0$, the property is obvious since $B_0^{(\alpha)}(x) = 1$ and thus $(D\circ\Omega_\alpha)(x^0) = 0 = (\Omega_\alpha\circ D)(x^0)$. For every $n \geq 1$, we have

$$(D\circ\Omega_\alpha)(x^n) = D(B_n^{(\alpha)}(x)) = \Omega_\alpha(nx^{n-1}) = (\Omega_\alpha\circ D)(x^n). \tag{19}$$

2. By using Lemma 1, for $\gamma = 1$, Equations (12) and (17), for all integers $n \geq 0$, it follows that

$$\begin{aligned}(\Omega_\alpha \circ \Delta)(x^n) &= \Omega_\alpha((x+1)^n - x^n)\\ &= B_n^{(\alpha)}(x+1) - B_n^{(\alpha)}(x)\\ &= \Delta(B_n^{(\alpha)}(x))\\ &= D(B_n^{(\alpha-1)}(x))\\ &= (D\circ\Omega_{\alpha-1})(x^n)\\ &= (\Omega_{\alpha-1}\circ D)(x^n).\end{aligned}$$

□

Abramowitz and Stegun [1] presented the well-known following properties of classical Bernoulli numbers and Bernoulli polynomials:

$$\begin{aligned}B_n(x+1) &= \sum_{k=0}^{n}\binom{n}{k}B_k(x)\\ &= B_n(x) + nx^{n-1},\ n \geq 1,\end{aligned} \tag{20}$$

$$\sum_{k=0}^{n}\binom{n}{k}B_k = B_n + \delta_{n,1}, \tag{21}$$

$$(-1)^n B_n = B_n + \delta_{n,1}, \tag{22}$$

where $\delta_{i,j}$ is Kronecker's symbol taking the value 1 if $i = j$ and 0 otherwise. Notice that Eq. (22) is equivalent to

$$B_1 = -\frac{1}{2} \quad \text{and} \quad B_{2n+1} = 0,\ n \geq 1. \tag{23}$$

By using the relations above, we can calculate the first Bernoulli numbers and Bernoulli polynomials.

3 Literature review

One notices that Relations (6) and (7) proven by Gessel, can be reformulated in a more useful form for this study. Changing k to $m-k$ in the right side of Relation (6), one obtains the following identity:

$$S^{(1)}_{n,l,r}(m,0,0) = (r+1)\sum_{k=1}^{m-1}\sum_{j=0}^{r+1}\binom{n+r}{j}\binom{\ell+r}{r+1-j}k^{n+r-j}(k-m)^{\ell+j-1}. \tag{24}$$

So, noting that for all integers r and s such that $r+s$ is even, we deduce the following property.

$$\sum_{k=1}^{m-1}(k^r(k-m)^s - k^s(k-m)^r) = 0, \tag{25}$$

Aïder and Bencherif [4] proposed a formula equivalent to Gessel's identity, which can be written as follows:

$$\sum_{k=0}^{n+r} m^{n+r-k}\binom{n+r}{k}\binom{n+k+r}{r}B_{n+k} = \sum_{k=1}^{m-1} p_r(n,m,k), \tag{26}$$

where

$$\begin{aligned} p_r(n,m,k) = {} & \frac{r+1}{2}\binom{n+r}{\frac{r+1}{2}}^2 (k(k-m))^{n+\frac{r-1}{2}} \\ & + (r+1)\sum_{j=0}^{\frac{r-1}{2}}\binom{n+r}{j}\binom{n+r}{r+1-j}k^{n+r-j}(k-m)^{n+j-1}. \end{aligned} \tag{27}$$

In what follows, we show that Gessel's relations (6) and (7) and their equivalent relations (24) and (26) generalize numerous identities involving Bernoulli numbers. Therefore, from Equations (21) and (22), Bencherif and Garici [10] stated the following well-known property:

$$\sum_{k=0}^{n}\binom{n}{k}B_k = (-1)^n B_n, \tag{28}$$

which can be written as

$$S^{(1)}_{n,0,0}(1,0,0) = 0. \tag{29}$$

Eq. (28) means that the sequence $((-1)^n B_n)_{n\geq 0}$ is self-dual (in other words, a Cesaro sequence) [10, 16, 18, 35, 46, 47, 52] and [33, p. 256]. Gould [24] also presented various explicit formulae for Bernoulli numbers. However, several authors have sought other recurrence relations that would make simpler Bernoulli numbers computation. Thus, in 1880 Lucas [32] proved, by using symbolic calculation, the following relation that he deemed important:

$$\sum_{k=0}^{n} \binom{n}{k} B_{\ell+k} + (-1)^{\ell+n+1} \sum_{k=0}^{\ell} \binom{\ell}{k} B_{n+k} = 0, \tag{30}$$

which can be written as

$$S^{(1)}_{n,\ell,0}(1,0,0) = 0. \tag{31}$$

This relation therefore constitutes a special case of Eq. (5). Lucas then noticed that for the special case where $\ell = n+1$, the Eq. (31) becomes

$$(n+1)S^{(1)}_{n,n+1,0}(1,0,0) = \sum_{k=0}^{n+1} \binom{n+1}{k} (n+k+1) B_{n+k} = 0, \tag{32}$$

furthermore, we deduce that for $n \geq 1$

$$\sum_{k=0}^{n} \binom{n+1}{k} (n+k+1) B_{n+k} = 0. \tag{33}$$

According to Lucas, Eq. (33) is a great use for the computation Bernoulli numbers. It permits the expression of the Bernoulli number B_{2n} just by using Bernoulli numbers B_j for $n \leq j \leq 2n-1$, which requires less computation than by applying the following relation deduced from Eq. (21)

$$B_{2n} = \frac{-1}{2n+1} \sum_{j=0}^{2n-1} \binom{2n+1}{j} B_j.$$

This latter assumes the knowledge of Bernoulli numbers B_j for $0 \leq j \leq 2n-1$. In fact, Property (33) was also discovered in 1827 by Von Ettingshausen [21] before being rediscovered in 1877 by Seidel [43]. Lucas proved Relation (30) using symbolic calculation or umbral calculus as shown in the articles of Agoh [2] and Gessel [22]. This relation has been the subject of intensive research. After Lucas, several authors have proved Eq. (33) again, using different methods. Then, Nielsen [37] proved the same relation again in 1923, as Kaneko [29] did in 1995. Kaneko provided two different proofs of Relation (33), one complicated one based on the theory of continued fractions applied to formal series, and the second much simpler one due to Zagier, based on an involutive transform of sequences. The relation was later proven again in 2000 by Satoh [42], then in 2001 by Chang and Ha [15, Corollary. 1 (a), p. 472], and afterward in 2009 by Cigler [19].

