The Multivariate Lah and Stirling Numbers

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Abstract
An ordered partition of \{1, 2, \ldots, n\} into \(k\) blocks \(B_1, B_2, \ldots, B_k\) is a partition where the order of blocks is considered. In the present paper, we consider the case that each block \(B_i\) has \(r_i\) copies. Using this extension of ordered set partitions, we introduce a new generalization of the Lah and Stirling numbers of both kinds which called multivariate Lah and Stirling numbers, respectively. We study several combinatorial properties such as explicit formulas, recurrence relations, generating functions, and some convolution identities.

1 Introduction
Let \([n] = \{1, \ldots, n\}\). A partition \(\pi\) of \([n]\) is a family of nonempty, pairwise disjoint subsets called blocks. A partition of \([n]\) into \(k\) blocks is denoted \(B_1/B_2/\cdots/B_k\) such that \(\min(B_1) < \min(B_2) < \cdots < \min(B_k)\).

For any \(n \geq k \geq 0\), let \(\{^n_k\}\), \([^n_k]\) and \([^n_k]\) be the Stirling numbers of the second kind, first kind and Lah numbers, respectively. The numbers \(\{^n_k\}\) count the number of set partitions of \([n]\) into \(k\) blocks, \([^n_k]\) count the number of partitions of \([n]\) into \(k\) cycles. Similarly, the Lah numbers \([^n_k]\) count the number of partitions of set \([n]\) into \(k\) nonempty lists.

The falling and rising factorials are defined, respectively by
\[
(x)_n = x(x + 1)(x + 2) \cdots (x + n - 1), \quad (x)_0 = 1,
\]
presents several properties of the multivariate Lah numbers by algebraic and combinatorial arguments. In the last section, we define the multivariate Stirling numbers of the first and second kinds and we provide an expression for the multivariate Lah numbers in terms of the multivariate Stirling numbers and multinomial Stirling numbers were introduced by Moak [10].

The Lah numbers have the following explicit formula \[ \binom{n}{k} = \frac{n!}{k!} \left( \frac{n-1}{1} \right) \cdot \cdot \cdot \], as follows:

An ordered partition \( \pi \) of \( [n] \) into \( k \) blocks is a partition where the order of blocks is important \( \pi_\sigma = B_{\sigma(1)}/B_{\sigma(2)}/\cdots/B_{\sigma(k)} \), where \( \sigma \) is a permutation of \([k] \). The number of ordered set partitions of \([n]\) into \( k \) blocks is given by \( k! \binom{n}{k} \), [9, p. 106], as follows:

\[
k! \binom{n}{k} = \sum_{r_1+r_2+\cdots+r_k=n} \binom{n}{r_1, r_2, \ldots, r_k},
\]

where \( \binom{n}{r_1, r_2, \ldots, r_k} = \frac{n!}{r_1!r_2!\cdots r_k!} \) is the multinomial coefficient. The coefficients \( \binom{n}{r_1, r_2, \ldots, r_k} \) have the following horizontal generating function

\[
(x_1 + x_2 + \cdots + x_k)^n = \sum_{r_1+r_2+\cdots+r_k=n} \binom{n}{r_1, r_2, \ldots, r_k} x_1^{r_1}x_2^{r_2} \cdots x_k^{r_k}.
\]

The relation (2) can be generalized to multivariate falling and rising factorials [7, p. 149]:

\[
\langle x_1 + x_2 + \cdots + x_k \rangle_n = \sum_{r_1+r_2+\cdots+r_k=n} \binom{n}{r_1, r_2, \ldots, r_k} \langle x_1 \rangle_{r_1}\langle x_2 \rangle_{r_2} \cdots \langle x_k \rangle_{r_k},
\]

and

\[
\langle x_1 + x_2 + \cdots + x_k \rangle_n = \sum_{r_1+r_2+\cdots+r_k=n} \binom{n}{r_1, r_2, \ldots, r_k} \langle x_1 \rangle_{r_1}\langle x_2 \rangle_{r_2} \cdots \langle x_k \rangle_{r_k}.
\]

Many authors have investigated the Stirling and Lah numbers; see, for instance, \([1, 2, 4, 8, 12]\).

The present paper is organized as follows: Section 2 introduces the multivariate Lah numbers \emph{multipartitions} set which generalize the set partitions and ordered set partitions. Section 3 presents several properties of the multivariate Lah numbers by algebraic and combinatorial arguments. In the last section, we define the multivariate Stirling numbers of the first and second kinds and we provide an expression for the multivariate Lah numbers in terms of the multivariate Stirling numbers and multinomial Stirling numbers were introduced by Moak [10].


