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New Series Identities with Cauchy, Stirling, and Harmonic Numbers, and Laguerre Polynomials

Khristo N. Boyadzhiev
Department of Mathematics
Ohio Northern University
Ada, Ohio 45810
USA
k-boyadzhiev@onu.edu

Abstract

We use an interplay between Newton series and binomial formulas to generate a number of series identities involving Cauchy numbers (also known as Gregory coefficients or Bernoulli numbers of the second kind), harmonic numbers, Laguerre polynomials, and Stirling numbers of the first kind.

1 Introduction

The Newton interpolation series is a classic tool in analysis with important applications [12, 17, 18, 19]. In this paper, we use the Newton series in order to construct two summation formulas: the first involving Cauchy numbers, and the second involving Stirling numbers of the first kind. We obtain a number of series identities—closed form evaluations of several series involving Cauchy numbers together with harmonic numbers. We also present similar series with harmonic numbers and Stirling numbers of the first kind. The results are close to some classical results about such series.

We often refer to sequence numbers in the *On-Line Encyclopedia of Integer Sequences* (OEIS) [22]. The Cauchy numbers c_n (OEIS sequence A006232 and A006233) are defined by the generating function

$$\frac{x}{\ln(x+1)} = \sum_{n=0}^{\infty} \frac{c_n}{n!}\, x^n \quad (|x|\ < 1)$$

and they can also be defined as an integral of the falling factorial (see Eq. (2) below). These numbers are called Cauchy numbers of the first kind by Comtet [11, p. 294]. The numbers $c_n/n!$ are called Bernoulli numbers of the second kind by some authors (see the comments in [3]). In Milne-Thomson's book the Cauchy numbers appear as $B_n^{(n)}(1)$ [17, p. 135]. Blagouchine in his recent papers [2, 3, 4] used the notation $G_n = c_n/n!$, and the name Gregory coefficients for G_n.

A detailed study of the Cauchy numbers c_n and several interesting identities involving these numbers were presented by Merlini et al. [16]. Cauchy numbers were also studied by Liu et al. [15] and Zhao [26].

The Cauchy numbers appear in some important formulas (such as the Laplace summation formula in [17, p. 181]), where they act like a counterpart to the Bernoulli numbers in the Euler-Maclaurin formula.

The Stirling numbers of the first kind $s(n,k)$ (OEIS A048994 and A008275) play a major role in this paper. They appeared for the first time in the works of James Stirling together with the Stirling numbers of the second kind $S(n,k)$ [6, 23]. They are related to the Cauchy numbers by the formula [11, p. 294]

$$c_n = \sum_{k=0}^{n} \frac{s(n,k)}{k+1}.$$

A detailed study of the Stirling numbers can be found in the books of Jordan [14], Comtet [11], and Graham et al. [13] (where the unsigned Stirling numbers of the first kind

$$(-1)^{n-k} s(n,k)$$

are considered). Informative comments and interesting series containing $s(n,k)$ can be found in Adamchik [1] and Blagouchine [2, Sec. 2]. Various results for series with Stirling numbers and harmonic numbers H_n were recently obtained by Choi [10] and Wang and Lyu [24].

To give the reader an idea of our identities, here are three samples:

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n c_n x^{n+1}}{(1+x)(1+2x)\cdots(1+nx)}$$

$$2\zeta(3) - 1 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} c_n (H_n^2 + H_n^{(2)})}{n!n}$$

$$\sum_{j=1}^{k} \zeta(j+1) = \sum_{n=k}^{\infty} \frac{(-1)^{n-k} s(n,k) H_n}{(n+1)!}.$$

Starting from the Newton interpolation series with simple nodes $n = 0, 1, 2, \ldots$, we derive two special representations (Proposition 1 and Proposition 8). Using these two propositions, with the help of several binomial identities we generate a number of expansions involving Cauchy, harmonic, and Stirling numbers. Our results are presented in the form of examples. In Section 3 we work mostly with Cauchy numbers and in Section 4 we present series with Stirling numbers of the first kind and harmonic numbers. Section 5 contains a formula for series with Stirling numbers and Laguerre polynomials. Particular cases of this formula provide series with Laguerre polynomials and harmonic numbers.