Recall that in 1971, Carlitz [13] already wondered whether Relation (30) could be deduced from only Eq. (21). A year later, in response to Carlitz's problem, Shannon [44] proved by using Eq. (23) by induction on m and n. In 2005, Vassilev and Missana [49] demonstrated the following relation:

$$\sum_{k=0}^{n-1}\binom{n}{k}B_{\ell+k}+(-1)^{\ell+n+1}\sum_{k=0}^{\ell-1}\binom{\ell}{k}B_{n+k}=0,\ \text{for } n\geq 1 \ \text{ and } \ell\geq 1, \tag{34}$$

which can be written as

$$S^{(1)}_{n,\ell,0}(1,0,0)-(1-(-1)^{\ell+n})B_{n+\ell}=0,\ \text{for } n\geq 1 \text{ and } \ell\geq 1, \tag{35}$$

which is equivalent to Eq. (30). Their proof refers to the symmetry of polynomials of two variables involving Bernoulli numbers, introduced in [48].

In 2007, Chu and Magli [18] again proved Eq. (30) by exploiting Eq. (28). The same identity had been shown in 2009 by Chen and Sun [17] via a computational algebraic approach, using an extension of Zeilberger's algorithm. In 2012, Bencherif and Garici [10] proved Eq. (30) by using Eq. (28). In 2014, Gould and Quaintance [23], by studying some properties of the binomial transform of a sequence, obtained the same result. In the same year, Prodinger [40] presented a short proof based on a special case of generating functions. A year later, Neto [36] managed to get the same result by exploiting some properties of the Zeon algebra. In the same year, Zekiri and Bencherif [51] attained a more general result, by applying the umbral calculus.

Recall that in 2000, Agoh [2, Eq. (4.3), p. 210] showed the generalization of Eq. (31)

$$S^{(1)}_{n,\ell,r}(1,0,0)=0 \tag{36}$$

by using congruences and umbral calculus. In the following year, Momiyama [34] proved a formula equivalent to Eq. (36) for $r=1$, by applying a q-adic method. Shortly after, in 2003, in his article on the applications of classical umbral calculus, Gessel [22] produced the generalized Eq. (32), which he called Kaneko's identity. First, Gessel began by examining the relation below, which is none other than Formula (24) when $r=0$:

$$S^{(1)}_{n,\ell,0}(m,0,0)=\sum_{k=1}^{m-1}((n+\ell)k-mn)k^{n-1}(k-m)^{\ell-1}. \tag{37}$$

Then, by using the following property,

$$(n+1)S^{(1)}_{n,n+1,0}(m,0,0)=\sum_{k=0}^{n+1}m^{n+1-k}\binom{n+1}{k}(n+k+1)B_{n+k}, \tag{38}$$

he obtained the following generalization of Kaneko's identity:

$$\sum_{k=0}^{n+1}m^{n+1-k}\binom{n+1}{k}(n+1+k)B_{n+k}=\sum_{k=1}^{m-1}p_1(m,n,k), \tag{39}$$

which is exactly Formula (26) when $r = 1$.

In 2009, Chen and Sun [17] proved, by using an extension of Zeilberger's algorithm, that

$$\sum_{k=0}^{n+3} m^{n+3-k}\binom{n+3}{k}\binom{n+k+3}{3}B_{n+k} = \sum_{k=1}^{m-1} q(m,n,k). \tag{40}$$

Then, Chen and Sun provided the expression $q(m,n,k)$, which can be written as follows:

$$q(m,n,k) = p_3(m,n,k) + \binom{n+3}{3}(3n+11)(k^{n+2}(k-m)^n - k^n(k-m)^{n+2}).$$

Recalling Eq. (25), one notices that

$$\sum_{k=1}^{m-1} q(m,n,k) = \sum_{k=1}^{m-1} p_3(m,n,k). \tag{41}$$

Equality (40) is equivalent to Formula (26) for $r = 3$.

4 Main result

The following theorem provides some simplified expressions of $S^{(\alpha)}_{n,\ell,r}(x,y,z)$ for $x+y+z-\alpha = s$ where s is a non-negative integer. Then it leads to interesting identities for $s = 0$ or $\alpha = 1$.