\section{Combinatorial definition of multivariate Lah numbers}

Let \( r_k := (r_1, r_2, \ldots, r_k) \) be a sequence of nonnegative integers. Now suppose that we have \( k \) categories of lists \((C_1, C_2, \ldots, C_k)\) such that \(|C_i| = r_i\). Let \( \mathcal{OP}_n^{r_k} \) be the set partitions of \([n]\) into \((r_1 + r_2 + \cdots + r_k)\)-lists. A partition \( \pi \in \mathcal{OP}_n^{r_k} \) is of the form \( \pi = B_1^{r_1}/B_2^{r_2}/\cdots/B_k^{r_k} \) where \( B_i = B_i/B_i/\cdots/B_i \) \( r_i \) times.

**Definition 1.** Let \( \pi = B_1^{r_1}/B_2^{r_2}/\cdots/B_k^{r_k} \) be a partition of the set \( \mathcal{OP}_n^{r_k} \). A *multipartition* is a permutation of the multiset \( \{B_1^{r_1}, B_2^{r_2}, \ldots, B_k^{r_k}\} \). We let \( \mathcal{G}_{n,r_k}(\pi) \) denote the set of all multipartitions of the multiset \( \{B_1^{r_1}, B_2^{r_2}, \ldots, B_k^{r_k}\} \).

**Example 2.** Let \( \pi = 1.2/3/4.5/6 \) be a partition of the set \([6]\) into \((2, 1, 1)\)-lists. The set \( \mathcal{G}_{6,3}(\pi) \) of multipartitions associated with the partition \( \pi \) is

\[
\begin{align*}
\frac{1.2}{B_1} & ; \frac{3}{B_2} ; \frac{4.5}{B_3} ; \frac{6}{B_3} ; \\
\frac{1.2}{B_1} & ; \frac{4.5}{B_2} ; \frac{3}{B_3} ; \\
\frac{4.5}{B_2} & ; \frac{1.2}{B_1} ; \frac{6}{B_3} ; \\
\frac{4.5}{B_2} & ; \frac{1.2}{B_1} ; \frac{3}{B_3} .
\end{align*}
\]

**Definition 3.** For any \( n, r_1, r_2, \ldots, r_k \geq 1 \), the multivariate Lah number, which we denote by \( \binom{n}{r_1, r_2, \ldots, r_k} \), is the number of multipartitions of the set \([n]\) into nonempty \( r_k \)-lists.

\[
\binom{n}{r_1, r_2, \ldots, r_k} = \sum_{\pi \in \mathcal{OP}_n^{r_k}} |\mathcal{G}_{n,r_k}(\pi)|.
\]

**Example 4.** There are 3 multipartitions of the set \([3]\) into \((1, 2)\)-lists

\[
1/2/3 ; \quad 2/1/3 ; \quad 2/3/1 .
\]

We have \( \binom{3}{1,1} = 12 \), so the corresponding multipartitions are

\[
1/2, 3 ; \quad 2, 3/1 ; \quad 1/3, 2 ; \quad 3, 2/1 ; \quad 1, 2/3 ; \quad 3, 1, 2 ; \\
2, 1/3 ; \quad 3, 2/1 ; \quad 1, 3/2 ; \quad 2/1, 3 ; \quad 3, 1/2 ; \quad 2/3, 1 .
\]

In the following theorem we provide an explicit formula for the multivariate Lah numbers.

**Theorem 5.** For any \( n, r_1, r_2, \ldots, r_k \geq 1 \), we have

\[
\binom{n}{r_1, r_2, \ldots, r_k} = \frac{(n-1)!}{(r_1 + r_2 + \cdots + r_k - 1)!} \left( \frac{n}{r_1, r_2, \ldots, r_k, n - \sum_{j=1}^{k} r_j} \right) .
\]
Proof. To construct a multipartition of \([n]\) into \(r_k\)-lists we can do the following. First, we select \(r_1\) elements from \([n]\), each element corresponding to the beginning of one list of category \(C_1\). There are \(\binom{n}{r_1}\) possibilities. Then, we choose \(r_2\) elements from the remaining \(n - r_1\) elements, which we place at the beginning of the lists of category \(C_2\) with \(\binom{n-r_1}{r_2}\) possible ways, and so on. We choose \(r_k\) of the remaining \(n - r_1 - \cdots - r_{k-1}\) elements, which we place at the start of the lists of category \(C_k\). There are \(\binom{n-r_1-\cdots-r_{k-1}}{r_k}\) possibilities. The remaining \(n-r_1-\cdots-r_k\) elements can be added with \((r_1+\cdots+r_k)(r_1+\cdots+r_k+1)\cdots(n-1)\) possibilities. This gives us,