2 The Newton interpolation series

We use finite differences. Given an appropriate function $f(x)$, by definition we have $\Delta f(x) = f(x+1) - f(x)$, $\Delta^n f(x) = \Delta(\Delta^{n-1} f(x))$ for $n = 1, 2, \ldots$ and $\Delta^0 f(x) = f(x)$.

The Newton interpolation series for a given function $f(x)$ is the representation of this function in terms of the polynomials

$$(z)_n = z(z-1)\cdots(z-n+1),\ (z)_0 = 1$$

[6, 12, 17, 18]. Namely,

$$f(z) = \sum_{n=0}^{\infty} \frac{\Delta^n f(0)}{n!}\, z(z-1)\cdots(z-n+1). \tag{1}$$

This is the case of simple nodes $0, 1, 2 \ldots$. The Newton series expansion is not as popular as the Taylor series. One reason for this is the difficulty of studying the remainder and determining the region of convergence. There are other difficulties, too. For instance, if we apply (1) to the function $f(x) = \sin(2\pi x)$, then $\Delta^n f(0) = 0$ for $n \geq 0$ and the series is identically zero.

Alexander Gelfond has done a thorough study of Newton interpolation series in [12, Ch. 2]. Another study can be found in Norlund [18, Ch. 5] (also [19] and [17, pp. 302–304]). The sufficient conditions for a function $f(z)$ to be represented by such a series are too elaborate to list here; however, we can say that if the function is analytic in a region of the form $\Re(z) > \lambda$ for some λ and has moderate growth in that region, then the representation (1) holds for $\Re(z) > \lambda$. Moreover, for any two numbers $\varepsilon, R > 0$ the series is uniformly convergent for $\Re(z) \geq \lambda + \varepsilon$ and $|z - \lambda| < R$. Boas and Buck showed [5, p. 34] that entire functions of exponential type less than $\ln 2$ have such representation.

The polynomials $(z)_n$ often appear in mathematics. For example, they are the ordinary generating functions for the Stirling numbers of the first kind $s(n,k)$ [11, p. 50]:

$$z(z-1)\cdots(z-n+1) = \sum_{k=0}^{n} s(n,k)\, z^k.$$

They also appear in the definition of the Cauchy numbers $c_n,\ n=0,1,\dots$ [11, p. 294]:

$$c_n = \int_0^1 z(z-1)\cdots(z-n+1)\,dz, \tag{2}$$

with $c_0 = 1,\ c_1 = \frac{1}{2},\ c_2 = -\frac{1}{6},\ c_3 = \frac{1}{4},\ \dots$ etc. For more details on these numbers see Merlini et al. [16] and also Liu et al. [15].

Integrating the series (1) term by term and using Eq. (2) we come to the most interesting formula,

$$\int_0^1 f(x)\,dx = \sum_{n=0}^{\infty} \frac{c_n}{n!}\,\Delta^n f(0), \tag{3}$$

which clearly demonstrates the importance of the Cauchy numbers. This representation appears in Jordan [14, Eq. (1), p. 277]. Note that in his book Jordan works with the numbers

$$b_n = \int_0^1 \binom{x}{n}\,dx = \frac{c_n}{n!}.$$

(A simpler version was used by Gregory in the 17$^{\text{th}}$ century.)

It is easy to compute that

$$\Delta^n f(x) = (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k f(x+k) \tag{4}$$

and in particular

$$\Delta^n f(0) = (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k f(k).$$

Replacing the finite differences by the binomial expression in (3) we come to the following proposition.