Theorem 3. *For all complex numbers α, λ, and for all non-negative integers l, n, r, and s, we have*

$$\begin{aligned}
&\sum_{k=0}^{n+r} \lambda^{n+r-k}\binom{n+r}{k}\binom{\ell+k+r}{r}B^{(\alpha)}_{\ell+k}(x)\\
&+(-1)^{\ell+n+r+1}\sum_{k=0}^{\ell+r} \lambda^{\ell+r-k}\binom{\ell+r}{k}\binom{n+k+r}{r}B^{(\alpha)}_{n+k}(\alpha+s-\lambda-x)\\
&=\Omega_{\alpha-1}\Big(\frac{D^{r+1}}{r!}\sum_{k=1}^{s}(x-k)^{\ell+r}(x+\lambda-k)^{n+r}\Big).
\end{aligned} \tag{42}$$

Proof. Let us consider the polynomial $P(x)$ defined by

$$P(x) = \sum_{k=1}^{s} P_k(x)$$

where

$$P_k(x) = \frac{D^r}{r!}\big((x-k)^{\ell+r}(x+\lambda-k)^{n+r}\big).$$

One notices that the right side of Eq. (42) is equivalent to $(\Omega_{\alpha-1} \circ D)(P(x))$, and from Lemma 2, one has $(\Omega_{\alpha-1} \circ D)(P(x)) = (\Omega_\alpha \circ \Delta)(P(x))$. To prove Theorem 3, it suffices to show that $(\Omega_\alpha \circ \Delta)(P(x))$ equals to the left side of Eq. (42). For this, one notes the equality $P_k(x+1) = P_{k-1}(x)$ and thus

$$(\Omega_\alpha \circ \Delta)(P(x)) = \Omega_\alpha\left(\sum_{k=1}^{s}(P_{k-1}(x) - P_k(x))\right) = \Omega_\alpha(P_0(x)) - \Omega_\alpha(P_s(x)). \tag{43}$$

One has

$$\begin{aligned} P_0(x) &= \frac{D^r}{r!} x^{\ell+r}(x+\lambda)^{n+r} \\ &= \frac{D^r}{r!} \sum_{k=0}^{n+r} \lambda^{n+r-k} \binom{n+r}{k} x^{\ell+k+r} \\ &= \sum_{k=0}^{n+r} \lambda^{n+r-k} \binom{n+r}{k} \binom{\ell+k+r}{r} x^{\ell+k} \end{aligned}$$

and

$$\begin{aligned} P_s(x) &= \frac{D^r}{r!}((x+\lambda-s)-\lambda)^{\ell+r}(x+\lambda-s)^{n+r} \\ &= \frac{D^r}{r!} \sum_{k=0}^{n+r} (-\lambda)^{\ell+r-k} \binom{\ell+r}{k} (x+\lambda-s)^{n+k+r} \\ &= (-1)^{\ell+n+r} \sum_{k=0}^{n+r} \lambda^{\ell+r-k} \binom{\ell+r}{k} \binom{n+k+r}{r} (-1)^{n+k}(x+\lambda-s)^{n+k}. \end{aligned}$$

Equations (13) and (15) yield

$$\Omega_\alpha(P_0(x)) = \sum_{k=0}^{n+r} \lambda^{n+r-k} n + rk \binom{\ell+k+r}{r} B_{\ell+k}^{(\alpha)}(x) \tag{44}$$

and

$$\begin{aligned} \Omega_\alpha(P_s(x)) &= (-1)^{\ell+n+r} \sum_{k=0}^{n+r} \lambda^{\ell+r-k} \binom{\ell+r}{k} \binom{n+k+r}{r} (-1)^{n+k} B_{n+k}^{(\alpha)}(x+\lambda-s) \\ &= (-1)^{\ell+n+r} \sum_{k=0}^{\ell+r} \lambda^{\ell+r-k} \binom{\ell+r}{k} \binom{n+k+r}{r} B_{n+k}^{(\alpha)}(\alpha+s-\lambda-x). \end{aligned} \tag{45}$$

By using Equations (43), (44), and (45), it follows that $(\Omega_\alpha \circ \Delta)(P(x))$ is indeed equal to the left side of Eq. (42). This completes our proof. $\square$

5 Applications

As already seen in the previous section, Theorem 3 is an extension of Gessel's formula, since it generalizes several identities involving Bernoulli numbers. In this section, we illustrate how Theorem 3 allows us to obtain more identities involving Bernoulli polynomials. For this purpose, noticing that by using Eq. (13), one has

$$B_{n+k}^{(\alpha)}(x) = (-1)^{n+k} B_{n+k}^{(\alpha)}(\alpha - x)$$

and thus $S_{n,l,r}^{(\alpha)}(x,y,z)$ given by Eq. (3) becomes as follows:

$$\begin{aligned} S_{n,l,r}^{(\alpha)}(x,y,z) = \sum_{k=0}^{n+r} \binom{n+r}{k} \binom{\ell+k+r}{r} x^{n+r-k} B_{\ell+k}^{(\alpha)}(y) \\ (-1)^{\ell+r+1} \sum_{k=0}^{\ell+r} (-1)^k \binom{\ell+r}{k} \binom{n+k+r}{r} x^{\ell+r-k} B_{n+k}^{(\alpha)}(\alpha - z). \end{aligned}$$

Also note that Eq. (42) can be formulated as follows:

$$S_{n,l,r}^{(\alpha)}(\lambda, x, \alpha + s - \lambda - x) = \Omega_{\alpha-1} \left(\frac{D^{r+1}}{r!} \sum_{k=1}^{s} (x-k)^{\ell+r} (x+\lambda-k)^{n+r} \right).$$

The left side of Eq. (42) can be written as

$$\begin{aligned} &\sum_{k=0}^{n+r} \lambda^{n+r-k} \binom{n+r}{k} \binom{\ell+k+r}{r} B_{\ell+k}^{(\alpha)}(x) \\ &- \sum_{k=0}^{\ell+r} (-1)^{\ell+r+k} \lambda^{\ell+r-k} \binom{\ell+r}{k} \binom{n+k+r}{r} B_{n+k}^{(\alpha)}(x+\lambda-s). \end{aligned}$$

By applying Leibniz's formula for $\alpha = 1$, the left side of Eq. (42) can be written as follows:

$$(r+1) \sum_{k=1}^{s} \sum_{j=0}^{r+1} \binom{n+r}{j} \binom{\ell+r}{r+1-j} (x-k)^{l+j-1} (x+\lambda-k)^{n+r-j}$$

and equals zero if $s = 0$.

