Journal of Integer Sequences, Vol. 23 (2020),
Article 20.4.6

# Symmetric Dellac Configurations

Ange Bigeni
Department of Mathematics
HSE University
Usacheva str. 6
Moscow 119048
Russia
ange.bigeni@gmail.com

Evgeny Feigin
Department of Mathematics
HSE University
Usacheva str. 6
Moscow 119048
Russia
and
Skolkovo Institute of Science and Technology
Nobelya Ulitsa 3
Moscow 121205
Russia
evgfeig@gmail.com

**Abstract**

We define symmetric Dellac configurations as the Dellac configurations that are symmetrical with respect to their centers. The even-length symmetric Dellac configurations coincide with the Fang–Fourier symplectic Dellac configurations. Symmetric Dellac configurations generate the Poincaré polynomials of (odd or even) symplectic or orthogonal versions of degenerate flag varieties. We give several combinatorial

interpretations of the Randrianarivony–Zeng polynomial extension of median Euler numbers in terms of objects that we call extended Dellac configurations. We show that the extended Dellac configurations generate symmetric Dellac configurations. As a consequence, the cardinalities of odd and even symmetric Dellac configurations are respectively given by two sequences $(1, 1, 3, 21, 267, \dots)$ and $(1, 2, 10, 98, 1594, \dots)$, defined as specializations of polynomial extensions of median Euler numbers.

# 1 Notation and vocabulary

For a pair of integers $n < m$, we denote the set $\{n, n+1, \dots, m\}$ by $[n, m]$. We denote the set $[1, n]$ by $[n]$.

In this paper, we call *tableau* (plural *tableaux*) a rectangular grid made up of a finite number of columns and rows. We call *box* the intersection of a column and a row. If a tableau $T$ has $n$ columns and $m$ rows, then for all $j \in [n]$ and for all $i \in [m]$, the $j$-th column of $T$ (from left to right) is denoted by $C_j^T$, and the $i$-th row of $T$ (from bottom to top) is denoted by $L_i^T$. The box $C_j^T \cap L_i^T$ is denoted by $(j : i)$. In the tableaux that we study, a box may be empty or contain a point $\bullet$. If the box $(j : i)$ contains a point referred to by a letter such as $p$, we also write $p = (j : i)$. If it is known that a row $L_i^T$ contains a single point, then this point is denoted by $p_i^T$.

# 2 Introduction

Let $N$ be a positive integer. Recall that a Dellac configuration [9] $D \in \mathrm{DC}_N$ is a tableau made up of $N$ columns and $2N$ rows, which contains $2N$ points one per row, two per column. In addition, all the points are located between the lines $y = x$ and $y = N + x$; in other words, if a box $(\ j\ :\ i\ )$ of $T$ contains a point, then $j \leq i \leq N + j$. The cardinality of $\mathrm{DC}_N$ is $h_N$ where $(h_N)_{N \geq 0} = (1, 1, 2, 7, 38, 295, \dots)$ is the sequence A000366 of normalized median Genocchi numbers [1, 3, 4, 5, 14, 15, 18, 19, 20]. For $D \in \mathrm{DC}_N$, let $\mathrm{inv}(D)$ be the number of pairs $(p_1, p_2)$ of points of $D$ such that $p_1$ is located to the left of $p_2$ and above $p_2$ (such a pair is named an *inversion* of $D$). For example, we depict in Figure 1 the elements of $(DC_N)_{N \in [3]}$ (from top to bottom), whose inversions are represented by segments connecting the points involved.

Given an $N$-dimensional vector space $E$, a basis $(e_1, e_2, \dots, e_N)$ of $E$ and an integer $k \in [N]$, let $\mathrm{pr}_k : E \to E$ be the projection along $e_k$ to the linear span of the rest basis vectors. Feigin [14] defined the degenerate flag variety $\mathcal{F}_N^a$ as the set of collections $(V_1, V_2, \dots, V_{N-1})$ of subspaces of $E$ such that $\dim(V_k) = k$ and $\mathrm{pr}_{k+1}(V_k) \subset V_{k+1}$ for all $k \in [N-1]$. Feigin [14, 16, 17] proved that the Poincaré polynomial $F_N$ of the degenerate flag variety $\mathcal{F}_N^a$ has the following combinatorial interpretation for all $N \geq 1$:

$$F_N(q) = \sum_{D \in \mathrm{DC}_N} q^{\mathrm{inv}(D)}.$$

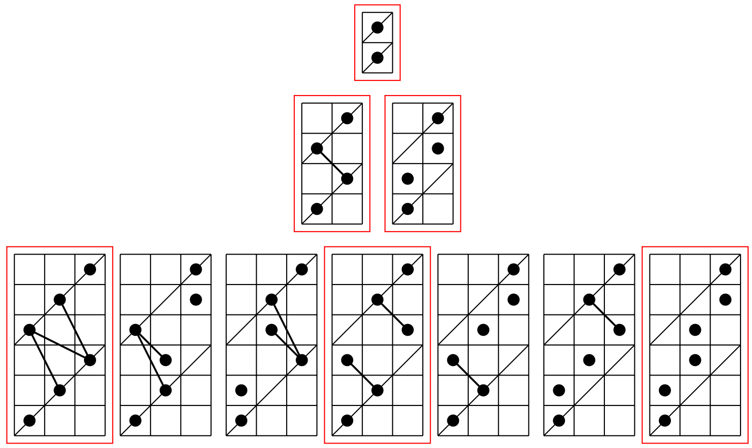

Figure 1: The elements of $\mathrm{DC}_1, \mathrm{DC}_2, \mathrm{DC}_3$ from top to bottom; the elements of $\mathrm{SDC}_1, \mathrm{SDC}_2, \mathrm{SDC}_3$ are framed in red.

For example, we can compute the Poincaré polynomials of $(\mathcal{F}^a_N)_{N\in[3]}$ from Figure 1:

$$F_1(q) = 1,$$
$$F_2(q) = q + 1,$$
$$F_3(q) = q^3 + 3q^2 + 2q + 1.$$

## 2.1 Symmetric Dellac configurations

For a positive integer $n$, consider a $2n$-dimensional vector space $E$, a basis $(e_1, e_2, \ldots, e_{2n})$ of $E$, and a non-degenerate skew-symmetric form defined on $E^2$ by $(e_i, e_{2n+1-i}) = 1$ for all $i \in [n]$ and $(e_i, e_j) = 0$ if $i + j \neq 2n$. Feigin et al. [17] defined the symplectic degenerate flag variety $\mathrm{SpF}^a_{2n}$, as the subvariety of $\mathcal{F}^a_{2n}$ consisting of collections $(V_1, \ldots, V_{2n-1})$ such that $V_k = V^{\perp}_{2n-k}$ for all $k \in [2n-1]$. Fang and Fourier [13] considered the action of an algebraic torus on $\mathcal{F}^a_{2n}$ [8, 7]. They parametrized the torus fixed points of $\mathrm{SpF}^a_{2n} \subset \mathcal{F}^a_{2n}$ (the points fixed by the action of the torus) using even symmetric Dellac configurations $D \in \mathrm{SDC}_{2n}$, which they introduced in their paper as *symplectic Dellac configurations*.

In a separate paper [6], we extended the definition of the symplectic degenerate flag varieties to $\mathrm{SpF}^a_N$ for all $N \geq 1$. We proved that the Poincaré polynomials $S_N$ of these varieties have the following combinatorial interpretation:

$$S_N(q) = \sum_{D\in \mathrm{SDC}_N} q^{\widetilde{\mathrm{inv}}(D)}.$$

Here $\widetilde{\mathrm{inv}}(D)$ is the number of inversions of $D$, modulo the rotation $r_\pi$. More precisely, we identify two inversions of $D$ if they are of the form $(p_1, p_2)$ and $(r_\pi(p_2), r_\pi(p_1))$. For example, below are the Poincaré polynomials of $(\mathrm{SpF}^a_N)_{N\in[4]}$ (the cases $N \in [3]$ are given by Figure 1):

$$\begin{aligned} S_1(q) &= 1, \\ S_2(q) &= q+1, \\ S_3(q) &= q^2+q+1, \\ S_4(q) &= q^4+3q^3+3q^2+2q+1. \end{aligned}$$

We also defined natural varieties $(\mathrm{SOF}^a_N)_{N\geq 1}$ [6], named *orthogonal degenerate flag varieties*, whose Poincaré polynomials $O_N$ have the following combinatorial interpretation:

$$O_N(q) = \sum_{D\in \mathrm{SDC}_N} q^{\overline{\mathrm{inv}}(D)},$$

where $\overline{\mathrm{inv}}(D)$ equals $\widetilde{\mathrm{inv}}(D)$ minus the number of inversions of $D$ of the kind $(p, r_\pi(p))$. For example, the Poincaré polynomials of $(\mathrm{SOF}^a_N)_{N\in[4]}$ (the cases $N \in [3]$ are given by Figure 1) are

$$\begin{aligned} O_1(q) &= 1, \\ O_2(q) &= 2, \\ O_3(q) &= 2q+1, \\ O_4(q) &= 4q^2+4q+2. \end{aligned}$$

The primary goal of this paper is to determine the cardinalities of the sets $(\mathrm{SDC}_N)_{N\geq 1}$. According to the parity of $N$, they appear in two sequences:

- $(l_n)_{n\geq 0} = (1, 1, 3, 21, \dots)$ A098278; and
- $(r_n)_{n\geq 0} = (1, 2, 10, 98, \dots)$ A098279

defined by Randrianarivony and Zeng [21] as specializations of polynomials interpolating median Euler numbers. We recall the necessary definitions below.

## 2.2 Extended median Euler numbers

Consider the family of polynomials $(D_n(x))_{n\geq 0}$ defined by

$$\begin{aligned} D_0(x) &= 1, \\ D_{n+1}(x) &= (x+1)(x+2)D_n(x+2) - x(x+1)D_n(x) \quad \forall n \geq 0. \end{aligned}$$

They were introduced by Randrianarivony and Zeng [21] as interpolations of the median Euler numbers $(L_n)_{n\geq 0} = (1, 1, 4, 46, \dots)$ A000657 and $(R_n)_{n\geq 0} = (1, 3, 24, 402, \dots)$ A002832, through the equalities $L_n = D_n(1/2)$ and $R_n = D_n(-1/2)$.

It is easy to check that for $n \geq 1$, the polynomial $D_n(x)$ is of the form $2^n(x+1)P_n(x+1)$ for some polynomial $P_n(x)$ of degree $n-1$ with positive integer coefficients. The polynomials $P_n(x)$ satisfy the following recursive formulas:

$$\begin{aligned} P_1(x) &= 1, \\ P_{n+1}(x) &= \frac{(x+2)(x+1)}{2} P_n(x+2) - \frac{x(x-1)}{2} P_n(x) \quad \forall n \geq 1. \end{aligned}$$

The first few such polynomials are

$$\begin{aligned} P_1(x) &= 1, \\ P_2(x) &= 2x + 1, \\ P_3(x) &= 6x^2 + 10x + 5, \\ P_4(x) &= 24x^3 + 84x^2 + 110x + 49, \\ P_5(x) &= 120x^4 + 720x^3 + 1758x^2 + 1954x + 797. \end{aligned}$$

For all $n \geq 1$, let $c_{n,n-1}x^{n-1} + c_{n,n-2}x^{n-2} + \cdots + c_{n,1}x + c_{n,0} = P_n(x)$. It is easy to check the following recursive formulas: $c_{1,0} = 1$ and

$$c_{n,0} = \sum_{i=0}^{n-2} 2^i c_{n-1,i}, \tag{1}$$

$$c_{n,k} = (k+1)c_{n-1,k-1} + \sum_{i=k}^{n-2} 2^{i-k} \left( \binom{i+1}{k} + 2\binom{i+1}{k-1} \right) c_{n-1,i}, \tag{2}$$

$$c_{n,n-1} = n c_{n-1,n-2} \tag{3}$$

for all $n \geq 2$ and $k \in [n-2]$. In particular, we have $c_{n,n-1} = n!$ for all $n \geq 1$.

The sequences $(l_n)_{n \geq 0} = (1, 1, 3, 21, \dots)$ and $(r_n)_{n \geq 0} = (1, 2, 10, 98, \dots)$ are defined by $l_n = D_n(0)/2^n$ and $r_n = D_n(1)/2^n$ for all $n \geq 0$, which, in terms of $P_n$, is equivalent to $l_0 = r_0 = 1$ and

$$\begin{aligned} l_n &= P_n(1), \\ r_n &= 2P_n(2) = 2P_{n+1}(0) \end{aligned}$$

for all $n \geq 1$. From Randrianarivony and Zeng [21, Théorème 24], it follows that

$$1 + \sum_{n \geq 1} x P_n(x) t^n = \cfrac{1}{1 - \cfrac{xt}{1 - \cfrac{(x+1)t}{1 - \cfrac{2(x+2)t}{1 - \cfrac{2(x+3)t}{1 - \cfrac{3(x+4)t}{1 - \cfrac{3(x+5)t}{\ddots}}}}}}}.$$

Afterwards, recall [10, 11, 12, 23] that a *surjective pistol* of size $n$ is a surjective map $f : [2n] \twoheadrightarrow \{2, 4, 6, \dots, 2n\}$ such that $f(j) \geq j$ for all $j \in [2n]$. Let $\mathrm{SP}_n$ be the set of the surjective pistols of size $n$. For all $f \in \mathrm{SP}_n$, we denote by $\max(f)$ the number of maximal points of $f$ (integers $j \in [2n-2]$ such that $f(j) = 2n$), and by $\mathrm{fd}(f)$ the number of doubled fixed points of $f$ (integers $j \in [2n-2]$ such that there exists $j' < j$ with $f(j') = f(j) = j$).