Proposition 1. *For appropriate functions $f(x)$ we have the representation*

$$\int_0^1 f(x)\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n c_n}{n!} \left(\sum_{k=0}^{n} \binom{n}{k} (-1)^k f(k) \right). \tag{5}$$

As we shall see, this formula is a valuable tool for obtaining identities with Cauchy numbers.

Definition 2. The *binomial transform* of one sequence $a_0,\ a_1,\dots$ is the new sequence defined by

$$b_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k$$

with inversion

$$a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k b_k.$$

The binomial transform of the sequence $\{f(k)\}$ is usually easier to compute, compared to the computation of $\Delta^n f(0)$, so that formula (5) often works better than (3). A table of binomial transform formulas can be found in [7]. Using such formulas we generate a number of examples in the next section.

Here is one simple demonstration of how (5) works. Taking the function $f(z) = z^p$, where $p \geq 0$ is an integer, we find by (5) that

$$\frac{1}{p+1} = \sum_{n=0}^{\infty} \frac{(-1)^n c_n}{n!} \left(\sum_{k=0}^{n} \binom{n}{k} (-1)^k k^p \right)$$

and we recall that the Stirling numbers of the second kind $S(p,n)$ can be represented as

$$S(p,n) = (-1)^n n! \sum_{k=0}^{n} \binom{n}{k} (-1)^k k^p.$$

Thus we get the identity

$$\frac{1}{p+1} = \sum_{n=0}^{p} c_n \, S(p,n).$$

(The infinite series terminates, as $S(p,n) = 0$ for $n > p$.) This is the Stirling inverse of the representation of the Cauchy numbers in terms of Stirling numbers of the first kind [11, p. 294], and also [14, p. 267])

$$c_n = \sum_{p=0}^{n} \frac{s(n,p)}{p+1}.$$

3 Series with Cauchy numbers and harmonic numbers

In this section we use Proposition 1, together with various binomial identities, to generate series identities involving Cauchy numbers and harmonic numbers.

Example 3. Let $y > 0$ be arbitrary. We use the function

$$f(z) = \frac{y}{y+z}, \quad \Re(z) + y > 0$$

together with the binomial identity [7, Entry (8.28)]

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{y}{y+k} = \binom{n+y}{n}^{-1}.$$

Equation (5) implies

$$y \ln\left(1 + \frac{1}{y}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n c_n}{n!} \binom{n+y}{n}^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n c_n}{(y+1)(y+2)\cdots(y+n)}$$

or

$$\ln\left(1+\frac{1}{y}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n c_n}{y(y+1)(y+2)\cdots(y+n)}$$

(cf. [25] and for a similar representation see [19, p. 244]). Replacing $1/y$ by x we find the following remarkable expansion of $\ln(1+x)$:

$$\ln(1+x)=\sum_{n=0}^{\infty}\frac{(-1)^n c_n x^{n+1}}{(1+x)(1+2x)\cdots(1+nx)}. \tag{6}$$

When $x=1$, this gives

$$\ln 2=\sum_{n=0}^{\infty}\frac{(-1)^n c_n}{(n+1)!}.$$

Using the asymptotic behavior of the Cauchy numbers at infinity [11, p. 294], namely

$$\frac{c_n}{n!}\sim\frac{(-1)^{n+1}}{n\ln^2 n}\quad(n\to\infty),$$

it is easy to verify (by the ratio test, for instance) that the series in (6) is convergent for all $x>0$.