5.1 Generalization of Gessel's formula to classical Bernoulli polynomials

For $\alpha = 1$, we have $B_n^{(1)}(x) = B_n(x)$ and $\Omega_{\alpha-1} = \Omega_0 = I$. Recalling Theorem 3, for $\alpha = 1$, $\lambda = m$, and $s = m - 1$ where m is a non-negative integer, leads us to the following corollary, which is a generalization of Gessel's theorem set in [8].

Corollary 4. *For every non-negative integers ℓ, n, r, and m, we have*

$$\begin{aligned}
&\sum_{k=0}^{n+r} m^{n+r-k}\binom{n+r}{k}\binom{\ell+k+r}{r}B_{\ell+k}(x)\\
&+(-1)^{\ell+n+r+1}\sum_{k=0}^{\ell+r} m^{\ell+r-k}\binom{\ell+r}{k}\binom{n+k+r}{r}B_{n+k}(-x)\\
&=(r+1)\sum_{k=1}^{m-1}\sum_{j=0}^{r+1}\binom{n+r}{j}\binom{\ell+r}{r+1-j}(x-k)^{\ell+j-1}(x+m-k)^{n+r-j}.
\end{aligned}\tag{46}$$

Replacing x by 0 in Eq. (46), we actually find Formula (6) thanks to Gessel. One notes that He [27] found, in 2014, the Gessel formula by using q-numbers and Bernoulli polynomials.

5.2 Nielsen's identity (1923)

First, let us consider the formula proposed by Nielsen [37] and published in 1923, where Bernoulli numbers and polynomials are differently defined but they match our notation $(-1)^{n-1}B_{2n}$ and $\frac{B_n(x+1)}{n!}$ respectively. Then, the equation in [37, Eq. (10), p. 182] can, with an adjusted notation, be stated as follows:

$$\begin{aligned}
&\sum_{k=0}^{n+r}(1-2\beta)^{n+r-k}\binom{n+r}{k}\binom{\ell+k+r}{r}B_{\ell+k}(x+\beta)\\
&-\sum_{k=0}^{\ell+r}(-1)^{\ell+r+k}(1-2\beta)^{\ell+r-k}\binom{\ell+r}{k}\binom{n+k+r}{r}B_{n+k}(x-\beta)\\
&=\frac{D^{r+1}}{r!}\Big((x+\beta-1)^{\ell+r}(x-\beta)^{n+r}\Big).
\end{aligned}\tag{47}$$

By using Eq. (13), Formula (47) can be written as

$$S^{(1)}_{n,\ell,r}(1-2\beta,x+\beta,1-x+\beta)=\frac{D^{r+1}}{r!}\Big((x+\beta-1)^{\ell+r}(x-\beta)^{n+r}\Big),$$

which is true according to Theorem 3.

5.3 Agoh's identities (2000)

In 2000, Agoh [2] investigated linear recurrences for Bernoulli numbers and Bernoulli polynomials. He derived many identities including the two following equations, [2, Eq. (3.2)(i),

p. 205] and [2, Eq. (3.4)(i), p. 207]:

$$\begin{aligned}
&\sum_{k=0}^{n} m^{n-k}\binom{n}{k}B_{\ell+k}(x)-\sum_{k=0}^{\ell}(-m)^{\ell-k}\binom{\ell}{k}B_{n+k}(x)\\
&=n\sum_{k=0}^{m-1}(x+k)^{n-1}(x-m+k)^{\ell}+\ell\sum_{k=0}^{m-1}(x+k)^{n}(x-m+k)^{\ell-1}
\end{aligned}\tag{48}$$

and

$$\begin{aligned}
&\sum_{k=0}^{n+r}\binom{n+r}{k}\binom{\ell+k+r}{r}B_{\ell+k}(x)\\
&+(-1)^{\ell+r+1}\sum_{k=0}^{\ell+r}(-1)^{k}\binom{\ell+r}{k}\binom{n+k+r}{r}B_{n+k}(x)\\
&=(n+r)\sum_{k=0}^{r}\binom{n+r-1}{k}\binom{\ell+r}{r-k}x^{n+r-k-1}(x-1)^{\ell+k}\\
&+(\ell+r)\sum_{k=0}^{r}\binom{\ell+r-1}{k}\binom{n+r}{r-k}x^{n+k}(x-1)^{\ell+r-k-1}.
\end{aligned}\tag{49}$$

Theorem 3 allows us to directly obtain these two identities. Indeed, according to this theorem, for all non-negative integers m, we have

$$S^{(1)}_{n,\ell,0}(m,x,-x)=D\Bigg(\sum_{k=1}^{m}(x-k)^{\ell}(x+m-k)^{n}\Bigg).\tag{50}$$

Changing k to $m-k$ in Eq. (50), one has

$$\begin{aligned}
D\Bigg(\sum_{k=1}^{m}(x-k)^{\ell}(x+m-k)^{n}\Bigg)&=D\Bigg(\sum_{k=0}^{m-1}(x+k)^{n}(x-m+k)^{\ell}\Bigg)\\
&=n\sum_{k=0}^{m-1}(x+k)^{n-1}(x-m+k)^{\ell}\\
&+\ell\sum_{k=0}^{m-1}(x+k)^{n}(x-m+k)^{\ell-1},
\end{aligned}$$

and this is just Eq. (48). For $\beta = 0$, and by noticing that the right side of Eq. (47) can be rewritten as

$$\begin{aligned}
&\frac{D^r}{r!}\Big((n+r)x^{n+r-1}(x-1)^{\ell+r} + (\ell+r)x^{n+r}(x-1)^{\ell+r-1}\Big)\\
&= (n+r)\sum_{k=0}^{r}\binom{n+r-1}{k}\binom{\ell+r}{r-k}x^{n+r-k-1}(x-1)^{\ell+k}\\
&+ (\ell+r)\sum_{k=0}^{r}\binom{\ell+r-1}{k}\binom{n+r}{r-k}x^{n+k}(x-1)^{\ell+r-k-1},
\end{aligned}$$