For example $\mathrm{SP}_2$ has three elements:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 4 & 4 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 4 & 4 \end{pmatrix},$$

whose numbers of maximal points are respectively 0,1 and 1, and whose numbers of doubled fixed points are respectively 1,0 and 0.

Still due to Randrianarivony and Zeng [21, Proposition 25], it is known that

$$P_n(x) = \sum_{f \in \mathrm{SP}_n} 2^{n-1-\max(f)-\mathrm{fd}(f)} x^{\max(f)} \tag{4}$$

for all $n \geq 1$. For example, considering the three elements of $\mathrm{SP}_2$ shown above, we indeed have the equality $P_2(x) = 2x + 1 = 2^{1-0-1}x^0 + 2^{1-1-0}x^1 + 2^{1-1-0}x^1$.

## 2.3 Even symmetric Dellac configurations

One of the goals of this paper is to give a new proof of the statement that the cardinality of $\mathrm{SDC}_{2n}$ (for $n \geq 1$) is the number $r_n$ from the sequence $(r_n)_{n\geq 0} = (1, 2, 10, 98, 1594, \dots)$ (another proof can be found in a previous paper [2]). The first step in this direction is to reduce $\mathrm{SDC}_{2n}$ to smaller elements that we name even extended Dellac configurations, whose set is denoted by $\mathcal{T}_n^e$. The main result of this section is Theorem 7, which interprets $P_n(x)$ combinatorially in terms of $\mathcal{T}_n^e$, and which implies the equality $|\mathrm{SDC}_{2n}| = 2P_n(2) = r_n$.

### 2.3.1 Generation of $\mathrm{SDC}_{2n}$

An *even extended Dellac configuration* $T \in \mathcal{T}_n^e$ is a tableau made up of $n$ columns and $2n$ rows, which contains $2n$ points (one per row, two per column) on or above the line $y = x$ (in other words, this is the definition of $\mathrm{DC}_n$, except that we remove the condition that the points must be under the line $y = n + x$). By considering the two points of the last column (from left to right) of a tableau $T \in \mathcal{T}_n^e$, it is easy to obtain the formula $|\mathcal{T}_n^e| = \frac{(n+1)n}{2} |\mathcal{T}_{n-1}^e| = (n+1)!\, n!\, / 2^n$. For example, in Figure 2 we represent the 3 elements of $T_2^e$. Note that the points located on or above the line $y = 2n - x$ (represented by a dashed line) are represented by stars; in general, we name such points *free points*, and we denote the number of free points of $T \in \mathcal{T}_n^e$ by $\mathrm{fr}(T)$. In Figure 2, the numbers of free points are 2, 1 and 2 from left to right.

There exists a one-to-one correspondence between $\mathrm{SDC}_{2n}$ and the set of tableaux $T \in \mathcal{T}_n^e$ whose free points are labeled with the number 0 or 1. For such a tableau $T$, the principle is:

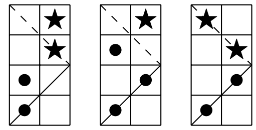

Figure 2: The 3 elements of $\mathcal{T}_2^e$.

- to draw an empty tableau $\widetilde{T}$ made up of $2n$ columns and $4n$ rows;
- for a box $(j:i)$ of $T$ containing a point $p$,
  - if $p$ is not free, or if $p$ is a free point whose label is 1, then we draw a point in the box $(j:i)$ of $\widetilde{T}$;
  - otherwise $p$ is free and its label is 0, and we draw a point in the box $(2n+1-j:i)$ of $\widetilde{T}$;
- to apply the central reflection with respect to the center $(n,2n)$ of $\widetilde{T}$ to the $2n$ points that have been drawn in $\widetilde{T}$, which indeed makes $\widetilde{T}$ an element of $\mathrm{SDC}_{2n}$.

For example, in Figure 3 we depict how a labeled tableau $T_0 \in \mathcal{T}_2^e$ is mapped to its corresponding element $\widetilde{T}_0 \in \mathrm{SDC}_4$.

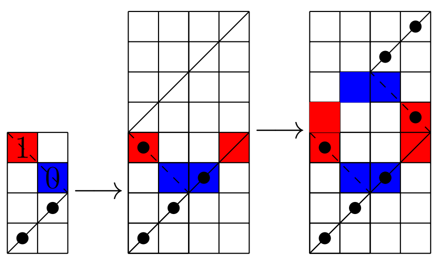


Figure 3: Construction of an even symmetric Dellac configuration.

This bijection between the tableaux $T \in \mathcal{T}_n^e$ whose free points are labeled with 0 or 1 and $\mathrm{SDC}_{2n}$ implies the following formula:

$$|\mathrm{SDC}_{2n}| = \sum_{T \in \mathcal{T}_n^e} 2^{\mathrm{fr}(T)}. \tag{5}$$

For example, for $n = 2$, the $10 = 2^2 + 2^1 + 2^2$ elements of $\mathrm{SDC}_4$ are generated by the 3 elements of $\mathcal{T}_2^e$ as depicted in Figure 4. The elements of $\mathrm{SDC}_4$ in the same frame under each tableau $T \in \mathcal{T}_2^e$ correspond to the $2^{\mathrm{fr}(T)}$ different labellings of the free points of $T$.

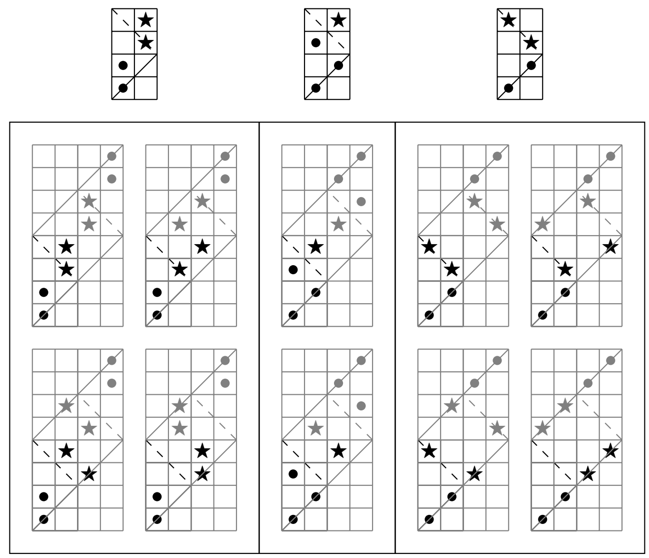

Figure 4: Construction of $\mathrm{SDC}_4$ from $\mathcal{T}_2^e$.

### 2.3.2 A combinatorial interpretation of $P_n(x)$ in terms of $\mathcal{T}_n^e$

**Definition 1.** Let $T \in \mathcal{T}_n^e$. For a given point $p_0 = (j : i)$ of $T$, we first define a path $(p_0, p_1, \dots)$ of points of $T$ as follows: for all $k \geq 0$, assume that the points $p_0, \dots, p_k$ are defined, and let $(j_k : i_k) = p_k$;

(a) if $j_k = n$, then $p_{k+1}$ is defined as $p_k$;

(b) if $j_k < n$ and $p_k$ is the upper point of its column, then $p_{k+1}$ is defined as $p^T_{2n-j_k}$ (the point of the $(2n - j_k)$-th row $L^T_{2n-j_k}$ of $T$, see Section 1);

(c) if $j_k < n$ and $p_k$ is the lower point of its column, then $p_{k+1}$ is defined as $p^T_{j_k}$.

We then define a finite subsequence $S_T(i) = (p_{k_0}, \dots, p_{k_f})$ of $(p_0, p_1, \dots)$ by $k_0 = 0$ and, if we assume that $k_0, \dots, k_q$ are defined for some $q \geq 0$,

- if $j_{k_q} < \max\{j_k : k \geq 0\}$, then $k_{q+1} = \min\{k > k_q : j_k > j_{k_q}\}$;
- otherwise $f = q$.

We finally define $B(T)$ as $S_T(n)$ and $R(T)$ as $S_T(2n)$. Let $\mathrm{b}(T) + 1$ and $\mathrm{r}(T) + 1$ be their respective number of elements.

For example, in Figure 5 we represent a tableau $T_1 \in \mathcal{T}_7^e$ such that $B(T_1)$ is the tuple $((2 : 7), (3 : 12), (7 : 11))$ and $R(T_1)$ is the tuple $((4 : 14), (7 : 10))$. In general, when depicting a tableau $T \in \mathcal{T}_n^e$, the first $\mathrm{b}(T)$ elements of $B(T)$ and the first $\mathrm{r}(T)$ elements of $R(T)$ are painted in blue and red respectively. The meaning of the green color in Figure 5 will be explained further (see Definition 4).

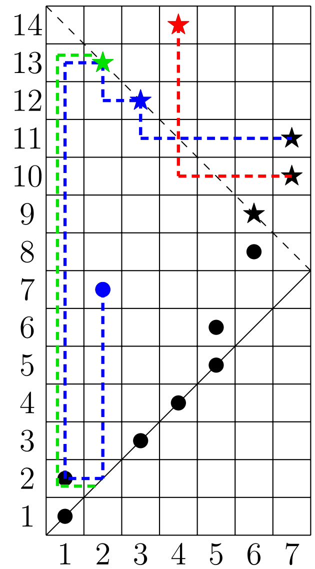


Figure 5: Extended Dellac configuration $T_1 \in \mathcal{T}_7^e$ such that $(\mathrm{b}(T_1), \mathrm{r}(T_1), \mathrm{g}(T_1)) = (2, 1, 1)$.

**Definition 2.** Let $j \in [n]$. We consider a tableau $T$ made up of $n$ columns (each of which contains at most two points) and $2n$ rows (each of which contains at most one point), such that each of the first $j-1$ rows (from bottom to top) of $T$ contains exactly one point on or above the line $y = x$, and each of the first $j-1$ columns (from left to right) contains exactly two points (for example, any tableau $T \in \mathcal{T}_n^e$ satisfies this condition). Let $i \in [j, 2n]$ be such that the first $j-1$ boxes of the row $L_i^T$ are empty (for example, any $i \in [j, 2n]$ such that the box $(j : i)$ of $T$ contains a point). We define a sequence $\mathcal{I}_T(j : i) = (i_0, i_1, \dots)$ of elements of $[2n]$ as follows: first of all, let $i_0$ be the integer $i$, afterwards, if the integers $i_0, \dots, i_k$ are defined for some $k \geq 0$:

1. if $i_k \in [j, 2n-j] \sqcup \{2n\}$, then $i_{k+1}$ is defined as $i_k$;
2. if $i_k \in [2n-j+1, 2n-1]$, let $j_k = 2n - i_k \in [j-1]$, then $i_{k+1}$ is defined by $(j_k : i_{k+1})$ being the upper point of $C_{j_k}^T$;
3. otherwise $i_k \in [j-1]$, and $i_{k+1}$ is defined by $(i_k : i_{k+1})$ being the lower point of the $C_{i_k}^T$.

For example, in Figure 5, one can read the sequences corresponding to the free points $p = (j : i)$ that appear in the red, blue or green path, by following the path in the opposite direction $(\dots, p_k, p_{k-1}, \dots)$. We have $\mathcal{I}_{T_1}(7 : 10) = (10, 14, 14, \dots), \mathcal{I}_{T_1}(7 : 11) = (11, 12, 13, 2, 7, 7, \dots)$, etc. The green path, as a subpath of the blue path, also gives $\mathcal{I}_{T_1}(2 : 13) = (13, 2, 2, \dots)$.

**Proposition 3.** *In the context of Definition 2, the sequence $\mathcal{I}_T(j : i) = (i_0, i_1, \dots)$ becomes stationary, i.e., there exists an integer $\mathrm{root}_T(j : i) \in [j, 2n-j] \sqcup \{2n\}$ (named the root of the* *box* $(j:i)$ *in* $T$*) and an integer* $k_0 \geq 0$ *such that* $i_k = \mathrm{root}_T(j:i)$ *for all* $k \geq k_0$*. Moreover, the map* $i \mapsto \mathrm{root}_T(j:i)$ *is a bijection from the set of integers* $i \in [j, 2n]$ *such that the first* $j-1$ *boxes of* $L_i^T$ *are empty, to* $[j, 2n-j] \sqcup \{2n\}$*.*

*Proof.* Since $[2n]$ is a finite set, there exist $0 \leq k_1 < k_2$ such that $i_{k_1} = i_{k_2}$. Suppose that $i_k \notin [j, 2n-j] \sqcup \{2n\}$ for all $k \geq 0$, i.e., the rule (i) of Definition 2 is never applied. Then, the sequence $(i_k)_{k \geq 0}$ is reversible: for all $k > 0$, let $j_{k-1} \in [n]$ such that $p_{i_k}^T = (j_{k-1} : i_k)$; if $(j_{k-1} : i_k)$ is the upper point of its column, then $i_{k-1} = 2n - j_{k-1}$, otherwise $i_{k-1} = j_{k-1}$. This reversibility and the equality $i_{k_2} = i_{k_1}$ imply $i_{k_2-k_1} = i_{k_1-k_1} = i_0 = i$. Since $k_2 - k_1 > 0$ and, for all $k > 0$, the point $p_{i_k}^T$ is of the kind $(j_{k-1} : i_k)$ for some $j_{k-1} \in [j-1]$, then the point $p = (j : i)$ equals $(j_{k_2-k_1-1} : i_{k_2-k_1})$ for some $j_{k_2-k_1-1} < j$, which is absurd, so $i_k$ belongs to $[j, 2n-j] \sqcup \{2n\}$ for $k$ big enough.