Before continuing further we recall that the harmonic numbers H_n are defined for $n=0,1,\ldots$ by

$$H_n=1+\frac{1}{2}+\cdots+\frac{1}{n}\quad(n>0),\ H_0=0$$

and it is often convenient to express them in the form

$$H_n=\psi(n+1)+\gamma$$

where $\psi(z)=\frac{d}{dz}\ln\Gamma(z)$ is the digamma function and $\gamma=-\psi(1)$ is Euler's constant. This formula provides an extension of the harmonic numbers as a holomorphic function H_z on $\Re(z)>-1$. We also use the generalized harmonic numbers $H_n^{(p)}$, where $H_0^{(p)}=0$ and for $n>0$ we have

$$H_n^{(p)}=1+\frac{1}{2^p}+\cdots+\frac{1}{n^p}=\zeta(p)+\frac{(-1)^{p-1}}{(p-1)!}\psi^{(p-1)}(n+1)\quad(p>1).$$

Example 4. Let $m>0$ be an integer. We take the function $f(z)=\frac{1}{(z+m)^2}$ and use the binomial identity

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{1}{(k+m)^2}=\frac{(m-1)!n!(H_{n+m}-H_{m-1})}{(n+m)!}$$

[7, Entry 8.38]. Equation (5) implies

$$\frac{1}{(m+1)!}=\sum_{n=0}^{\infty}\frac{(-1)^n c_n(H_{n+m}-H_{m-1})}{(n+m)!}. \tag{7}$$

For $m = 1,\ m = 2,$ and $m = 3$ we find correspondingly

$$\frac{1}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n c_n H_{n+1}}{(n+1)!}$$
$$\frac{1}{6} = \sum_{n=0}^{\infty} \frac{(-1)^n c_n (H_{n+2} - 1)}{(n+2)!}$$
$$\frac{1}{12} = \sum_{n=0}^{\infty} \frac{(-1)^n c_n (2H_{n+3} - 3)}{(n+3)!},$$

etc.

Example 5. Here we exploit the binomial formula [7, Entry 9.16]

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k H_k^{(3)} = -\frac{1}{2n} (H_n^2 + H_n^{(2)}) \tag{8}$$

$(n = 1, 2 \dots)$ where

$$H_k^{(3)} = 1 + \frac{1}{2^3} + \cdots \frac{1}{k^3} = \zeta(3) + \frac{1}{2}\psi''(k+1).$$

We apply (5) to the function $f(z) = \zeta(3) + \frac{1}{2}\psi''(z+1)$. In this case

$$\int_0^1 \zeta(3) + \frac{1}{2}\psi''(z+1)\,dz = \zeta(3) + \frac{1}{2}\ \psi'(z+1)|_0^1 = \zeta(3) + \frac{1}{2}(\psi'(2) - \psi'(1)) = \zeta(3) - \frac{1}{2}$$

and therefore,

$$2\zeta(3) - 1 = \sum_{n=0}^{\infty} \frac{(-1)^n c_n}{n!} \left(\sum_{k=0}^{n} \binom{n}{k} (-1)^k H_k^{(3)} \right)$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^n c_n}{n!} \left(\sum_{k=0}^{n} \binom{n}{k} (-1)^k H_k^{(3)} \right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} c_n (H_n^2 + H_n^{(2)})}{n!n}$$

(the term for $n = 0$ in the first sum is zero and we can start the summation from $n = 1$). Thus

$$2\zeta(3) - 1 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} c_n (H_n^2 + H_n^{(2)})}{n!n}. \tag{9}$$

It can be seen from the digamma definition of the harmonic numbers that

$$\lim_{p \to 0} \frac{1}{p}(H_p^2 + H_p^{(2)}) = 2\zeta(3)$$

which can be considered the zero term in the sum. With this agreement, starting the summation from $n = 0$ we can write

$$-1 = \sum_{n=0}^{\infty} \frac{(-1)^{n-1} c_n (H_n^2 + H_n^{(2)})}{n!n}.$$

Example 6. For completeness we now reprove two known identities. First, the classical representation

$$\gamma = \sum_{n=0}^{\infty} \frac{(-1)^{n-1} c_n}{n!n} \tag{10}$$

obtained by Lorenzo Mascheroni in 1790. Informative historical notes on this series can be found in Blagouchine [2, p. 406].