one obtains, with an adjusted notation, Eq. (49). It should be noted that for $r \in \{0,1\}$, Eq. (49) becomes the following formulae [3, Cor. 3.4, Eq. (3.9), Eq. (3.10), p. 163] which is what Agoh found in 2017, through a different method:

$$\sum_{k=0}^{n}\binom{n}{k}B_{\ell+k}(x) - \sum_{k=0}^{\ell}(-1)^{\ell-k}\binom{\ell}{k}B_{n+k}(x) = \big((n+\ell)x - n\big)x^{n-1}\big(x-1\big)^{\ell-1}$$

and

$$\begin{aligned}
&\sum_{k=0}^{n+1}\binom{n+1}{k}(\ell+k+1)B_{\ell+k}(x) - \sum_{k=0}^{\ell+1}(-1)^{\ell+1-k}\binom{\ell+1}{k}(n+k+1)B_{n+k}(x)\\
&= \big((n+\ell+2)(n+\ell+1)x^2 - 2(n+1)(n+\ell+1)x + (n+1)n\big)x^{n-1}\big(x-1\big)^{\ell-1}.
\end{aligned}$$

5.4 Chang and Ha's identity (2001)

In 2001, Chang and Ha [15] obtained a class of recurrence relations for the Bernoulli numbers that includes the Kaneko formula (33). Among these identities, we can find the following relations due to Chang and Ha [15, Cor. 1.b, p. 472]:

$$\sum_{k=n}^{2n}\binom{n+1}{k-n}(k+1)\frac{B_k}{2^k} = (-1)^n\frac{n+1}{2^{2n+1}}, n \geq 1. \tag{51}$$

Below, we show that Theorem 3 allows us to retrieve Eq. (51). From Theorem 3

$$S^{(1)}_{n,n,r}\big(2-2\beta, \beta+x, \beta-x\big) = \big(r+1\big)\frac{D^{r+1}}{(r+1)!}\big(x^2 - (\beta-1)^2\big)^{n+r}.$$

For $x = 0$ and r odd, one deduces that

$$\begin{aligned}
&2\sum_{k=0}^{n+r}(2-2\beta)^{n+r-k}\binom{n+r}{k}\binom{n+k+r}{r}B_{n+k}(\beta)\\
&\qquad\qquad = (-1)^{n+\frac{r-1}{2}}(r+1)\binom{n+r}{\frac{r+1}{2}}(\beta-1)^{2n+r-1}.
\end{aligned}$$

Then for $\beta = 0$, we have

$$\sum_{k=0}^{n+r} \binom{n+r}{k} \binom{n+k+r}{r} \frac{B_{n+k}}{2^{n+k}} = \frac{(-1)^{n+\frac{r-1}{2}}(r+1)}{2^{2n+r+1}} \binom{n+r}{\frac{r+1}{2}}. \tag{52}$$

For $r = 1$, changing k to $k - n$ in Eq. (52), we obtain Eq. (51).

5.5 Sun's identities (2003)

In 2003, Sun [47] derived a general combinatorial identity in terms of polynomials with dual sequences of coefficients as he deduced combinatorial identities involving Bernoulli polynomials, including following relations: [47, Thm. 1.2, Eq. (1.15), Eq. (1.16), p. 712] and [47, Remark. 1.2, Eq. (1.18) p. 713], where $z = 1 - x - y$.

$$(-1)^n \sum_{k=0}^{n} \binom{n}{k} x^{n-k} B_{\ell+k}(y) = (-1)^\ell \sum_{k=0}^{\ell} \binom{\ell}{k} x^{\ell-k} B_{n+k}(z) = 0, \tag{53}$$

$$\begin{aligned}
&(-1)^n \sum_{k=0}^{n} \binom{n+1}{k} (\ell+k+1) x^{n+1-k} B_{\ell+k}(y) \\
&+ (-1)^\ell \sum_{k=0}^{\ell} \binom{\ell+1}{k} (n+k+1) x^{\ell+1-k} B_{n+k}(z) \\
&= (-1)^n (n+\ell+2)\big((B_{n+\ell+1}(x+y)) - B_{n+\ell+1}(y)\big),
\end{aligned} \tag{54}$$

$$\sum_{k=0}^{n} \binom{n+1}{k} (n+k+1)(1-2x)^{n-k+1} B_{n+k}(x) = -2(n+1) B_{2n+1}(x). \tag{55}$$

Theorem 3 allows us to deduce the three identities above. Indeed, the theorem states that $S^{(1)}_{n,\ell,r}(x,y,z) = 0$ for $x + y + z = 1$. And for $r = 0$, it gives Eq. (53). Chen and Sun [17, Thm. 5.1, p. 2121] proved this equation. If $r = 1$, it becomes a relation from which Eq. (54) can be derived. One notices that

$$B_{n+k}(z) = B_{n+k}(1-x-y) = (-1)^{n+k} B_{n+k}(x+y).$$

Finally, by Theorem 3, we also have $\frac{1}{2} S^{(1)}_{n,n,1}(1-2x, x, x) = 0$, i.e., Eq. (55). Note that, in 2016, Pita [39] also proved Eq. (53).

5.6 Wu, Sun, and Pan's identities (2004)

In 2004, by studying some formal power series, Wu, Sun, and Pan [50] obtained the following identities [50, Thm. 2, Eq. (6), Eq. (8), p. 3]:

$$(-1)^n \sum_{k=0}^{n} \binom{n}{k} B_{\ell+k}(x) = (-1)^\ell \sum_{k=0}^{\ell} \binom{\ell}{k} B_{n+k}(-x), \tag{56}$$

and

$$(-1)^n \sum_{k=0}^{n} \binom{n+1}{k} (\ell+k+1) B_{\ell+k}(x) + (-1)^\ell \sum_{k=0}^{\ell} \binom{\ell+1}{k} (n+k+1) B_{n+k}(-x)$$
$$= (-1)^n (n+\ell+2)(n+\ell+1) x^{n+\ell}. \tag{57}$$

These identities can be obtained promptly by using Theorem 3, especially if we have $S^{(1)}_{n,\ell,r}(1,x,-x) = 0$. Replacing r by 0 in Theorem 3, we obtain Eq. (56); however, if $r=1$, Equations (13), (20), and Theorem 3 lead to Eq. (57).