Let $k_{\min}$ be the smallest integer $k \geq 0$ such that $i_k = \mathrm{root}_T(j:i)$. As stated in the previous paragraph, the sequence $(i = i_0, i_1, \ldots, i_{k_{\min}})$ is reversible because it never involves the rule (i) of Definition 2, so the map $i \mapsto \mathrm{root}_T(j:i)$ is injective. Finally, the number of integers $i \in [j, 2n]$ such that the first $j-1$ boxes of $L_i^T$ are empty is exactly the cardinality $2(n-j+1)$ of the set $[j, 2n-j] \sqcup \{2n\}$ (hence the map $i \mapsto \mathrm{root}_T(j:i)$ is bijective): since the first $j-1$ rows of $T$ contain exactly $j-1$ points, and the first $j-1$ columns of $T$ contain exactly $2(j-1)$ points, then, among the $2n-j+1$ upper rows of $T$ (the rows $C_j^T, C_{j+1}^T, \ldots, C_{2n}^T$), exactly $j-1$ of them contain a point in one of their first $j-1$ boxes, so $2n-j+1-(j-1) = 2(n-j+1)$ of them have their first $j-1$ boxes empty. □

**Definition 4.** We define $\mathcal{G}(T)$ as the set of the free points $p = (j:i)$ of $T$ such that:

- the point other than $p$ in $C_j^T$ is an element of $B(T)$;
- the integer $\mathrm{root}_T(j:i)$ equals $j$.

The cardinality of $\mathcal{G}(T)$ is denoted by $\mathrm{g}(T)$ and its elements are painted in green in the graphical representation of $T$.

For example, in Figure 5, we have $\mathcal{G}(T_1) = \{(2:13)\}$.

**Definition 5.** Let $n \geq 1$ and $T \in \mathcal{T}_n^e$. We denote by $\mathrm{Omax}(T)$ (whose cardinality is $\mathrm{omax}(T)$) the set of the elements of $B(T), R(T)$ and $\mathcal{G}(T)$. Note that $\mathrm{omax}(T) \geq 2$. The set $\mathrm{Max}(T)$ (whose cardinality is $\max(T)$) is defined as $\mathrm{Omax}(T)$ from which we removed the last element of $B(T)$ and the last element of $R(T)$. Hence

$$\max(T) = b(T) + r(T) + g(T) = \mathrm{omax}(T) - 2.$$

*Remark* 6. For all $T \in \mathcal{T}_n^e$, the $\mathrm{b}(T)$ last points of $B(T)$, the $\mathrm{r}(T)+1$ points of $R(T)$, and the $\mathrm{g}(T)$ points of $\mathcal{G}(T)$, are pairwise distinct free points of $T$, so

$$\mathrm{fr}(T) \geq \max(T) + 1.$$

The first main result of this paper is the following.

**Theorem 7.** *For all $n \geq 1$ we have*

$$P_n(x) = \sum_{T \in \mathcal{T}_n^e} 2^{\mathrm{fr}(T)-1-\max(T)} x^{\max(T)}.$$

For example, the three elements of $\mathcal{T}_2^e$ shown in Figure 6 give

$$P_2(x) = 2^{2-1-1-0-0} x^{1+0+0} + 2^{1-1-0-0-0} x^{0+0+0} + 2^{2-1-0-1-0} x^{0+1+0} = 2x + 1.$$

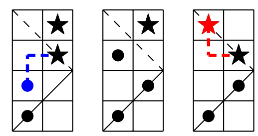

Figure 6: The three elements of $\mathcal{T}_2^e$.

The cardinality of $\mathrm{SDC}_{2n}$ being $r_n = 2P_n(2)$ is then obtained by setting $x = 2$ in Theorem 7 in view of Formula (5). Section 3 is dedicated to the proof of Theorem 7 (by induction, using the recursive formulas (1), (2) and (3)).

*Remark* 8. An independent proof of $|\mathrm{SDC}_{2n}| = r_n$ is given in a previous paper [2]. The proof is based on a surjection $\varphi : \mathcal{T}_n^e \twoheadrightarrow \mathrm{SP}_n$ such that

$$2^{n-\mathrm{fd}(f)} = \sum_{T \in \varphi^{-1}(f)} 2^{\mathrm{fr}(T)} \tag{6}$$

for all $f \in \mathrm{SP}_n$, which proves $|\mathrm{SDC}_{2n}| = 2P_n(2) = r_n$ in view of Formula (4) and Formula (5). Afterwards, one can check that for all $f \in \mathrm{SP}_n$ and $T \in \varphi^{-1}(f)$,

1. $\mathrm{g}(T)$ is the number of integers $j \in \{2, 4, \ldots, 2n-2\}$ such that $f(j-1) = f(j) = 2n$ and $\min\{j' < j-1 : f(j') = j\}$ is even;

2. $\mathrm{r}(T)$ is the number of all other integers $j \in \{2, 4, \ldots, 2n-2\}$ such that $f(j) = 2n$;

3. $\mathrm{b}(T)$ is the number of integers $j \in \{1, 3, \ldots, 2n-3\}$ such that $f(j) = 2n$.

It follows that $\max(f) = \max(T)$, which proves Theorem 7 by Formula (4) and Formula (6).

## 2.4 Odd symmetric Dellac configurations

A second goal of this paper is to prove that the cardinality of $\mathrm{SDC}_{2n-1} (n \geq 1)$ is the number $l_n$ from the sequence $(l_n)_{n\geq 0} = (1, 1, 3, 21, 267, \ldots)$ introduced in Section 2.2. It is obviously true for $n = 1$ (see Figure 1). Afterwards, as we did for the even case, we reduce $\mathrm{SDC}_{2n-1} (n \geq 2)$ to smaller elements called odd extended Dellac configurations, whose set is denoted by $\mathcal{T}_{n-1}^o$. In Theorem 10, we give a combinatorial interpretation of $P_n(x)$ in terms of $\mathcal{T}_{n-1}^o$, which implies the equality $|\mathrm{SDC}_{2n-1}| = P_n(1) = l_n$.

### 2.4.1 Generation of $\mathrm{SDC}_{2n+1} (n \geq 1)$

We define *odd extended Dellac configurations* $T \in \mathcal{T}_n^o$ $(n \geq 1)$ as the tableaux $T$ of size $n \times (2n+1)$ with the following conditions:

- each column of $T$ contains exactly two points;
- each row of $T$ contains at most one point;
- each of the first $n$ rows of $T$ contains exactly one point.

In other words, a tableau $T \in \mathcal{T}_n^o$ is like an element of $\mathcal{T}_n^e$, except that it has $2n+1$ rows instead of $2n$, and exactly one of its $n+1$ upper rows is empty (i.e., has no points). There is a natural surjection $\phi : \mathcal{T}_n^o \twoheadrightarrow \mathcal{T}_n^e$ which consists in deleting the empty row of the elements of $\mathcal{T}_n^o$, and it is clear that $|\phi^{-1}(T)| = n+1$ for all $T \in \mathcal{T}_n^e$. Hence $|\mathcal{T}_n^o| = (n+1)|\mathcal{T}_n^e| = ((n+1)!\,)^2/2^n$. For example, the 9 elements of $\mathcal{T}_2^o$ are surjected into the 3 elements of $\mathcal{T}_2^e$ as shown in Figure 7.

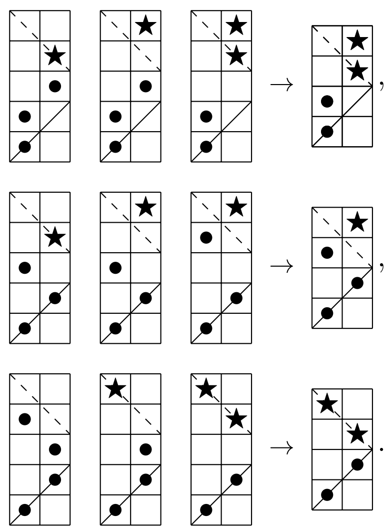

Figure 7: A surjection from $\mathcal{T}_2^o$ to $\mathcal{T}_2^e$.

For a tableau $T \in \mathcal{T}_n^o$, we represent with stars the points of $T$ located on or above the line $y = 2n+1-x$ (dashed line). We call these points *free points* of $T$ and denote their number by $\mathrm{fr}(T)$.

Similarly as for $\mathrm{SDC}_{2n}$, the odd symmetric Dellac configurations $D \in \mathrm{SDC}_{2n+1}$ are in one-to-one correspondence with the elements of $\mathcal{T}_n^o$ whose free points are labeled with the integers 0 or 1: for such a tableau $T$, we construct the corresponding element of $\mathrm{SDC}_{2n+1}$ as follows:

- we draw an empty tableau $\overline{T}$ made up of $2n+1$ columns and $4n+2$ rows;

- for a box $(j:i)$ of $T$ containing a point $p$,
  - if $p$ is not free, or if $p$ is a free point whose label is 1, then we draw a point in the box $(j:i)$ of $\overline{T}$;
  - otherwise $p$ is free and its label is 0, and we draw a point in the box $(2n+2-j:i)$ of $\overline{T}$;
- we draw a point in the box $(n+1:i_0)$ where $L_{i_0}^T$ is its unique empty row;
- we apply the central reflexion with respect to the center $(n+1/2, 2n+1)$ of $\overline{T}$ to the $2n+1$ points that have been drawn in $\overline{T}$, which indeed makes $\overline{T}$ an element of $\mathrm{SDC}_{2n+1}$.

For example, we depict in Figure 8 how a labeled tableau $T_2 \in \mathcal{T}_2^o$ is mapped to its corresponding element $\overline{T_2} \in \mathrm{SDC}_5$.

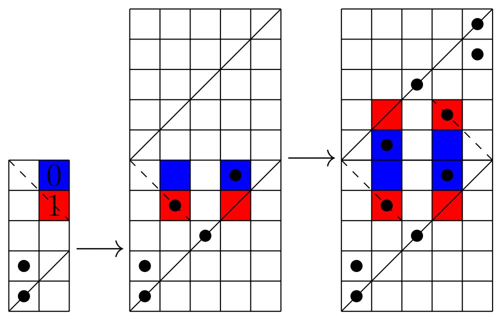


Figure 8: Construction of an odd symmetric Dellac configuration.

This bijection between the tableaux $T \in \mathcal{T}_n^o$ whose free points are labeled with 0 or 1 and $\mathrm{SDC}_{2n+1}$ implies the following formula for all $n \geq 1$:

$$|\mathrm{SDC}_{2n+1}| = \sum_{T \in \mathcal{T}_n^o} 2^{\mathrm{fr}(T)}. \tag{7}$$

For example, for $n = 1$, the $3 = 2^0 + 2^1$ elements of $\mathrm{SDC}_3$ are generated by the 2 elements of $\mathcal{T}_2^o$ as depicted in Figure 9, where the elements of $\mathrm{SDC}_3$ in the same frame under every $T \in \mathcal{T}_1^o$ correspond to the $2^{\mathrm{fr}(T)}$ different labellings of the free points of $T$.

### 2.4.2 A combinatorial interpretation of $P_n(x)$ in terms of $\mathcal{T}_{n-1}^o$

We adapt Definition 1, Definition 2, Proposition 3 and Definition 4 to tableaux $T \in \mathcal{T}_n^o$ as follows:

- first of all, we replace all the occurrences of $2n$ with $2n+1$, which does not change the proof of Proposition 3;

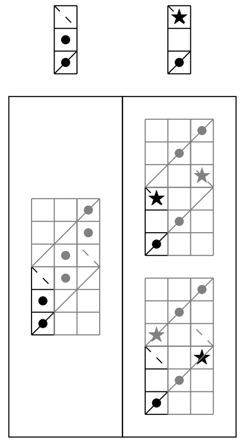

Figure 9: Construction of $\mathrm{SDC}_3$ from $\mathcal{T}_1^o$.

- regarding $S_T(i)$ from Definition 1, if $L_i^T$ is its unique empty row, then $S_T(i)$ is defined as the empty sequence; otherwise, if the points $p_0, \ldots, p_{k_0}$ are defined for some $k_0 \geq 0$, and if the rule b) or the rule c) involves the unique empty row of $T$, then we define $p_k$ as $p_{k_0}$ for all $k > k_0$;
- in Definition 1, instead of $R(T)$ and $B(T)$, we define a single finite sequence $V(T) = S_T(2n+1)$ (possibly empty), whose number of elements is denoted by $\mathrm{v}(T)$;
- the set $\mathcal{G}(T)$ is defined by replacing $B(T)$ with $V(T)$ in Definition 4.

For example, in Figure 10 we represent a tableau $T_3 \in \mathcal{T}_6^o$ such that $V(T_3)$ is the tuple $((2:13),(3:11),(4:10))$ and $\mathcal{G}(T_3) = \{(2:12)\}$. In general, when depicting a tableau $T \in \mathcal{T}_n^o$, the $\mathrm{v}(T)$ elements of $V(T)$ and the $\mathrm{g}(T)$ elements of $\mathcal{G}(T)$ are painted in purple and green respectively.