We take $f(z) = \psi(z+1) + \gamma,\ f(k) = H_k$ and use the simple identity [7, Identity (9.3a)]

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k H_k = -\frac{1}{n}.$$

Clearly

$$\int_0^1 \psi(z+1) + \gamma dz = \ln \Gamma(z+1)|_0^1 + \gamma = \gamma$$

and (5) implies (10) immediately.

Next, we prove the representation

$$\frac{\pi^2}{6} - 1 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} c_n H_n}{n!n}, \tag{11}$$

which can be found in [16], and also [2, Eq. 44, p. 413]. We take $f(z) = \zeta(2) - \psi'(z+1)$, so that $f(k) = H_k^{(2)} = 1 + \frac{1}{2^2} + \cdots + \frac{1}{k^2},\ f(0) = 0$. We also need the identity [7, 9.4b]

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k H_k^{(2)} = -\frac{H_n}{n}.$$

Then we compute

$$\int_0^1 \zeta(2) - \psi'(z+1)\, dz = \ \zeta(2) - \psi(z+1)|_0^1 = \zeta(2) - \psi(2) + \psi(1) = \zeta(2) - 1.$$

and (11) follows from (5).

As in the previous example, we notice that

$$\lim_{p \to 0} \frac{H_p}{p} = \lim_{p \to 0} \frac{\psi(p+1) + \gamma}{p} = \zeta(2) = \frac{\pi^2}{6}, \tag{12}$$

and if we agree to write

$$\frac{\pi^2}{6} = \left.\frac{H_n}{n}\right|_{n=0},$$

then Eq. (11) becomes (with summation from $n = 0$)

$$-1 = \sum_{n=0}^{\infty} \frac{(-1)^{n-1} c_n H_n}{n! n}.$$

Merlini, Sprugnoli, and Verri [16] found (10) and (11) by using a symbolic Laplace summation formula. Blagouchine derived (11) from a special series transformation formula [2, p. 412]. With the help of his formula, he also obtained several more interesting representations of this kind. Two finite sums connecting Cauchy numbers and harmonic numbers can be found in [26].

Example 7. In this example we use the constant

$$M_1 = \int_1^2 \frac{\psi(1+t) + \gamma}{t}\, dt \approx 0.86062,$$

and we prove that

$$M_1 = \frac{1}{2} + \sum_{n=0}^{\infty} \frac{(-1)^{n-1} c_n H_n}{(n+1)!}. \tag{13}$$

For this purpose we take the function $f(z) = \dfrac{\psi(z+1) + \gamma}{z+1}$ and apply the binomial formula

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{H_k}{k+1} = -\frac{H_n}{n+1} \tag{14}$$

[7, Entry 9.32]. The left hand side in (5) becomes

$$\int_0^1 \frac{\psi(z+1) + \gamma}{z+1} dz = \int_1^2 \frac{\psi(x) + \gamma}{x} dx = \int_1^2 \frac{\psi(x+1) - 1/x + \gamma}{x} dx$$

$$= M_1 - \int_1^2 \frac{1}{x^2} dx = M_1 - \frac{1}{2}$$

and therefore,

$$M_1 - \frac{1}{2} = \sum_{n=0}^{\infty} \frac{(-1)^{n-1} c_n H_n}{(n+1)!}.$$

The constant M_1 is important, because it represents the value of some interesting series. It was shown in [8] that

$$M_1 = \sum_{n=1}^{\infty} \frac{1}{n} \ln\left(1 + \frac{1}{n+1}\right)$$

$$M_1 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(n - \zeta(2) - \zeta(3) - \cdots - \zeta(n)\right)$$

the first term in this series is 1. Also

$$M_1 = \sum_{n=1}^{\infty} H_n^- \left(\zeta(n+1) - 1\right)$$

where

$$H_n^- = 1 - \frac{1}{2} + \frac{1}{3} + \ldots + \frac{(-1)^{n-1}}{n}$$

are the skew harmonic numbers. Now we have one more series in this list.