5.7 Chen's identity (2007)

A sequence $(b_n)_{n\geq 0}$ is called a "binomial transform" of another sequence $(a_n)_{n\geq 0}$ if and only if $b_n = \sum_{k=0}^{n} \binom{n}{k} a_k$.

In 2007, Chen [16] proposed a general identity for such pairs of sequences from which he deduced several identities for Bernoulli numbers and polynomials. One of these identities is the relation [16, Thm. 5.3, Eq. (5.7), p. 149]:

$$\sum_{k=0}^{n+r} \binom{n+r}{k} \binom{\ell+k+r}{r} y^{n+r-k} B_{\ell+k}(x)$$
$$= \sum_{k=0}^{\ell+r} \binom{\ell+r}{k} \binom{n+k+r}{r} (-y)^{\ell+r-k} B_{n+k}(x+y), \tag{58}$$

which can be written as $S^{(1)}_{n,\ell,r}(y,x,1-x-y) = 0$. This can be obtained directly from Theorem 3. Note that, in 2010, He and Zhang [26] also proved Eq. (58).

5.8 Neto's identity (2015)

In 2015, Neto [36] provided a short proof of Eq. (30) using Zeon algebra, and then proved the analogous identity for the Bernoulli numbers of higher order:

$$\sum_{k=0}^{n} x^{n-k} \binom{n}{k} B^{(x)}_{\ell+k} = (-1)^{n+\ell} \sum_{k=0}^{\ell} x^{\ell-k} \binom{\ell}{k} B^{(x)}_{n+k}. \tag{59}$$

This identity can be written as $S^{(x)}_{n,\ell,0}(x,0,0) = 0$, which can be obtained immediately from Theorem 3. In fact, it is interesting to express Relation (59) by using Stirling polynomials. Indeed, Stirling polynomials $\sigma_n(x)$ are defined in [25] by

$$\left(\frac{te^t}{e^t-1}\right)^x = x \sum_{n=0}^{\infty} \sigma_n(x) t^n.$$

One has

$$n!x\sigma_n(x) = (-1)^n B_n^{(x)}. \tag{60}$$

$B_n^{(x)}$ are the generalized Bernoulli numbers, which are also called Nörlund polynomials [9].

More generally, from Theorem 3, we have

$$S_{n,\ell,r}^{(x)}(x,0,0) = 0, \tag{61}$$

for all non-negative integers r. By Equations (60) and (61), one has

$$\sum_{k=0}^{n+r}(-1)^k\binom{n+r}{k}(\ell+k+r)!x^{n+1+r-k}\sigma_{\ell+k}(x)$$
$$+(-1)^{r+1}\sum_{k=0}^{\ell+r}(-1)^k\binom{\ell+r}{k}(n+k+r)!x^{\ell+1+r-k}\sigma_{n+k}(x) = 0.$$

For r odd and $n=\ell$, one obtains

$$\sum_{k=0}^{n+r}(-1)^k\binom{n+r}{k}(n+k+r)!x^{n+r-k}\sigma_{n+k}(x) = 0.$$

6 Identities for other polynomials

In the previous paragraphs, we have shown that Theorem 3 generalizes several well-known identities involving Bernoulli numbers and Bernoulli polynomials proven by numerous authors through various methods. The following theorem is useful because similarly to Theorem 3, it allows us to establish analogous identities for particular sequences such as Fibonacci and Lucas polynomial sequences.

Theorem 5. *Let $(w_n)_{n\geq 0}$ be a sequence such that exponential generating function $S_w(t) = \sum_{n=0}^{\infty} w_n \frac{t^n}{n!}$ satisfies $S_w(t) = \varepsilon e^{\lambda t} S_w(-t)$ where $\varepsilon = \pm 1$ and $\lambda \in \mathbb{C}$. Then for all non-negative integers n, ℓ, and r, we have*

$$\sum_{k=0}^{n+r}(-1)^k\binom{n+r}{k}\binom{\ell+k+r}{r}\lambda^{n+r-k}w_{\ell+k}$$
$$+(-1)^{r+1}\varepsilon\sum_{k=0}^{\ell+r}(-1)^k\binom{\ell+r}{k}\binom{n+k+r}{r}\lambda^{\ell+r-k}w_{n+k} = 0. \tag{62}$$

Proof. Let $\mathcal{L}_w$ be a linear transformation defined over polynomial space in x specifying its action on monomials as follows:

$$\mathcal{L}_w\left(x^k\right) = w_k.$$

The relation $S_w(t) = \varepsilon S_w(-t)$ is equivalent to

$$w_n = \varepsilon \sum (-1)^k \binom{n}{k} \lambda^{n-k} w_k, n \geq 0, \tag{63}$$

which can be written as

$$x^n - \varepsilon(\lambda - x)^n \in \ker \mathcal{L}_w,\ \ n \geq 0.$$

By linearity, any polynomial $P(x)$, by the previous conditions satisfies

$$P(x) - \varepsilon P(\lambda - x) \in \ker \mathcal{L}_w.$$

Let us consider $P(x) = x^{\ell+r}(\lambda - x)^{n+r}$ and note that $P(\lambda - x) = x^{n+r}(\lambda - x)^{\ell+r}$. Then one obtains

$$\frac{D^r}{r!}\big(P(x)\big) = \sum_{k=0}^{n+r} (-1)^k \binom{n+r}{k} \binom{\ell+k+r}{r} \lambda^{n+r-k} x^{\ell+k}$$

and

$$\frac{D^r}{r!}\big(P(\lambda - x)\big) = \sum_{k=0}^{\ell+r} (-1)^k \binom{\ell+r}{k} \binom{n+k+r}{r} \lambda^{\ell+r-k} x^{n+k}.$$