*Remark* 9. For all $T \in \mathcal{T}_{n-1}^o$, the $\mathrm{v}(T)$ elements of $V(T)$ and the $\mathrm{g}(T)$ elements of $\mathcal{G}(T)$ are pairwise distinct free points, so $\mathrm{fr}(T) \geq \mathrm{v}(T) + \mathrm{g}(T)$.

The second main result of this paper is the following.

**Theorem 10.** *For all $n \geq 2$,*

$$P_n(x) = \sum_{T \in \mathcal{T}_{n-1}^o} 2^{\mathrm{fr}(T) - g(T)} x^{\mathrm{v}(T)} (1+x)^{g(T)}.$$

For example, the two elements of $\mathcal{T}_1^o$ shown in Figure 11 give

$$P_2(x) = 2^{0-0} x^0 (1+x)^0 + 2^{1-0} x^1 (1+x)^0 = 2x + 1,$$

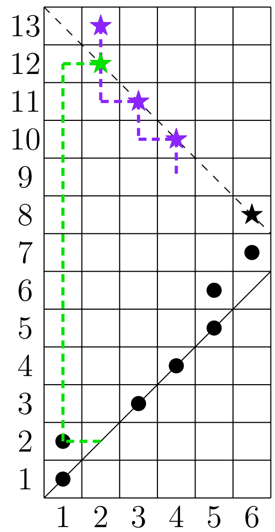


Figure 10: Odd extended Dellac configuration $T_3 \in \mathcal{T}_6^o$ such that $(\mathrm{v}(T_3), \mathrm{g}(T_3)) = (3, 1)$.

and the nine elements of $\mathcal{T}_2^o$ shown in Figure 12 give

$$\begin{aligned} P_3(x) &= 2^{1-0}x^{0}(1+x)^{0} + 2^{1-0}x^{1}(1+x)^{0} + 2^{2-1}x^{1}(1+x)^{1} \\ &\quad + 2^{1-0}x^{0}(1+x)^{0} + 2^{1-0}x^{1}(1+x)^{0} + 2^{1-0}x^{1}(1+x)^{0} \\ &\quad + 2^{0-0}x^{0}(1+x)^{0} + 2^{1-0}x^{1}(1+x)^{0} + 2^{2-0}x^{2}(1+x)^{0} \\ &= 2 + 2x + 2x(1+x) + 2 + 2x + 2x + 1 + 2x + 4x^2 \\ &= 6x^2 + 10x + 5. \end{aligned}$$

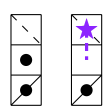

Figure 11: The two elements of $\mathcal{T}_1^o$.

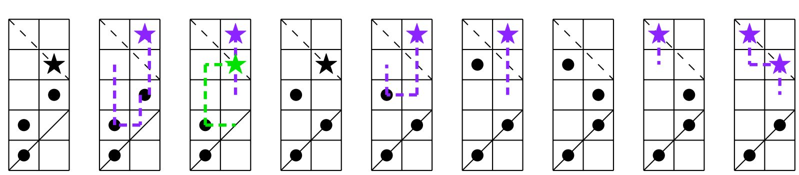

Figure 12: The nine elements of $\mathcal{T}_2^o$.

The cardinality of $\mathrm{SDC}_{2n-1}$ being $P_n(1) = l_n$ for all $n \geq 2$ is then obtained by setting $x = 1$ in Theorem 10, in view of Formula (7). Section 4 is dedicated to the proof of Theorem 10.

# 3 Proof of Theorem 7

For all $n \geq 1$, let

$$E_n(x) = \sum_{T \in \mathcal{T}_n^e} 2^{\mathrm{fr}(T)-1-\max(T)} x^{\max(T)}.$$

By Remark 6 and the inequality $\mathrm{fr}(T) \leq n$ for all $T \in \mathcal{T}_n^e$ (the free points of $T$ are of the kind $(j : i)$ such that $2n \geq i \geq 2n+1-j \geq n+1$, and hence are located in the $n$ top rows), the polynomial $E_n(x)$ has positive integers coefficients and its degree is at most $n-1$. Moreover, if we define $T_{n,\max} \in \mathcal{T}_n^e$ as the tableau whose points are $(j : j)$ and $(j : 2n+1-j)$ for all $j \in [n]$ (see Figure 13 for the case $n = 4$), then $\mathrm{b}(T_{n,\max}) = 0, \mathrm{r}(T_{n,\max}) = n-1$ and $\mathrm{g}(T_{n,\max}) = 0$, so $\max(T_{n,\max}) = n-1$, which makes the degree of $E_n(x)$ exactly $n-1$.

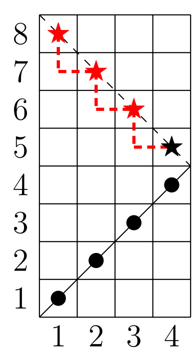


Figure 13: The tableau $T_{4,\max} \in \mathcal{T}_4^e$.

For all $k \in [0, n-1]$, let $\mathcal{T}_{n,k}^e$ be the set of the tableaux $T \in \mathcal{T}_n^e$ such that $\max(T) = k$, and let $e_{n,k} = \sum_{T \in \mathcal{T}_{n,k}^e} 2^{\mathrm{fr}(T)-1-k}$, so that

$$E_n(x) = \sum_{k=0}^{n-1} e_{n,k} x^k.$$

Theorem 7 states that $E_n(x) = P_n(x)$, i.e., that $e_{n,k} = c_{n,k}$ for all $k \in [0, n-1]$. It is clear that $e_{1,0} = 1 = c_{1,0}$. We will prove the rest by checking that $(e_{n,k})_{0 \leq k \leq n}$ satisfies the recursive formulas (1) ,(2),(3). To do so, we first define in the next section a surjection from $\mathcal{T}_n^e$ to $\mathcal{T}_{n-1}^e$.

### 3.1 A surjection $\Pi : \mathcal{T}_n^e \twoheadrightarrow \mathcal{T}_{n-1}^e (n \geq 2)$

Consider $T \in \mathcal{T}_n^e$ and the integers $(j_{n-1}, j_{n+1}) \in [n]^2$ such that the points of $L_{n-1}^T$ and $L_{n+1}^T$ are located in $C_{j_{n-1}}^T$ and $C_{j_{n+1}}^T$ respectively. We define $(i_{\min}, i_{\max})$ as $(n-1, n+1)$ if $j_{n-1} \leq j_{n+1}$, as $(n+1, n-1)$ otherwise.

We define two new paths

$$(B'(T), R'(T)) = \begin{cases} (S_T(i_{\min}), S_T(i_{\max})), & \text{if } C_{n-1}^T \text{ contains an element of } R(T); \\ (S_T(i_{\max}), S_T(i_{\min})), & \text{if } C_{n-1}^T \text{ contains an element of } \mathcal{G}(T); \\ (S_T(n+1), S_T(n-1)), & \text{otherwise and if } p_{n+1}^T \text{ is free}; \\ (S_T(n-1), S_T(n+1)), & \text{otherwise.} \end{cases}$$

We also define a new set $\mathcal{G}'(T)$: the set of the free points of $T$ of the kind $p = (j : i)$ such that the other point of $C_j^T$ is an element of $B'(T)$, and such that $\text{root}_T(j : i) = j$. Their respective numbers of elements located in $C_1^T, C_2^T, \dots, C_{n-1}^T$ are denoted by $\text{b}'(T)$, $\text{r}'(T)$ and $\text{g}'(T)$. Note that $S_T(n+1) = ((n : n+1))$ for all tableau $T$ in which $p_{n+1}^T$ is free, so $r'(T)$ or $b'(T)$ may equal 0. However, the first element of $S_T(n-1)$, the point $p_{n-1}^T$, is always of the kind $(j : n-1)$ for some $j < n$, so $\text{b}'(T) + \text{r}'(T) \geq 1$ in general. The set of the elements of $B'(T), R'(T)$ or $\mathcal{G}'(T)$ is denoted by $\text{Max}'(T)$ whose cardinality is $\text{max}'(T) = \text{b}'(T) + \text{r}'(T) + \text{g}'(T)$. For example, consider the tableau $T_1 \in \mathcal{T}_7^e$ from Figure 5, we depict in Figure 14 the paths $B'(T_1) = ((5 : 6), (6 : 9))$ and $R'(T_1) = ((6 : 8))$ by painting them in light blue and orange respectively (there the set $\mathcal{G}'(T_1)$ is empty).

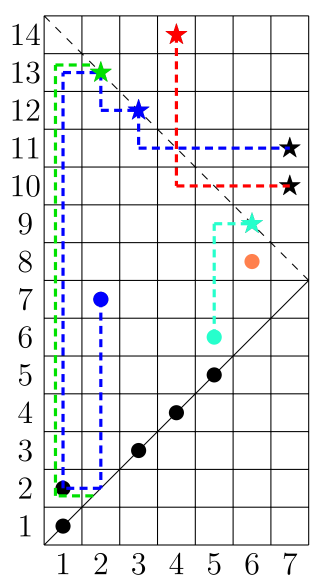


Figure 14: The tableau $T_1 \in \mathcal{T}_7^e$, with $\text{b}'(T_1) = 2, \text{r}'(T_1) = 1$ and $\text{g}'(T_1) = 0$.

In order to construct an element $\Pi(T)$ of $\mathcal{T}_{n-1}^e$ from $T \in \mathcal{T}_n^e$, we "project" all points of $T$ that are not elements of $\text{Max}(T)$ or $\text{Max}'(T)$ in a tableau $(n-1) \times (2n-2)$, then we construct $\mathrm{Max}(\Pi(T))$ as a "union" of $\mathrm{Max}(T)$ and $\mathrm{Max}'(T)$, in a meaning that we will make explicit.

**Definition 11.** Consider an integer $j \in [n]$ and a tableau $T$ that satisfies the conditions of Definition 2. Let $i_f \in [j, 2n-j] \sqcup \{2n\}$ and let $i_0 \in [j, 2n]$ be the unique integer such that $\mathrm{root}_T(j : i_0) = i_f$ in view of Proposition 3. We define the insertion of a point in the box $(j : i_f)$ of $T$ as the plotting of a point in the box $(j : i_0)$.

Note that if $L^T_{i_f}$ has its first $j-1$ boxes empty, then $i_0 = i_f$.

For example, if $T$ is a $7 \times 14$ tableau whose first column is the same as in Figure 14, then the insertions of points in the boxes $(2 : 2)$ and $(2 : 7)$ of $T$ result in plotting points in the boxes $(2 : 13)$ and $(2 : 7)$ of $T$ respectively.

In general, let us now label the points of $\mathrm{Max}(T)$ and $\mathrm{Max}'(T)$ with letters $\rho, \beta$ or $\gamma$. The points labeled with $\beta$ (respectively $\rho, \gamma$) will correspond to the elements of $B(\Pi(T))$ (respectively $R(\Pi(T)), \mathcal{G}(\Pi(T))$).

**Definition 12.** Let $T \in \mathcal{T}^e_n (n \geq 2)$. We label the elements of $\mathrm{Max}(T)$ and $\mathrm{Max}'(T)$ as follows.

- If a point $p_1$ belongs to $B(T)$ (respectively $R'(T)$) and the other point $p_2$ in the same column belongs to $B'(T)$ (respectively $R(T)$), then we label $p_1$ with $\beta$ and $p_2$ with $\gamma$.
- If a point $p$ belongs to $B(T)$ or $B'(T)$ (respectively $R(T)$ or $R'(T)$, $\mathcal{G}(T)$ or $\mathcal{G}'(T)$) and $p$ doe not meet the first rule, then we label $p$ with $\beta$ (respectively $\rho, \gamma$).

**Definition 13** (Map $\Pi : \mathcal{T}^e_n \to \mathcal{T}^e_{n-1}$)**.** Let $T \in \mathcal{T}^e_n$. We first define a tableau $\Pi_0(T)$, made up of $n-1$ columns and $2n-2$ rows, such that for all points $p = (j : i)$ of $T$ that does not belong to $\mathrm{Max}(T)$ or $\mathrm{Max}'(T)$ (in particular $j < n$ and $i \notin \{n, n+1\}$), the tableau $\Pi_0(T)$ contains a point in its box $\left(j : \widetilde{i}\right)$ where the integer $\widetilde{i}$ is defined by

$$\widetilde{i} = \begin{cases} i, & \text{if } i < n; \\ i-2, & \text{if } i > n+1. \end{cases}$$

For example, for the tableau $T_1 \in \mathcal{T}^e_7$ depicted in Figure 14, we obtain the tableau $\Pi_0(T_1)$ of Figure 15. Afterwards, let $X_T$ be the empty set. For $j$ from 1 to $n-1$, if $C^T_j$ contains a point $p$ labeled with $\beta$ (respectively $\rho, \gamma$), we insert a point in the box $(j : n-1)$ of $\Pi_0(T)$ (respectively the box $(j : 2n-2)$, the box $(j : j)$), following Definition 11. We say that *we have inserted $p$ in $\Pi_0(T)$*. Also, if $p \in \mathrm{Max}(T)$, let $p_{\mathrm{new}}$ be the point that we just plotted in $\Pi_0(T)$ following the latter insertion, then we add $p_{\mathrm{new}}$ to $X_T$.