4 Series with Stirling numbers of the first kind and harmonic numbers

In this section we assume that $f(z)$ is a function analytic in some half plane $\Re(z) > \lambda$, where $\lambda < 0$, and has moderate growth. Being analytic in a neighborhood of the origin, $f(z)$ is represented by a Taylor series centered at $z = 0$.

Now we manipulate the representation (1) in a different way. We replace $(z)_n$ by

$$z(z-1)\cdots(z-n+1) = \sum_{m=0}^{n} s(n,m)\, z^m$$

and change the order of summation to get

$$f(z) = \sum_{m=0}^{\infty} z^m \left(\sum_{n=0}^{\infty} \frac{s(n,m)}{n!} \Delta^n f(0) \right).$$

Comparing this to the Taylor series

$$f(z) = \sum_{m=0}^{\infty} z^m \frac{f^{(m)}(0)}{m!}$$

we conclude that

$$\frac{f^{(m)}(0)}{m!} = \sum_{n=0}^{\infty} \frac{s(n,m)}{n!} \Delta^n f(0).$$

Applying Eq. (4), we now come to an important identity:

Proposition 8. *Under the initial assumption on the function $f(z)$, for every $m \geq 0$ we have the representation*

$$\frac{f^{(m)}(0)}{m!} = \sum_{n=0}^{\infty} \frac{(-1)^n s(n,m)}{n!} \left(\sum_{k=0}^{n} \binom{n}{k} (-1)^k f(k) \right). \tag{15}$$

(The summation, in fact, starts from $n = m$ since $s(n, m) = 0$ for $n < m$.)

When $m = 1$ we have $s(n, 1) = (-1)^{n-1}(n-1)!$ and the formula takes the form

$$f'(0) = \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{k=0}^{n} \binom{n}{k} (-1)^{k-1} f(k) \right).$$

This evaluation was obtained by a different method in [9]. Formula (15) is an efficient mechanism for producing various series with Stirling numbers of the first kind together with special numbers. It resembles formula (5), where the Cauchy numbers are replaced by Stirling numbers and on the left hand side we have now derivatives. Most examples from the previous section can be repeated with the same function $f(z)$ and the same binomial identity, but now using formula (15) instead of (5).

Example 9. First we prove a classical result. Taking $f(z) = \psi(z+1) + \gamma$, $f(k) = H_k$ we have

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k f(k) = -\frac{1}{n}$$

(see Example 6). The Taylor series of $\psi(z+1) + \gamma$ is well-known

$$\psi(z+1) + \gamma = \sum_{k=1}^{\infty} (-1)^{k-1} \zeta(k+1)\, z^k, \quad |z| < 1 \tag{16}$$

and (15) produces the representation ($k \geq 1$)

$$\zeta(k+1) = \sum_{n=k}^{\infty} \frac{(-1)^{n-k} s(n,k)}{n!n}.$$

This is almost a folklore mathematical result. In a more general form it can be found in Jordan [14, p. 166]. Comments and extensions can be found in Adamchik [1, Section 5], and also in Blagouchine [2, p. 412].

Note that the numbers

$$\left[\begin{matrix} n \\ k \end{matrix} \right] = (-1)^{n-k} s(n,k)$$

are called the *unsigned Stirling numbers of the first kind*. They are thoroughly studied in the classical book of Graham, Knuth, and Patashnik [13, Chapter 6]. These numbers appear in the examples that follow.