Thus, by combining the previous formulae, we find Relation (62) of Theorem 5. □

6.1 Identities for Fibonacci and Lucas polynomials

The bivariate polynomials of Fibonacci and Lucas $\big(u_n(x,y)\big)_{n\geq 0}$ and $\big(v_n(x,y)\big)_{n\geq 0}$ are defined in [6] by

$$u_n(x,y) = x u_{n-1}(x,y) + y u_{n-2}(x,y) \text{ and } v_n(x,y) = x v_{n-1}(x,y) + y v_{n-2}(x,y)$$

for $n \geq 2$ with $u_0(x,y) = 0$, $u_1(x,y) = 1$, $v_0(x,y) = 2$, $v_1(x,y) = x$. It is well-known that we have

$$u_n(x,y) = \frac{\big(\alpha(x,y)\big)^n - \big(\beta(x,y)\big)^n}{\alpha(x,y) - \beta(x,y)},\ \ v_n(x,y) = \big(\alpha(x,y)\big)^n + \big(\beta(x,y)\big)^n$$

where

$$\alpha(x,y) = \frac{1}{2}\left(x + \sqrt{x^2+4y}\right) \text{ and } \beta(x,y) = \frac{1}{2}\left(x - \sqrt{x^2+4y}\right).$$

Let us consider the sequences $u^* = \big(u_{sn}(x,y)\big)_n$ and $v^* = \big(v_{sn}(x,y)\big)_n$ where s is a non-negative integer. This implies that

$$S_{u^*}(t) = -e^{v_s(x,y)t} S_{u^*}(-t) \text{ and } S_{v^*}(t) = e^{v_s(x,y)t} S_{v^*}(-t).$$

Applying Theorem 5 leads to the following corollary:

Corollary 6. *Let n, ℓ, r, and s be non-negative integers. Then*

$$\sum_{k=0}^{n+r}(-1)^k\binom{n+r}{k}\binom{\ell+k+r}{r}\big(v_s(x,y)\big)^{n+r-k}u_s(l+k)(x,y)$$
$$+(-1)^r\sum_{k=0}^{\ell+r}(-1)^k\binom{\ell+r}{k}\binom{n+k+r}{r}\big(v_s(x,y)\big)^{\ell+r-k}u_{s(n+k)}(x,y)=0, \tag{64}$$
$$\sum_{k=0}^{n+r}(-1)^k\binom{n+r}{k}\binom{\ell+k+r}{r}\big(v_s(x,y)\big)^{n+r-k}v_s(l+k)(x,y)$$
$$-(-1)^r\sum_{k=0}^{\ell+r}(-1)^k\binom{\ell+r}{k}\binom{n+k+r}{r}\big(v_s(x,y)\big)^{\ell+r-k}v_{s(n+k)}(x,y)=0. \tag{65}$$

Corollary 6 generalizes identities involving Fibonacci and Lucas numbers [30] mentioned in [10] and [18]. As it provides identities for several sequences of integers, such as the Fibonacci sequence $(F_n)_{n\geq 0}=(u_n(1,1))_{n\geq 0}$, the Lucas sequence $(L_n)_{n\geq 0}=(v_n(1,1))_{n\geq 0}$, the sequence of Pell numbers $(P_n)_{n\geq 0}=(u_n(2,1))_{n\geq 0}$, the sequence of Pell-Lucas numbers $(Q_n)_{n\geq 0}=(v_n(2,1))_{n\geq 0}$, the sequence of Jacobsthal numbers $(J_n)_{n\geq 0}=(v_n(1,2))_{n\geq 0}$, and the sequence Jacobsthal-Lucas numbers $(j_n)_{n\geq 0}=(v_n(1,2))_{n\geq 0}$ appear respectively in the OEIS (*On-Line Encyclopedia of Integer Sequences*) [45] as A000045, A000032, A000129, A002203, A001045 and A014551 and also the sequences $(F_{2n})_{n\geq 0}$, $(L_{2n})_{n\geq 0}$, $(P_{2n})_{n\geq 0}$, $(Q_{2n})_{n\geq 0}$, $(J_{2n})_{n\geq 0}$, and $(j_{2n})_{n\geq 0}$ respectively as A001906, A005248, A001542, A003499, A002450, and A052539 in the OEIS.

6.2 Identities for Chebyshev polynomials

We can also apply Theorem 5 to Chebyshev polynomials of the first kind $T(x)=(T_n(x))_{n\geq 0}$ and to Chebyshev polynomials of the second kind $U(x)=(U_n(x))_{n\geq 0}$ defined recursively in [31, 7] by

$$T_n(x)=2xT_{n-1}(x)-T_{n-2}(x) \text{ and } U_n(x)=2xU_{n-1}(x)-U_{n-2}(x)$$

for $n\geq 2$ with $T_0(x)=1$, $T_1(x)=x$, $U_0(x)=1$, and $U_1(x)=2x$. Clearly

$$T_n(x)=\frac{1}{2}v_n(2x,-1) \text{ and } U_n(x)=u_{n+1}(2x,-1).$$

By applying Corollary 6, we deduce the following relations:

$$\sum_{k=0}^{n+r}(-1)^k\binom{n+r}{k}\binom{\ell+k+r}{r}(2T_s(x))^{n+r-k}T_s(\ell+k)(x)$$
$$-(-1)^r\sum_{k=0}^{\ell+r}(-1)^k\binom{\ell+r}{k}\binom{n+k+r}{r}(2T_s(x))^{\ell+r-k}T_s(n+k)(x)$$
$$=0. \tag{66}$$

$$
\begin{aligned}
&\sum_{k=0}^{n+r}(-1)^k\binom{n+r}{k}\binom{\ell+k+r}{r}\big(2T_s(x)\big)^{n+r-k}U_{s(\ell+k)-1}(x)\\
&+(-1)^r\sum_{k=0}^{\ell+r}(-1)^k\binom{\ell+r}{k}\binom{n+k+r}{r}\big(2T_s(x)\big)^{\ell+r-k}U_{s(n+k)-1}(x)\\
&=0,
\end{aligned}
\tag{67}
$$

where $n,\ell,$ and s are positive integers.