Let $\Pi(T)$ be the tableau obtained at the end of this algorithm. It is straightforward by Definition 11 that $\Pi(T)$ is an element of $\mathcal{T}^e_{n-1}$, and that $X_T \subsetneq \mathrm{Omax}(\Pi(T))$ (it is not equal because the point $p^{\Pi(T)}_{n-1}$ is an element of $\mathrm{Omax}(\Pi(T))$ that results from the insertion in $\Pi_0(T)$ of the point $p^T_{n-1}$, but $p^T_{n-1}$ belongs to $\mathrm{Max}'(T)$, not to $\mathrm{Max}(T)$. Hence $p^{\Pi(T)}_{n-1} \notin X_T$). The set $X_T$ will be useful to reconstruct $T$ from $\Pi(T)$ (see Proposition 20).

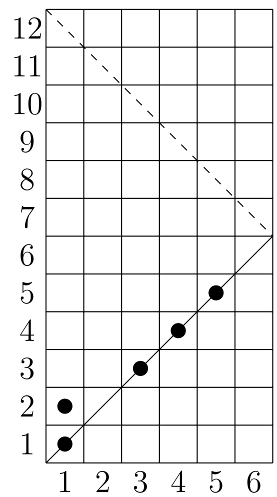


Figure 15: The tableau $\Pi_0(T_1)$.

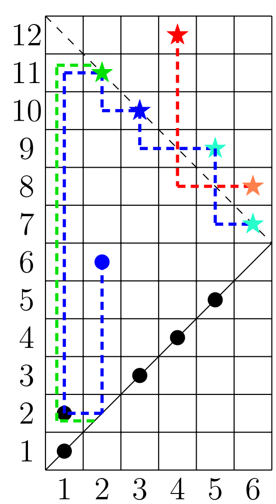


Figure 16: The tableau $\Pi(T_1) \in \mathcal{T}_6^e$.

For example, the tableau $\Pi(T_1) \in \mathcal{T}_6^e$ is as depicted in Figure 16, and $X_{T_1}$ is the tuple $\{(2:6),(2:11),(3:10),(4:12)\}$.

Note that $(\mathrm{b}(\Pi(T_1)), \mathrm{r}(\Pi(T_1)), \mathrm{g}(\Pi(T_1))) = (3,1,1)$.

*Remark* 14. For all $T \in \mathcal{T}_n^e$, we have

$$\max(\Pi(T)) = \max(T) + \max'(T) - 2.$$

Indeed, the insertions in $\Pi_0(T)$ of the two points of $C_{n-1}^T$, which both belong to $\mathrm{Max}(T)$ or $\mathrm{Max}'(T)$ (because their roots belong to $\{n-1, n, n+1, 2n\}$), produce the two points of $C_{n-1}^{\Pi(T)}$, in other words, the last element of $B(\Pi(T))$ and the last element of $R(\Pi(T))$, which are not counted by the statistics b and r.

In particular, since $\max'(T) \geq 1$ in general (the set $\mathrm{Max}'(T)$ always contains $p_{n-1}^T$), we have

$$\max(\Pi(T)) \in [\max(T) - 1, n-2].$$

For example, for the tableau $T_1 \in \mathcal{T}_7^e$, we obtain $\max(\Pi(T_1)) = (2+1+1)+(2+1)-2 = 5$.

We now intend to construct $\Pi^{-1}(T_0)$ for all $T_0 \in \mathcal{T}_{n-1}^e$. To do so, we first need to define labels for the elements of $\mathrm{Omax}(T_0)$, given a subset $X \subsetneq \mathrm{Omax}(T_0)$.

**Definition 15.** For a tableau $T_0 \in \mathcal{T}_{n-1}^e$ and a subset $X \subsetneq \mathrm{Omax}(T_0)$, let us define labels of the elements of $\mathrm{Omax}(T_0)$ as follows.

- For all points $p \in \mathcal{G}(T_0)$, let $q \neq p$ be the other point of the column of $p$ (the point $q$ is then an element of $B(T_0)$).
  - If neither $p$ nor $q$ belong to $X$, we label $p$ and $q$ with the respective letters $\mathfrak{g}'$ and $\mathfrak{b}'$.
  - If only $q$ belongs to $X$, we label $p$ and $q$ with the respective letters $\mathfrak{b}'$ and $\mathfrak{b}$.
  - If only $p$ belongs to $X$, we label $p$ and $q$ with the respective letters $\mathfrak{r}$ and $\mathfrak{r}'$.
  - If both $p$ and $q$ belong to $X$, we label $p$ and $q$ with the respective letters $\mathfrak{g}$ and $\mathfrak{b}$.
- For all points $p$ among the first $b_0(T)$ elements of $B(T_0)$ (respectively among the first $r(T_0)$ elements of $R(T_0)$) that are not covered by the first case, if $p \notin X$, then we label $p$ with $\mathfrak{b}'$ (respectively $\mathfrak{r}'$), otherwise we label $p$ with $\mathfrak{b}$ (respectively $\mathfrak{r}$).
- Let $p^b$ (respectively $p^r$) be the last element of $B(T_0)$ (respectively of $R(T_0)$). Recall that $p^b$ and $p^r$ are the two points of $C_{n-1}^{T_0}$.
  - If $p^r \notin X$, we label $p^r$ with the letter $\mathfrak{r}'$. Afterwards, we label $p^b$ with the letter $\mathfrak{b}'$ if $p^b \notin X$, with the letter $\mathfrak{b}$ otherwise.

– Otherwise, let $j_{\min}$ be the smallest integer $j \in [n-1]$ such that $C_j^{T_0}$ contains an element of $\mathrm{Omax}(T_0)\backslash X$. If $j_{\min} = n-1$, then $\mathrm{Omax}(T_0) = \{p^b, p^r\}$ and hence $X = \{p^r\}$, and we label $p^b$ and $p^r$ with the respective letters $\mathfrak{b}'$ and $\mathfrak{r}$. If $j_{\min} < n-1$, all the elements of $\mathrm{Omax}(T_0)$ that appear in $C_{j_{\min}}^{T_0}$ have already been labeled. We then have two cases:

  * If $C_{j_{\min}}^{T_0}$ contains an element labeled with $\mathfrak{b}'$, then we label $p^r$ with the letter $\mathfrak{r}$. Afterwards, we label $p^b$ with the letter $\mathfrak{b}'$ if $p^b \notin X$, with the letter $\mathfrak{b}$ otherwise.
  * Otherwise, we label $p^b$ and $p^r$ with the respective letters $\mathfrak{r}'$ and $\mathfrak{r}$ if $p^b \notin X$, with the respective letters $\mathfrak{b}$ and $\mathfrak{g}$ otherwise.

*Remark* 16. In the context of Definition 15, the elements of $\mathrm{Omax}(T_0)$ that are labeled with $\mathfrak{b}, \mathfrak{r}$ or $\mathfrak{g}$ are exactly the elements of $X$.

**Definition 17.** Let $i \in [0, n-2]$, and $T_0 \in \mathcal{T}_{n-1,i}^e$. We consider a subset $X \subsetneq \mathrm{Omax}(T_0)$, whose cardinality $|X|$ is denoted by $k$. Note that $\mathrm{omax}(T_0) = i+2$ hence $k \in [0, i+1]$. We intend to define a set $\mathfrak{T}(T_0, X) \subset \mathcal{T}_n^e$ whose cardinality is either 1 or 2 following three situations described below. Consider the labels introduced in Definition 15.

1. Let $\mathcal{S}_1$ be the situation where $k \leq i$ and no point of the last column $C_{n-1}^{T_0}$ of $T_0$ is labeled with $\mathfrak{r}$ or $\mathfrak{g}$.

2. Let $\mathcal{S}_2$ be the situation where:

   - either $k = i+1$;
   - or one of the points of $C_{n-1}^{T_0}$ is labeled with $\mathfrak{r}$ or $\mathfrak{g}$, and there are no two elements of $\mathrm{Omax}(T_0)\backslash X$ labeled with $\mathfrak{b}'$ and $\mathfrak{r}'$.

3. Let $\mathcal{S}_3$ be the remaining situation, i.e., where $k \leq i$, one of the points of $C_{n-1}^{T_0}$ is labeled with $\mathfrak{r}$ or $\mathfrak{g}$, and there exist two elements of $\mathrm{Omax}(T_0)\backslash X$ labeled with $\mathfrak{b}'$ and $\mathfrak{r}'$.

In any situation of the three above situations, first consider a $n \times 2n$ tableau $\mathfrak{T}_0(T_0)$ containing points only in its boxes $(j : \bar{i})$ for all points $(j : i)$ of $T_0$ that does not belong to $\mathrm{Omax}(T_0)$, where $\bar{i}$ is defined as $i$ if $i < n$, as $i+2$ otherwise. Let us set $T^1 = T^2 = T^{3/} = T^{3\backslash} = \mathfrak{T}_0(T_0)$. For $j$ from 1 to $n-1$, assume that $C_j^{T_0}$ contains an element $p \in \mathrm{Omax}(T_0)$ (otherwise $T^1, T^2, T^{3/}, T^{3\backslash}$ already contain two points in their $j$-th columns and we do nothing).

- If the label of $p$ is $\mathfrak{b}$ (respectively $\mathfrak{r}, \mathfrak{g}, \mathfrak{g}'$), we insert a point in the box $(j : n)$ (respectively $(j : 2n), (j : j), (j : j)$) of $T^1, T^2, T^{3/}$ and $T^{3\backslash}$.

- If the label of $p$ is $\mathfrak{b}'$ (respectively $\mathfrak{r}'$),

  – in the situation $\mathcal{S}_1$, we insert a point in the box $(j : n-1)$ (respectively $(j : n+1)$) of $T^1$;

- in the situation $\mathcal{S}_2$, we insert a point in the box $(j : n-1)$ of $T^2$;
- in the situation $\mathcal{S}_3$,
  - if the points of the last column of $T_0$ have the labels $\mathfrak{r}'$ and $\mathfrak{r}$, or $\mathfrak{b}$ and $\mathfrak{g}$, then we insert a point in the box $(j : n+1)$ (respectively $(j : n-1)$) of $T^{3/}$, and we insert a point in the box $(j : n-1)$ (respectively $(j : n+1)$) of $T^{3\backslash}$;
  - otherwise, we insert a point in the box $(j : n-1)$ (respectively $(j : n+1)$) of $T^{3/}$, and we insert a point in the box $(j : n+1)$ (respectively $(j : n-1)$) of $T^{3\backslash}$.

We say that *we inserted $p$ in* $T^1$ (respectively $T^2$, $T^{3/}$ and $T^{3\backslash}$) in the situation $\mathcal{S}_1$ (respectively $\mathcal{S}_2, \mathcal{S}_3$). At the end of this algorithm, we finally insert two points in the boxes $(n : n)$ and $(n : 2n)$ of $T^1$ (respectively $T^2$, $T^{3/}$ and $T^{3\backslash}$) in the situation $\mathcal{S}_1$ (respectively $\mathcal{S}_2, \mathcal{S}_3$).

The choice of $T_0$ and $X \subsetneq \mathrm{Omax}(T_0)$ results in one of the three situations $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$, depending on whether $T^1$, or $T^2$, or both $T^{3/}$ and $T^{3\backslash}$ belong to $\mathcal{T}_n^e$ at the end of the above algorithm. Let $\mathfrak{T}(T_0, X) \subset \mathcal{T}_n^e$ be defined as $\{T^1\}$ (respectively $\{T^2\}, \{T^{3/}, T^{3\backslash}\}$) in the situation $\mathcal{S}_1$ (respectively $\mathcal{S}_2, \mathcal{S}_3$).

*Remark* 18. If $T \in \mathfrak{T}(T_0, X)$, then by construction the cardinality of $\mathrm{Max}(T)$ is the number of points of $\mathrm{Omax}(T_0)$ labeled with $\mathfrak{b}, \mathfrak{r}$ or $\mathfrak{g}$, in other words it equals the cardinality $k$ of $X$ by Remark 16, and $\mathfrak{T}(T_0, X) \subset \mathcal{T}_{n,k}^e$.

*Remark* 19. In the context of Definition 17, if a tableau is defined in the situation $\mathcal{S}_1$ or $\mathcal{S}_3$, then the point $p_{n+1}^T$ is not free, in other words the box $(n : n+1)$ of $T$ is empty, whereas in the situation $\mathcal{S}_2$ the point $p_{n+1}^T$ is $(n : n+1)$ hence is free. Indeed, in the situation $\mathcal{S}_1$, the points of $C_{n-1}^{T_0}$ have the labels $\mathfrak{b}'$ and $\mathfrak{r}'$, so the points of $C_{n-1}^T$ are obtained by inserting two points in the boxes $(n-1 : n-1)$ and $(n-1 : n+1)$ of $T^1$, which implies that $p_{n+1}^T$ is located in the first $n-1$ columns of $T$ and $p_{n+1}^T$ is then not free; in the situation $\mathcal{S}_3$, since there exists two elements of $\mathrm{Omax}(T_0)\backslash X$ with the labels $\mathfrak{b}'$ and $\mathfrak{r}'$, by construction we insert a point in boxes $(j_1 : n-1)$ and $(j_2 : n+1)$ for some $j_1, j_2 < n$ in both tableaux $T^{3/}$ and $T^{3\backslash}$, so $p_{n+1}^T$ is also not free; finally, in the situation $\mathcal{S}_2$, we never insert a point in any box $(j : n+1)$ for all $j < n$, so $p_{n+1}^T$ is necessarily located in $C_n^T$ hence is free.