Example 10. Working with the function $f(z) = \zeta(2) - \psi'(z+1)$, $f(k) = H_k^{(2)}$ we have for $n \geq 1$

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k f(k) = -\frac{H_n}{n}$$

and also from (16)

$$\zeta(2) - \psi'(z+1) = \zeta(2) + \sum_{k=0}^{\infty}(-1)^{k+1}(k+1)\zeta(k+2)\, z^k$$
$$= \sum_{k=1}^{\infty}(-1)^{k+1}(k+1)\zeta(k+2)\, z^k.$$

This way for $k \geq 1$ we have

$$\zeta(k+2) = \frac{1}{k+1}\sum_{n=k}^{\infty}\frac{(-1)^{n-k}s(n,k)H_n}{n!n}. \tag{17}$$

The equality also holds for $k = 0$ under the agreement (12).

Continuing further in this direction we can use identity (8) from Example 5 with the same function $f(z) = \zeta(3) + \frac{1}{2}\psi''(z+1), f(k) = H_k^{(3)}$. Again from (16)

$$f(z) = \frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k-1}(k+1)(k+2)\zeta(k+3)z^k$$

and (15) implies

$$\zeta(k+3) = \frac{1}{(k+1)(k+2)}\sum_{n=k}^{\infty}\frac{(-1)^{n-k}s(n,k)}{n!n}\left(H_n^2 + H_n^{(2)}\right) \tag{18}$$

for any $k \geq 1$.

The identities (17) and (18) were found independently by a different method by Wang and Lyu in [24, p. 171].

Example 11. Again let $k \geq 1$. Applying (15) to the function

$$f(z) = \frac{\psi(z+1)+\gamma}{z+1},\ \Re(z) > -1$$

and the binomial identity (14), we find [21, p. 28]

$$\sum_{j=1}^{k}\zeta(j+1) = \sum_{n=k}^{\infty}\frac{(-1)^{n-k}s(n,k)H_n}{(n+1)!}. \tag{19}$$

To prove this, we use the Taylor series (16) and notice that for $|z| < 1$ we have

$$\frac{\psi(z+1)+\gamma}{z+1} = \frac{1}{z+1}\sum_{k=1}^{\infty}(-1)^{k-1}\zeta(k+1)\, z^k = \sum_{k=1}^{\infty}\left((-1)^{k-1}\sum_{j=1}^{k}\zeta(j+1)\right)z^k.$$

Then (15) implies (19) immediately.

Example 12. Here we use the identity

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{(-1)^k}{(k+1)^2} = \frac{H_{n+1}}{n+1},$$

together with the function $f(z) = (z+1)^{-2}$, $\Re(z) > -1$. For $|z| < 1$ we have the Taylor series

$$\frac{1}{(z+1)^2} = \sum_{k=0}^{\infty} (-1)^k (k+1) z^k.$$

Therefore, from (15) we find the curious companion to Eq. (19)

$$k+1 = \sum_{n=k}^{\infty} \frac{(-1)^{n-k} s(n,k) H_{n+1}}{(n+1)!} \tag{20}$$

(see [21, p. 29]). More generally, using the identity from Example 4

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{1}{(k+m)^2} = \frac{(m-1)!n!(H_{n+m} - H_{m-1})}{(n+m)!}$$

with the function

$$f(z) = (z+m)^{-2} = \sum_{k=0}^{\infty} \frac{(-1)^k (k+1) z^k}{m^{k+2}}$$

we find

$$\frac{k+1}{m^{k+2}(m-1)!} = \sum_{n=k}^{\infty} \frac{(-1)^{n-k} s(n,k) (H_{n+m} - H_{m-1})}{(n+m)!}, \tag{21}$$

where $m = 1$ gives Eq. (20). In the same line, using the identity

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{1}{(k+m)^3} = \frac{(m-1)!n!}{2(n+m)!} \left((H_{n+m} - H_{m-1})^2 + H_{n+m}^{(2)} - H_{m-1}^{(2)} \right)$$

[7, 8.42] together with the function

$$f(z) = (z+m)^{-3} = \sum_{k=1}^{\infty} \frac{(-1)^k (k+1)(k+2) z^k}{m^{k+3}}$$

we come to the series

$$\frac{(k+1)(k+2)}{m^{k+3}(m-1)!} = \sum_{n=k}^{\infty} \frac{(-1)^{n-k} s(n,k)}{2(n+m)!} \left((H_{n+m} - H_{m-1})^2 + H_{n+m}^{(2)} - H_{m-1}^{(2)} \right). \tag{22}$$