6.3 Identities for generalized Euler polynomials

Generalized Euler polynomials $E_n^{(\alpha)}(x)$ [41, p. 102] are defined, for $\alpha\in\mathbb{C}$, by

$$S_{E^{(\alpha)}}(t)=\sum_{n=0}^{\infty}E_n^{(\alpha)}(x)\frac{t^n}{n!}=\left(\frac{2}{e^t+1}\right)^{\alpha}e^{tx}.$$

One has

$$S_{E^{(\alpha)}}(t)=e^{(2x-\alpha)t}S_{E^{(\alpha)}}(-t).$$

By applying Theorem 5, one obtains the following identity:

$$
\begin{aligned}
&\sum_{k=0}^{n+r}(-1)^k\binom{n+r}{k}\binom{\ell+k+r}{r}(2x-\alpha)^{n+r-k}E_{\ell+k}^{(\alpha)}(x)\\
&+(-1)^{r+1}\sum_{k=0}^{\ell+r}(-1)^k\binom{\ell+r}{k}\binom{n+k+r}{r}(2x-\alpha)^{\ell+r-k}E_{n+k}^{(\alpha)}(x)\\
&=0.
\end{aligned}
\tag{68}
$$

6.4 Identity for generalized Genocchi polynomials

Generalized Genocchi polynomials $G_n^{(m)}(x)$ are defined, for $m\in\mathbb{N}$, by

$$\sum_{n=0}^{\infty}G_n^{(m)}(x)\frac{t^n}{n!}=\left(\frac{2t}{e^t+1}\right)^{m}e^{tx}.$$

One has

$$S_{G^{(m)}}(t)=(-1)^m e^{(2x-m)t}S_{E^{(\alpha)}}(-t).$$

Theorem 5 leads to

$$\begin{aligned}
&\sum_{k=0}^{n+r}(-1)^k\binom{n+r}{k}\binom{\ell+k+r}{r}(2x-m)^{n+r-k}G^{(m)}_{\ell+l}(x)\\
&+(-1)^{m+r+1}\sum_{k=0}^{\ell+r}(-1)^k\binom{\ell+r}{k}\binom{n+k+r}{r}(2x-m)^{\ell+r-k}G^{(m)}_{n+k}(x)\\
&=0. \qquad (69)
\end{aligned}$$

6.5 Identities for generalized Stirling numbers of the second kind

The Stirling numbers of the second kind appear in the OEIS as A008277, are defined by the explicit formula [25]

$$S(n,m)=\frac{(-1)^m}{m!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}j^n.$$

The first generalization of these numbers was provided by d'Ocagne [38] and Carlitz [14]:

$$S^{(\alpha)}(n,m)=\frac{(-1)^m}{m!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}(\alpha+j)^n.$$

These numbers are connected with generalized Bernoulli polynomials [11]:

$$S^{(\alpha)}(n+m,m)=\binom{n+m}{m}B_n^{(-m)}(\alpha).$$

We are interested in the generalized Stirling numbers [12], which are defined by

$$S^{(\alpha)}(n,m,s)=\frac{(-1)^m}{m!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}(\alpha+sj)^n.$$

Note that

$$S^{(\alpha)}(n,m,1)=S^{(\alpha)}(n,m)\ \text{ and }\ S^{(0)}(n,m,1)=S(n,m).$$

For m and s fixed, let us consider the sequence $(w_{m,s})_n=(S^{(\alpha)}(n,m,s))_n$, for which the generating function is

$$\sum_{n=0}^{\infty}S^{(\alpha)}(n,m,s)\frac{t^n}{n!}=\frac{1}{m!}e^{\alpha t}(e^{st}-1)^m.$$

One has

$$S_{w_{m,s}}(t)=(-1)^m e^{(2x+ms)t}S_{w_{m,s}}(-t).$$

Applying Theorem 5, we obtain the following identity:

$$\begin{aligned}
&\sum_{k=0}^{n+r}(-1)^k\binom{n+r}{k}\binom{\ell+k+r}{r}(2x+ms)^{n+r-k}S^{(\alpha)}(\ell+k,m,s)\\
&+(-1)^{m+r+1}\sum_{k=0}^{\ell+r}(-1)^k\binom{\ell+r}{k}\binom{n+k+r}{r}(2x+ms)^{\ell+r-k}S^{(\alpha)}(n+k,m,s)\\
&=0. \qquad (70)
\end{aligned}$$

7 Conclusion

The study permits the development of two important results. The first one proposes an identity allowing the generalization of various relations related to classical Bernoulli numbers and Bernoulli polynomials, in addition to Gessel's recent identity. The second result allows the deduction of similar identities for Fibonacci, Lucas, and Chebyshev polynomials as well as for the generalized Euler, Genocchi polynomials, and generalized Stirling numbers.

8 Acknowledgments

The authors would like to thank the referee and the editor for their valuable comments and suggestions.

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2010 *Mathematics Subject Classification*: Primary 11B68; Secondary 05A10, 11B65.
Keywords: Bernoulli polynomial, Bernoulli number, identity.

(Concerned with sequences A000032, A000045, A000129, A001045, A001542, A001906, A002203, A002450, A003499, A005248, A008277, A014551, and A052539.)

Received June 30 2020; revised versions received July 19 2020; October 14 2020; November 16 2020; November 18 2020. Published in *Journal of Integer Sequences*, November 24 2020.

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