Let us now compare the number $\mathrm{fr}(T_0)$ to the numbers $\mathrm{fr}(T)$ for all $T \in \mathfrak{T}(T_0, X)$. First of all, note that $\mathrm{fr}(T_0)$ equals the number of free points located in $\mathfrak{T}_0(T_0)$ (which we denote by $\mathrm{fr}_0(T_0)$), and that $\mathrm{omax}(T_0) - 1 = i + 1$ in view of Remark 6. Likewise, if $T \in \mathfrak{T}(T_0, X)$, by construction

$$\mathrm{fr}(T) = \mathrm{fr}_0(T_0) + (\max(T) + 1) + \max'(T) - \eta$$

where $\eta$ is the number of non-free points of $\mathrm{Max}'(T)$, i.e., $\eta = 1$ if $p_{n+1}^T$ is free (in which case $p_{n-1}^T$ is the only non-free point of $\mathrm{Max}'(T)$), which happens exactly in the situation $\mathcal{S}_2$, or $\eta = 2$ otherwise, in the situations $\mathcal{S}_1$ and $\mathcal{S}_3$. Since $\max(T) = k$ (in view of Remark 18) and $\max'(T) = \mathrm{omax}(T_0) - |X| = i + 2 - k$ by construction, we then have

$$\begin{cases} \mathrm{fr}(T) = \mathrm{fr}_0(T_0) + i + 3 - 2 = \mathrm{fr}(T_0), & \text{in the situation } \mathcal{S}_1 \text{ or } \mathcal{S}_3; \\ \mathrm{fr}(T) = \mathrm{fr}_0(T_0) + i + 3 - 1 = \mathrm{fr}(T_0) + 1, & \text{in the situation } \mathcal{S}_2. \end{cases}$$

**Proposition 20.** *For all $i \in [0, n-2]$ and $T_0 \in \mathcal{T}_{n-1,i}$, we have $\Pi^{-1}(T_0) \subset \bigsqcup_{k=0}^{i+1} \mathcal{T}_{n,k}^e$, and for all $k \in [0, i+1]$,*

$$\Pi^{-1}(T_0) \cap \mathcal{T}_{n,k}^e = \bigsqcup_{\substack{X \subset \mathrm{Omax}(T_0),\\ |X|=k}} \mathfrak{T}(T_0, X). \tag{8}$$

*Therefore $\Pi$ is surjective.*

*Proof.* Let $T \in \mathcal{T}_n^e$, if $\Pi(T) = T_0$, then $i \in [\max(T)-1, n-2]$ by Remark 6. Hence we have $\Pi^{-1}(T_0) \subset \bigsqcup_{k=0}^{i+1} \mathcal{T}_{n,k}^e$.

Let us now prove that the elements $T \in \mathfrak{T}(T_0, X)$ for all $X \subsetneq \mathrm{Omax}(T_0)$ are indeed mapped to $T_0$ by $\Pi$. It is straightforward that $\Pi_0(T)$ is obtained by erasing all the elements of $\mathrm{Omax}(T_0)$ from $T_0$. Afterwards, for a point $p \in \mathrm{Omax}(T_0)$, let $\bar{p} \in \mathrm{Max}(T) \sqcup \mathrm{Max}'(T)$ be the point of $T$ obtained by inserting $p$ in $\mathfrak{T}_0(T_0)$. One can check that the label of $\bar{p}$ as defined by Definition 12 is $\beta$ (respectively $\rho, \gamma$) if and only if $p \in B(T_0)$ (respectively $p \in R(T_0), \mathcal{G}(T_0)$), so we obtain the equality $\Pi(T) = T_0$ and the inclusion $\mathfrak{T}(T_0, X) \subset \Pi^{-1}(T_0)$.

Reciprocally, for all $T \in \Pi^{-1}(T_0)$, we show that $T \in \mathfrak{T}(T_0, X_T)$. It is straightforward that $\mathfrak{T}_0(T_0)$ is obtained by erasing all the elements of $\mathrm{Omax}(T)$ and $\mathrm{Max}'(T)$ from $T$. Afterwards, for a point $p \in \mathrm{Max}(T) \sqcup \mathrm{Max}'(T)$, let $\tilde{p} \in \mathrm{Omax}(T_0)$ be the point of $T_0$ that was obtained by inserting $p$ in in $\Pi_0(T)$. One can check that if $p \in B(T)$ (respectively $p \in R(T), \mathcal{G}(T), B'(T), R'(T), \mathcal{G}'(T)$), then the label of $\tilde{p}$ as defined by Definition 15 with $X = X_T$ (recall that $X_T$ is defined at the same time as $\Pi(T)$ in Definition 13) is $\mathfrak{b}$ (respectively $\mathfrak{r}, \mathfrak{g}, \mathfrak{b}', \mathfrak{r}', \mathfrak{g}'$). Consequently, if $\mathfrak{T}(T_0, X_T)$ is defined in the situation $\mathcal{S}_1$ or $\mathcal{S}_2$, then $\mathfrak{T}(T_0, X_T) = \{T\}$. If $\mathfrak{T}(T_0, X_T)$ is defined in the situation $\mathcal{S}_3$, then $\mathfrak{T}(T_0, X_T) = \{T^{3/}, T^{3\backslash}\}$ where $T = T^{3/}$ if $i_{\min}^T = n-1$, otherwise $T = T^{3\backslash}$.

So $\Pi^{-1}(T_0) = \bigsqcup_{X \subsetneq \mathrm{Omax}(T_0)} \mathfrak{T}(T_0, X)$. Formula (8) then follows from Remark 18. □

**Lemma 21.** *Let $T_0 \in \mathcal{T}_{n-1}^e, X \subsetneq \mathrm{Omax}(T_0)$ and $T \in \mathfrak{T}(T_0, X)$. The following assertions are equivalent.*

*(i) The set $\mathfrak{T}(T_0, X)$ is defined in the situation $\mathcal{S}_1$ of Definition 17.*

*(ii) The point $p_{n+1}^T$ is not free and no point of $C_{n-1}^T$ is an element of $R(T)$ or $\mathcal{G}(T)$.*

*(iii) The set $X$ does not contain the last element $p^r$ of $R(T_0)$, and $|X| \leq \max(T_0)$.*

*Proof.* The equivalence (i) $\Leftrightarrow$ (iii) is straightforward by the third point of Definition 15.

Let us prove that (i) implies (ii). Suppose that $\mathfrak{T}(T_0, X)$ is defined in the situation $\mathcal{S}_1$ of Definition 17, and hence $\mathfrak{T}(T_0, X) = \{T\}$. We know that $\max(T) = |X| \leq \max(T_0)$ by hypothesis, so $\max'(T) = \mathrm{omax}(T_0) - |X| \geq 2$. Now, as stated in the proof of Proposition 20, the hypothesis that no point of $C_{n-1}^{T_0}$ has the label $\mathfrak{r}$ or $\mathfrak{g}$ implies that no point of $C_{n-1}^T$ belongs to $R(T)$ or $\mathcal{G}(T)$. It remains to prove that $p_{n+1}^T$ is not free. In general, by Proposition 3, the roots of the points of $C_{n-1}^T$ belong to $\{n-1, n, n+1, 2n\}$. Since no point of $C_{n-1}^T$ belongs to $R(T)$ by hypothesis, their roots cannot be $2n$. If a point $p$ of $C_{n-1}^T$ has the root $n+1$, then obviously $p_{n+1}^T$ is located in one of the first $n-1$ columns of $T$, so $p_{n+1}^T$ is not free.

Otherwise, one point $p \in C_{n-1}^T$ has the root $n-1$ and the other point $q$ of $C_{n-1}^T$ has the root $n$. It implies that $q$ is an element of $B(T)$. Since no point of $C_{n-1}^T$ belongs to $R(T)$ or $\mathcal{G}(T)$, in particular $p \notin \mathcal{G}(T)$, so $p$ cannot be free by definition of $\mathcal{G}(T)$. It follows that $p = (n-1 : n-1)$ and $S_T(n-1) = (p)$. Since $\max'(T) \geq 2$, the sequence $S_T(n+1)$ has at least one element located in the first $n-1$ columns of $T$, in other words $p_{n+1}^T$ (the first element of $S_T(n+1)$) is not free. So (i) implies (ii).

Let us now prove that (ii) implies (i). If $p_{n+1}^T$ is not free and no point of $C_{n-1}^T$ belongs to $R(T)$ or $\mathcal{G}(T)$, as stated in the proof of Proposition 20, no point of $C_{n-1}^T$ has the label $\mathfrak{r}$ or $\mathfrak{g}$. Also, since $\max'(T) \geq 2$ because $\mathrm{Max}(T)$ contains $p_{n-1}^T$ in general, and $p_{n+1}^T$ in this case (because $p_{n+1}^T$ is not free), we have $|X| = \max(T) = \mathrm{omax}(T_0) - \max'(T) = \max(T_0) + 2 - \max'(T) \leq \max(T_0)$. So (ii) implies (i). □

**Lemma 22.** *Let $T_0 \in \mathcal{T}_{n-1}^e$, $X \subsetneq \mathrm{Omax}(T_0)$ and $T \in \mathfrak{T}(T_0, X)$. The following assertions are equivalent.*

- *(i) The set $\mathfrak{T}(T_0, X)$ is defined in the situation $\mathcal{S}_2$ of Definition 17.*
- *(ii) The point $p_{n+1}^T$ is free.*
- *(iii) If $|X| < \max(T_0) + 1$, then the last element $p^r$ of $R(T_0)$ belongs to $X$, and no two elements of $\mathrm{Omax}(T_0) \backslash X$ are labeled with $\mathfrak{b}'$ and $\mathfrak{r}'$.*

*Proof.* The equivalence (i) ⇔ (iii) is straightforward by the third point of Definition 15.

Let us prove that (i) implies (ii). Suppose that $\mathfrak{T}(T_0, X)$ is defined in the situation $\mathcal{S}_2$ of Definition 17, and hence $\mathfrak{T}(T_0, X) = \{T\}$. If $|X| = \max(T_0) + 1$, then $\max'(T) = \mathrm{omax}(T_0) - |X| = 1$, so $\mathrm{Max}'(T) = \{p_{n-1}^T\}$ and $p_{n+1}^T$ is free (because $p_{n+1}^T$ does not belong to $\mathrm{Max}'(T)$). If $|X| < \max(T_0) + 1$, one of the points of $C_{n-1}^{T_0}$ is labeled with $\mathfrak{r}$ or $\mathfrak{g}$, so one of the points of $C_{n-1}^T$ is labeled with $\rho$ or $\gamma$. Assume now that $p_{n+1}^T$ is not free, then one of the two points $p_{n-1}^T$ and $p_{n+1}^T$ is an element of $B'(T)$ and the other is an element of $R'(T)$, which by Definition 12 have labels $\beta$ and $\rho$, and their insertions in $\Pi_0(T)$ plot two points labeled with $\mathfrak{b}'$ and $\mathfrak{r}'$, which contradicts (i). So (i) implies (ii).

Let us prove that (ii) implies (i). Suppose that $p_{n+1}^T$ is free. Recall that the set $X = X_T$ has the cardinality $\max(T)$ by Remark 18, so if $\max(T) = \max(T_0) + 1$ then we indeed have (i). Suppose now that $\max(T) < \max(T_0) + 1$. In general, by Proposition 3, the roots of the points of $C_{n-1}^T$ belong to $\{n-1, n, n+1, 2n\}$. Since $p_{n+1}^T$ is free, these roots cannot be $n+1$. Since $\max'(T) = \mathrm{omax}(T_0) - \max(T) \geq 2$, the point $p_{n-1}^T$ cannot be located in the box $(n-1 : n-1)$ (otherwise $\mathrm{Max}'(T)$ would be $\{p_{n-1}^T\}$ because $p_{n+1}^T$ is free, whereas its cardinality exceeds 2). Consequently, if the roots of the points of $C_{n-1}^T$ are $n-1$ and $n$, then the point $p$ of $C_{n-1}^T$ whose root is $n-1$ is free, which implies that $p$ is an element of $\mathcal{G}(T)$ (because the point whose root is $n$ belongs to $B(T)$), and hence $p$ is labeled with $\gamma$ and its insertion in $\Pi_0(T)$ plots a point labeled with $\mathfrak{g}$ in $C_{n-1}^{T_0}$. Otherwise, the roots of the points of $C_{n-1}^T$ are $n$ and $2n$, in particular the point $q$ of $C_{n-1}^T$ whose root is $2n$ belongs to $R(T)$, so $q$ is labeled with $\rho$ and its insertion in $\Pi_0(T)$ plots a point labeled with $\mathfrak{r}$ in $C_{n-1}^{T_0}$. In both cases, we obtain (i). □

The following lemma is a consequence of Lemma 21 and Lemma 22.