Example 13. Here we use the identity

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k H_{k+1} = -\frac{1}{n(n+1)}$$

together with the function $f(z) = \psi(z+2) + \gamma, \;\; f(k) = H_{k+1}$. We write

$$f(z) = \psi(z+2) + \gamma = \psi(z+1+1) + \gamma = \frac{1}{1+z} + \psi(z+1) + \gamma$$

for $\Re(z) > -1$. From (16)

$$f(z) = \sum_{k=1}^{\infty} (-1)^{k-1} (\zeta(k+1) - 1)\, z^k, \;\; |z| < 1$$

and (15) produces the representation (see [14, p. 339])

$$\zeta(k+1) - 1 = \sum_{n=k}^{\infty} \frac{(-1)^{n-k} s(n,k)}{(n+1)!n} \quad (k \geq 1). \tag{23}$$

Example 14. Now we present a series connecting the Stirling numbers $s(n,k)$ and the central binomial coefficients $\binom{2n}{n}$. Starting from the identity in [7, Entry 8.43].

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{1}{2k+1} = \frac{4^n}{2n+1} \binom{2n}{n}^{-1}$$

we introduce the function $f(z) = (2z+1)^{-1}, \; \Re(z) > -\frac{1}{2}$ with Taylor series for $|z| < 1/2$

$$\frac{1}{2z+1} = \sum_{k=0}^{\infty} (-1)^k 2^k z^k.$$

From (15) we have the strange representation

$$2^k = \sum_{n=k}^{\infty} \frac{(-1)^{n-k} 4^n s(n,k)}{n!(2n+1)} \binom{2n}{n}^{-1}. \tag{24}$$

All these series with positive terms are very slowly convergent.

5 Series with Laguerre polynomials and harmonic numbers

In this section we present series with the classical Laguerre polynomials $L_n(x)$, which have the representation

$$L_n(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{x^k}{k!}.$$

Let $x > 0$ and consider the function $f(z) = \frac{x^z}{\Gamma(z+1)}$ so that $f(k) = \frac{x^k}{k!}$. Proposition 8 implies the formula

$$\left(\frac{d}{dz}\right)^m \frac{x^z}{\Gamma(z+1)}\bigg|_{z=0} = \sum_{n=m}^{\infty} \frac{(-1)^n s(n,m)}{n!} L_n(x). \tag{25}$$

This way for every $m = 1,\ 2, \ldots$, we obtain different series with Laguerre polynomials.

Example 15. When $m = 1$, $(-1)^n s(n,1) = -(n-1)!$ and we come to the known series

$$-\gamma - \ln x = \sum_{n=1}^{\infty} \frac{1}{n} L_n(x). \tag{26}$$

This is an entry in the reference book [20, Entry 5.11.1(3)]. Interestingly, this equation resembles [4, Eq. 101].

For $m = 2$ we have $(-1)^n s(n,2) = (n-1)! H_{n-1}$ and this provides the new series

$$\frac{\gamma^2}{2} + \gamma \ln x - \frac{\pi^2}{12} + \frac{\ln^2 x}{2} = \sum_{n=1}^{\infty} \frac{1}{n} H_{n-1} L_n(x). \tag{27}$$

For $m > 2$ the derivatives in (25) become very long. For $m = 3$ we just write

$$\left(\frac{d}{dz}\right)^3 \frac{x^z}{\Gamma(z+1)}\bigg|_{z=0} = \frac{1}{2} \sum_{n=k}^{\infty} \frac{1}{n} \left(H_{n-1}^2 - H_{n-1}^{(2)}\right) L_n(x) \tag{28}$$

etc.

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