**Lemma 23.** *Let $T_0 \in \mathcal{T}^e_{n-1}$, $X \subsetneq \mathrm{Omax}(T_0)$ and $T \in \mathfrak{T}(T_0, X)$. The following assertions are equivalent.*

1. *The set $\mathfrak{T}(T_0, X)$ is defined in the situation $\mathcal{S}_3$ of Definition 17.*
2. *The point $p^T_{n+1}$ is not free and one point of $C^T_{n-1}$ is an element of $R(T)$ or $\mathcal{G}(T)$.*
3. *The set $X$ contains the last element $p^r$ of $R(T_0)$, $|X| \leq \max(T_0)$, and no two elements of $\mathrm{Omax}(T_0)\backslash X$ are labeled with $\mathfrak{b}'$ and $\mathfrak{r}'$.*

**Proposition 24.** *For all $i \in [0, n-2]$ and $T_0 \in \mathcal{T}_{n-1,i}$, the set $\Pi^{-1}(T_0) \subset \bigsqcup_{k=0}^{i+1} \mathcal{T}^e_{n,k}$ is partitioned as follows:*

- *$\Pi^{-1}(T_0) \cap \mathcal{T}^e_{n,0}$ has one single element, which has $\mathrm{fr}(T_0)$ free points;*
- *$\Pi^{-1}(T_0) \cap \mathcal{T}^e_{n,i+1}$ has $i+2$ elements, each of which has $\mathrm{fr}(T_0)+1$ free points;*
- *for all $k \in [i]$, let $N_k(T_0)$ be the number of elements $T \in \Pi^{-1}(T_0) \cap \mathcal{T}^e_{n,k}$ such that $p^T_{n+1}$ is free. All these tableaux have $\mathrm{fr}(T_0)+1$ free points. Afterwards, in addition to these tableaux, the set $\Pi^{-1}(T_0) \cap \mathcal{T}^e_{n,k}$ is also composed of*
  - *$\binom{i+1}{k}$ tableaux whose $(n-1)$-th column contains no element of $R(T)$ or $\mathcal{G}(T)$, each of which has $\mathrm{fr}(T_0)$ free points;*
  - *$2\left(\binom{i+1}{k-1} - N_k(T_0)\right)$ other tableaux, each of which has $\mathrm{fr}(T_0)$ free points.*

*Proof.* By Formula (8), the set $\Pi^{-1}(T_0) \cap \mathcal{T}^e_{n,0}$ is $\mathfrak{T}(T_0, \emptyset)$. By Lemma 21, the set $\mathfrak{T}(T_0, \emptyset)$ is defined in the situation $\mathcal{S}_1$ of Definition 17. Hence $\mathfrak{T}(T_0, \emptyset)$ has one single element $T^1$ such that $\mathrm{fr}(T^1) = \mathrm{fr}(T_0)$ in view of Remark 19.

Still by Formula (8), the set $\Pi^{-1}(T_0) \cap \mathcal{T}^e_{n,i+1}$ is

$$\bigsqcup_{\substack{X \subset \mathrm{Omax}(T_0),\\ |X|=i+1}} \mathfrak{T}(T_0, X),$$

and for every of the $\binom{i+2}{i+1} = i+2$ subsets $X \subset \mathrm{Omax}(T_0)$ whose cardinalities are $i+1$, the set $\mathfrak{T}(T_0, X)$ is defined in the situation $\mathcal{S}_2$ of Definition 17 in view of Lemma 22, so $\mathfrak{T}(T_0, X)$ contains one single element $T^2$ such that $\mathrm{fr}(T^2) = \mathrm{fr}(T_0)+1$ in view of Remark 19.

Let $k \in [i]$. Consider the tableaux $T \in \Pi^{-1}(T_0) \cap \mathcal{T}^e_{n,k}$ such that $p^T_{n+1}$ is not free and no point of $C^T_{n-1}$ is an element of $R(T)$ or $\mathcal{G}(T)$. By Formula (8) and Lemma 21, the set of these tableaux is $\bigsqcup_X \mathfrak{T}(T_0, X)$ where the union is over the subsets $X \subset \mathrm{Omax}(T_0)\backslash\{p^r\}$ (where $p^r$ is the last element of $R(T_0)$) such that $|X| = k$. Since $\mathrm{omax}(T_0) = i+2$, there are $\binom{i+1}{k}$ such subsets, and for every of these subsets $X$, the set $\mathfrak{T}(T_0, X)$ is defined in the situation $\mathcal{S}_1$ of Definition 17, and hence $\mathfrak{T}(T_0, \emptyset)$ has one single element $T^1$ such that $\mathrm{fr}(T^1) = \mathrm{fr}(T_0)$ in view of Remark 19.

Consider now the $N_k(T_0)$ tableaux $T \in \Pi^{-1}(T_0) \cap \mathcal{T}^e_{n,k}$ such that $p^T_{n+1}$ is free. By Formula (8) and Lemma 22, the set of these tableaux is the union $\bigsqcup_X \mathfrak{T}(T_0, X)$ over the subsets $X \subset \mathrm{Omax}(T_0)$ such that $p^r \in X, |X|= k$ and no two elements of $\mathrm{Omax}(T_0)\backslash X$ are labeled with $\mathfrak{b}'$ and $\mathfrak{r}'$. For every of these subsets $X$, the set $\mathfrak{T}(T_0, X)$ is defined in the situation $\mathcal{S}_2$ of Definition 17, and hence $\mathfrak{T}(T_0, X)$ has one single element $T^2$ such that $\mathrm{fr}(T^2) = \mathrm{fr}(T_0) + 1$ in view of Remark 19. Consequently, the number of subsets $X \subset \mathrm{Omax}(T_0)$ that contain $p^r$, whose cardinality is $k$, and such that no two elements of $\mathrm{Omax}(T_0)\backslash X$ are labeled with $\mathfrak{b}'$ and $\mathfrak{r}'$, is exactly $N_k(T_0)$.

Finally, consider the tableaux $T \in \Pi^{-1}(T_0) \cap \mathcal{T}^e_{n,k}$ such that $p^T_{n+1}$ is not free and one of the points of $C^T_{n-1}$ is an element of $R(T)$ or $\mathcal{G}(T)$. In view of the above paragraphs, they are exactly the tableaux $T \in \Pi^{-1}(T_0) \cap \mathcal{T}^e_{n,k}$ defined in the situation $\mathcal{S}_3$ of Definition 17. By Lemma 23, the set of these tableaux is $\bigsqcup_X \mathfrak{T}(T_0, X)$ where the union is over the subsets $X \subset \mathrm{Omax}(T_0)$ such that $p^r \in X, |X|= k$, and there exist two elements of $\mathrm{Omax}(T_0)\backslash X$ labeled with $\mathfrak{b}'$ and $\mathfrak{r}'$. Since there exist $\binom{i+1}{k-1}$ subsets $X \subset \mathrm{Omax}(T_0)$ that contain $p^r$ and whose cardinality is $k$, in view of the previous paragraph, the number of these subsets $X$ such that there exist two elements of $\mathrm{Omax}(T_0)\backslash X$ labeled with $\mathfrak{b}'$ and $\mathfrak{r}'$ is exactly $\binom{i+1}{k-1} - N_k(T_0)$. For every of these subsets $X$, the set $\mathfrak{T}(T_0, X)$ is defined in the situation $\mathcal{S}_3$ of Definition 17. Hence $\mathfrak{T}(T_0, X)$ has two elements $T^{3/}$ and $T^{3\backslash}$ such that $\mathrm{fr}(T^{3/}) = \mathrm{fr}(T^{3\backslash}) = \mathrm{fr}(T_0)$ in view of Remark 19, so the total number of tableaux $T \in \Pi^{-1}(T_0) \cap \mathcal{T}^e_{n,k}$ defined in the situation $\mathcal{S}_3$ is exactly $2\left(\binom{i+1}{k-1} - N_k(T_0)\right)$. □

**Corollary 25.** *The integers $(e_{n,k})_{0\leq k\leq n-1}$ satisfy the recursive formulas (1), (2), and (3). This proves Theorem 7 in view of $e_{1,0} = 1 = c_{1,0}$.*

*Proof.* Let $n \geq 2$. By Proposition 20, we have $\mathcal{T}^e_{n,k} \subset \bigsqcup_{i=k-1}^{n-2} \Pi^{-1}(\mathcal{T}^e_{n-1,i})$ for all $k \in [n-1]$, and:

- For all $i \in [0, n-2]$, each tableau $T_0 \in \mathcal{T}^e_{n-1}$ gives birth to one tableau $T \in \mathcal{T}^e_{n,0}$, such that $\mathrm{fr}(T) = \mathrm{fr}(T_0)$. Hence

$$e_{n,0} = \sum_{T_0\in\mathcal{T}^e_{n-1}} 2^{\mathrm{fr}(T_0)-1} = \sum_{i=0}^{n-2} 2^i e_{n-1,i},$$

  which is Formula (1).

- For all $k \in [n-2]$,

  1. each tableau $T_0 \in \mathcal{T}^e_{n-1,k-1}$ generates $k+1$ tableaux $T \in \mathcal{T}^e_{n,k}$, such that $\mathrm{fr}(T) = \mathrm{fr}(T_0) + 1$. Hence

$$\sum_{\substack{T\in\mathcal{T}^e_{n,k},\\ \Pi(T)\in\mathcal{T}^e_{n-1,k-1}}} 2^{\mathrm{fr}(T)-1-k} = (k+1) \sum_{T_0\in\mathcal{T}^e_{n-1,k-1}} 2^{\mathrm{fr}(T_0)-1-(k-1)} = (k+1)e_{n-1,k-1};$$

2. for all $i \in [k, n-2]$, each tableau $T_0 \in \mathcal{T}^e_{n-1,i}$ gives birth to $\binom{i+1}{k}+2\left(\binom{i+1}{k-1} - N_k(T_0)\right)$ tableaux $T \in \mathcal{T}^e_{n,k}$ such that $\mathrm{fr}(T) = \mathrm{fr}(T_0)$, and $N_k(T_0)$ tableaux $T \in \mathcal{T}^e_{n,k}$ such that $\mathrm{fr}(T) = \mathrm{fr}(T_0) + 1$. Hence the sum $\sum_{\substack{T\in\mathcal{T}^e_{n,k},\\ \Pi(T)\in\mathcal{T}^e_{n-1,i}}} 2^{\mathrm{fr}(T)-1-k}$ equals

$$\begin{aligned}
&\sum_{T_0 \in\mathcal{T}^e_{n-1,i}} \left(\binom{i+1}{k}+2\left(\binom{i+1}{k-1}-N_k(T_0)\right)\right)2^{\mathrm{fr}(T_0)-1-k}+N_k(T_0)2^{\mathrm{fr}(T_0)-k}\\
&= 2^{i-k}\left(\binom{i+1}{k}+2\binom{i+1}{k-1}\right)\sum_{T_0\in\mathcal{T}^e_{n-1,i}} 2^{\mathrm{fr}(T_0)-1-i}\\
&= 2^{i-k}\left(\binom{i+1}{k}+2\binom{i+1}{k-1}\right)e_{n-1,i}.
\end{aligned}$$

The equalities of (i) and (ii) give

$$\sum_{T\in\mathcal{T}^e_{n,k}} 2^{\mathrm{fr}(T)-1-k} = (k+1)e_{n-1,k-1} + 2^{i-k}\left(\binom{i+1}{k}+2\binom{i+1}{k-1}\right)e_{n-1,i},$$

which is Formula (2).

- Finally note that $\Pi(\mathcal{T}^e_{n,n-1}) \subset \mathcal{T}^e_{n-1,n-2}$, and each tableau $T_0 \in \mathcal{T}^e_{n-1,n-2}$ generates $n$ tableaux $T \in \mathcal{T}^e_{n,n-1}$, such that $\mathrm{fr}(T) = \mathrm{fr}(T_0) + 1$. Hence

$$\begin{aligned}
e_{n,n-1} = \sum_{T\in\mathcal{T}^e_{n,n-1}} 2^{\mathrm{fr}(T)-1-(n-1)} &= n\sum_{T_0\in\mathcal{T}^e_{n-1,k-1}} 2^{\mathrm{fr}(T_0)-1-(n-2)}\\
&= ne_{n-1,n-2},
\end{aligned}$$

which is Formula (3).

This completes the proof. □

# 4 Proof of Theorem 10

In this section, we construct a surjection $\mathcal{P} : \mathcal{T}^e_n \twoheadrightarrow \mathcal{T}^o_{n-1} (n \geq 2)$ such that

$$\sum_{T\in\mathcal{P}^{-1}(T_0)} 2^{\mathrm{fr}(T)-1-\max(T)}x^{\max(T)} = 2^{\mathrm{fr}(T_0)-\mathrm{g}(T_0)}x^{\mathrm{v}(T_0)}(1+x)^{\mathrm{g}(T_0)} \tag{9}$$

for all $T_0 \in \mathcal{T}^o_{n-1}$, which proves Theorem 10 in view of Theorem 7.

**Definition 26.** Let $T \in \mathcal{T}^e_n$. We label the elements of $B(T)$ and $R(T)$ with the letter $\nu$, and the elements of $\mathcal{G}(T)$ with the letter $\gamma$.

**Definition 27** (Map $\mathcal{P} : \mathcal{T}_n^e \to \mathcal{T}_{n-1}^o$)**.** Let $T \in \mathcal{T}_n^e$. We first define a tableau $\mathcal{P}_0(T)$, made up of $n-1$ columns and $2n-1$ rows, containing a point in its box $\left(j : \overset{\approx}{i}\right)$ for all points $p = (j : i)$ of $T$ that does not belong to $\mathrm{Max}(T)$ (in particular $j < n$), where

$$\overset{\approx}{i} = \begin{cases} i, & \text{if } i < n; \\ i-1, & \text{otherwise.} \end{cases}$$

For example, for the tableau $T_1 \in \mathcal{T}_7^e$ depicted in Figure 5, we obtain the tableau $\mathcal{P}_0(T_1)$ of Figure 17.

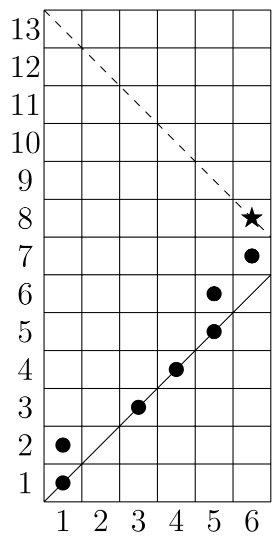


Figure 17: The tableau $\mathcal{P}_0(T_1)$.

Afterwards, for $j$ from 1 to $n-1$, if $C_j^T$ contains a point $p$ labeled with $\nu$ (respectively $\gamma$), we insert a point in the box $(j : 2n-1)$ of $\mathcal{P}_0(T)$ (respectively the box $(j : j)$), following Definition 11.

Let $\mathcal{P}(T)$ be the tableau obtained at the end of this algorithm. It is straightforward by Definition 11 that $\mathcal{P}(T)$ is an element of $\mathcal{T}_{n-1}^o$.

For example, for the tableau $T = T_1 \in \mathcal{T}_7^e$ (see Figure 5), the tableau $\mathcal{P}(T_1) \in \mathcal{T}_6^o$ is as depicted in Figure 10.

Let now $T_0 \in \mathcal{T}_{n-1}^o$. To construct $\mathcal{P}^{-1}(T_0)$, we consider four words $b, r, br, bg$ with which we will label the columns of $T_0$ that contain elements of $V(T_0)$ (recall that if a point of a column $C$ of $T_0$ belongs to $\mathcal{G}(T_0)$, then the other point of $C$ belongs to $V(T_0)$). Let

$$J(T_0) = \{j \in [n-1] : C_j^{T_0} \text{ contains an element of } V(T_0)\},$$
$$J_g(T_0) = \{j \in [n-1] : C_j^{T_0} \text{ contains an element of } \mathcal{G}(T_0)\}.$$

We have $J_g(T_0) \subset J(T_0), |J(T_0)| = v(T_0), |J_g(T_0)| = g(T_0)$. Consider the set $L(T_0)$ of the functions $l : J(T_0) \to \{b, r, bg, br\}$ such that $l(j) \in \{b, r\}$ if $j \notin J_g(T_0)$, otherwise $l(j) \in \{br, bg, r\}$. The cardinality of $L(T_0)$ is $|L(T_0)| = 2^{v(T_0)-g(T_0)} 3^{g(T_0)}$.

**Definition 28.** For all $l \in L(T_0)$, we define a tableau $U^l(T_0) \in \mathcal{T}_n^e$ as follows. First consider a tableau $U_0^l(T_0)$ made up of $n$ columns and $2n$ rows, that contains points in its boxes $\left(j : \bar{\bar{i}}\right)$ for all points $(j : i)$ of $T_0$ that does not belong to $V(T_0)$ or $\mathcal{G}(T_0)$, where $\bar{\bar{i}}$ is defined as $i$ if $i < n$, as $i + 1$ otherwise. Let us set $T = U_0^l(T_0)$. For $j$ from 1 to $n - 1$, assume that $C_j^{T_0}$ contains an element of $V(T_0)$ (otherwise $C_j^T$ already contains two points and we do nothing).

- If no point of $C_j^{T_0}$ belongs to $\mathcal{G}(T_0)$, we have $j \in J(T_0) \backslash J_g(T_0)$, and $l(j) \in \{b, r\}$. If $l(j) = b$ (respectively $l(j) = r$), then we insert a point in the box $(j : n)$ (respectively $(j : 2n)$) of $T$.
- Otherwise $j \in J_g(T_0)$ and $l(j) \in \{br, bg, r\}$. If $l(j) = br$ (respectively $bg, r$), then we insert two points in the boxes $(j : n)$ and $(j : 2n)$ (respectively $(j : n)$ and $(j : j)$, $(j : 2n)$ and $(j : j)$) of $T$.

At the end of the algorithm, we insert two points in the boxes $(n : n)$ and $(n : 2n)$ of $T$. By Definition 11, the tableau $T$ has become an element of $\mathcal{T}_n^e$, which we denote by $U^l(T_0)$.

For example, the tableau $T_1 \in \mathcal{T}_7^e$ depicted in Figure 5 is $U^l(\mathcal{P}(T_1))$, where recall that $\mathcal{P}(T_1) \in \mathcal{T}_6^o$ is the tableau depicted in Figure 10, and where $l = \begin{pmatrix} 2 & 3 & 4 \\ bg & b & r \end{pmatrix}$.

**Proposition 29.** *The set* $\mathcal{P}^{-1}(T_0)$ *is exactly* $\{U^l(T_0) : l \in L(T_0)\}$ *(so* $\mathcal{P}$ *is surjective).*

*Proof.* Let $T \in \mathcal{P}^{-1}(T_0)$. By Definition 27, for all $j \in [n-1]$, we have $j \in J(T_0)$ (respectively $j \in J_g(T_0)$) if and only if one of the points of $C_j^T$ belongs to $\mathrm{Max}(T)$ (respectively to $\mathcal{G}(T)$). We then define $l_T \in L(T_0)$ by

$$l_T(j) = \begin{cases} b, & \text{if } C_j^T \text{ has a point of } B(T) \text{ and no point of } R(T) \text{ or } \mathcal{G}(T); \\ br, & \text{if } C_j^T \text{ has a point of } B(T) \text{ and a point of } R(T); \\ bg, & \text{if } C_j^T \text{ has a point of } B(T) \text{ and a point of } \mathcal{G}(T); \\ r, & \text{if } C_j^T \text{ has a point of } R(T) \text{ and no point of } B(T); \end{cases}$$

for all $j \in J(T_0)$. It is easy to check that $T = U^{l_T}(T_0)$, and that $\mathcal{P}(U^l(T_0)) = T_0$ for all $l \in L(T_0)$. □

**Proposition 30.** *Formula* (9) *is true.*

*Proof.* Let $l \in L(T_0)$ and $T = U^l(T_0) \in \mathcal{P}^{-1}(T_0)$. Let us determine $\mathrm{fr}(T)$ and $\max(T)$ in terms of $T_0$ and $l$. First of all, note that in general all the elements of $V(T_0)$ and $\mathcal{G}(T_0)$ are free points of $T_0$. Let $\mathrm{fr}_0(T_0) = \mathrm{fr}(T_0) - v(T_0) - g(T_0)$. By Definition 28, $\mathrm{fr}_0(T_0)$ is the number of free points of $T$ that were initially in $U_0^l(T_0)$. Let $\delta_l \in [0, g(T_0)]$ be the number of elements $j \in J_g(T_0)$ such that $l(j) = r$. By construction of $T$, for all of these $\delta_l$ integers $j$, the two points of $C_j^{T_0}$ are an element of $V(T_0)$ and an element of $\mathcal{G}(T_0)$ whose insertions in $U_0^l(T_0)$ plot two free points, the first of which belongs to $R(T)$, the second of which does not belong to $\text{Max}(T)$. Consequently, by Remark 6, we have

$$\begin{aligned} b(T) + r(T) &= v(T_0), \\ g(T) &= g(T_0) - \delta_l, \\ \max(T) &= b(T) + r(T) + g(T) \\ &= v(T_0) + g(T_0) - \delta_l, \\ \text{fr}(T) &= \text{fr}_0(T_0) + (\max(T) + 1) + \delta_l \\ &= \text{fr}(T_0) + 1, \end{aligned}$$

so that

$$2^{\text{fr}(T)-1-\max(T)} x^{\max(T)} = 2^{\text{fr}(T_0)-v(T_0)-g(T_0)+\delta_l} x^{v(T_0)+g(T_0)-\delta_l}. \tag{10}$$

Now, for all $\delta \in [0, g(T_0)]$, the number of elements $l \in L(T_0)$ such that $\delta_l = \delta$ equals $\binom{g(T_0)}{\delta} 2^{g(T_0)-\delta} 2^{v(T_0)-g(T_0)} = \binom{g(T_0)}{\delta} 2^{v(T_0)-\delta}$, so, by Proposition 29 and Formula (10), the sum $\sum_{T \in \mathcal{P}^{-1}(T_0)} 2^{\text{fr}(T)-1-\max(T)} x^{\max(T)}$ equals

$$\begin{aligned} &\sum_{\delta=0}^{g(T_0)} \binom{g(T_0)}{\delta} 2^{v(T_0)-\delta} 2^{\text{fr}(T_0)-v(T_0)-g(T_0)+\delta} x^{v(T_0)+g(T_0)-\delta} \\ &= 2^{\text{fr}(T_0)-g(T_0)} x^{v(T_0)+g(T_0)} \sum_{\delta=0}^{g(T_0)} \binom{g(T_0)}{\delta} x^{-\delta} \\ &= 2^{\text{fr}(T_0)-g(T_0)} x^{v(T_0)+g(T_0)} \left(1 + \frac{1}{x}\right)^{g(T_0)}, \end{aligned}$$

hence Formula (9). □

This proves Theorem 10.

# 5 Acknowledgment

The work of EF was partially supported by the grant RSF 19-11-00056.

# References

[1] D. Barsky and D. Dumont, Congruences pour les nombres de Genocchi de 2e espèce, in *Groupe de Travail d'Analyse Ultramétrique*, Tome 7–8, 1979–1981, Exposé no. 34.

[2] A. Bigeni, Enumerating the symplectic Dellac configurations, preprint, 2017. Available at http://arxiv.org/abs/1705.03804.

[3] A. Bigeni, A generalization of the Kreweras triangle through the universal $sl_2$ weight system, *J. Combin. Theory Ser. A* **161** (2019), 309–326.

[4] A. Bigeni, Combinatorial interpretations of the Kreweras triangle in terms of subset tuples, *Electron. J. Combin.* **25** (2018), P4.44.

[5] A. Bigeni, Combinatorial study of the Dellac configurations and the q-extended normalized median Genocchi numbers, *Electron. J. Combin.* **21** (2014), P2.32.

[6] A. Bigeni and E. Feigin, Symmetric Dellac configurations and symplectic/orthogonal flag varieties, *Linear Algebra Appl.* **573** (2019), 54–79.

[7] G. Cerulli Irelli, M. Lanini, and P. Littelmann, Degenerate flag varieties and Schubert varieties: a characteristic free approach, *Pacific J. Math* **284** (2016), 283–308.

[8] G. Cerulli Irelli, E. Feigin, and M. Reineke, Quiver Grassmannians and degenerate flag varieties, *Algebra Number Theory* **6** (2012), 165–194.

[9] H. Dellac, Problem 1735, *L'Intermédiaire des Mathématiciens* **7** (1900), 9–10.

[10] D. Dumont, Interprétations combinatoires des nombres de Genocchi, *Duke Math. J.* **41** (1974), 305–318.

[11] D. Dumont and A. Randrianarivony, Dérangements et nombres de Genocchi, *Discrete Math.* **132** (1994), 37–49.

[12] D. Dumont and J. Zeng, Further results on Euler and Genocchi numbers, *Aequationes Math.* **47** (1994), 31–42.

[13] X. Fang and G. Fourier, Torus fixed points in Schubert varieties and normalized median Genocchi numbers, *Sém. Lothar. Combin.* **75** (2015), Art. B75f.

[14] E. Feigin, Degenerate flag varieties and the median Genocchi numbers, *Math. Res. Lett.* **18** (2011), 1163–1178.

[15] E. Feigin, The median Genocchi numbers, q-analogues and continued fractions, *European J. Combin.* **33** (2012), 1913–1918.

[16] E. Feigin, $\mathbb{G}_a^M$ degeneration of flag varieties, *Selecta Math.* **18** (2012), 513–537.

[17] E. Feigin, M. Finkelberg, and P. Littelmann, Symplectic degenerate flag varieties, *Canad. J. Math.* **66** (2014), 1250–1286.

[18] G. Han and J. Zeng, On a $q$-sequence that generalizes the median Genocchi numbers, *Ann. Sci. Math. Québec* **23** (1999), 63–72.

[19] G. Han and J. Zeng, $q$-Polynômes de Gandhi et statistique de Denert, *Discrete Math.* **205** (1999), 119–143.

[20] G. Kreweras, Sur les permutations comptées par les nombres de Genocchi de 1-ière et 2-ième espèce, *European J. Combin.* **18** (1997), 49–58.

[21] A. Randrianarivony and J. Zeng, Une famille de polynômes qui interpole plusieurs suites classiques de nombres, *Adv. in Appl. Math.* **17** (1996), 1–26.

[22] N. J. A. Sloane et al., *The On-Line Encyclopedia of Integer Sequences*, published electronically at `https://oeis.org`, 2019.

[23] J. Zeng and J. Zhou, A q-analog of the Seidel generation of Genocchi numbers, *European J. Combin.* **27** (2006), 364–381.

---

2010 *Mathematics Subject Classification*: Primary 05A15, Secondary 14M15.
*Keywords:* Dellac configuration, median Euler number, flag variety.

---

(Concerned with sequences A000366, A000657, A002832, A098278, and A098279.)

---

Received April 19 2019; revised versions received April 11 2020; April 14 2020. Published in *Journal of Integer Sequences*, April 15 2020.

---

Return to Journal of Integer Sequences home page.