\[
\binom{n}{r_1}\binom{n-r_1}{r_2}\cdots\binom{n-r_1-\cdots-r_{k-1}}{r_k}\prod_{i=0}^{n-1-r_1-\cdots-r_{k-1}}(n-1-i) = \frac{(n-1)!}{(r_1+r_2+\cdots+r_k-1)!}\left(\frac{n}{r_1, r_2, \ldots, r_k, n-\sum_{j=1}^{k} r_j}\right),
\]

which completes the proof. \(\square\)

As particular cases of the multivariate Lah numbers, when \(k = 1\) we obtain the classical Lah numbers and for \(r_1 + r_2 + \cdots + r_k = n\) we get the multinomial coefficient

\[
\binom{n}{r_1, r_2, \ldots, r_k} = \binom{n}{r_1, r_2, \ldots, r_k}.
\]

Also, when \(r_i = 1\) for all \(i \in [k]\) we obtain the ordered Lah numbers,

\[
\binom{n}{1, 1, \ldots, 1, \text{k-times}} = n!\left(\frac{n-1}{k-1}\right).
\]

Let \(\sigma(1), \sigma(2), \ldots, \sigma(k)\) be a permutation of \([k]\). Then

\[
\binom{n}{r_1, r_2, \ldots, r_k} = \binom{n}{r_{\sigma(1)}, r_{\sigma(2)}, \ldots, r_{\sigma(k)}}.
\]

From Relation (6), we deduce an expression for the multivariate Lah numbers in terms of classical Lah numbers.

**Corollary 6.** For any \(n, r_1, \ldots, r_k \geq 1\), we have

\[
\binom{n}{r_1, r_2, \ldots, r_k} = \binom{r_1 + r_2 + \cdots + r_k}{r_1, r_2, \ldots, r_k} \binom{n}{r_1 + r_2 + \cdots + r_k}.
\]
3 Fundamental properties of multivariate Lah numbers

In this section, we provide some fundamental properties of the multivariate Lah numbers. We start by giving the exponential generating function.

**Theorem 7.** The exponential generating function of the multivariate Lah numbers is

\[
\sum_{n \geq 0} \left[ \prod_{i=1}^{k} \frac{1}{r_i!} \left( \frac{t}{1-t} \right)^{r_i} \right] \frac{t^n}{n!} = \sum_{n \geq 0} (n-1)! \left( \prod_{i=1}^{k} \frac{1}{r_i!} \left( \frac{t}{1-t} \right)^{r_i} \right) \left( \prod_{i=1}^{k} \frac{1}{r_i!} \left( \frac{t}{1-t} \right)^{r_i} \right) \left( \prod_{i=1}^{k} \frac{1}{r_i!} \left( \frac{t}{1-t} \right)^{r_i} \right)
\]

which completes the proof.

In the following theorem, we give the multivariate exponential generating function for the multivariate Lah numbers.

**Theorem 8.** We have

\[
\sum_{n \geq 0} \left[ \prod_{i=1}^{k} \frac{1}{r_i!} \left( \frac{t}{1-t} \right)^{r_i} \right] \frac{t^n}{n!} = \exp \left( \frac{t}{1-t} \left( z_1 + z_2 + \cdots + z_k \right) \right).
\]
Theorem 9. For any \( n \geq 1 \), we have
\[
(x_1 + x_2 + \cdots + x_k)_n = \sum_{r_1+r_2+\cdots+r_k \leq n} \binom{n}{r_1,r_2,\ldots,r_k} (x_1)^{r_1} (x_2)^{r_2} \cdots (x_k)^{r_k},
\] (10)
and
\[
\langle x_1 + x_2 + \cdots + x_k \rangle_n = \sum_{r_1+r_2+\cdots+r_k \leq n} (-1)^{n-r_1-\cdots-r_k} \binom{n}{r_1,r_2,\ldots,r_k} (x_1)^{r_1} (x_2)^{r_2} \cdots (x_k)^{r_k}.
\] (11)

Proof. We have
\[
\left(1 + \frac{t}{1-t}\right)^{x_1+x_2+\cdots+x_k} = \left(1 + \frac{t}{1-t}\right)^{x_1} \left(1 + \frac{t}{1-t}\right)^{x_2} \cdots \left(1 + \frac{t}{1-t}\right)^{x_k}
\]
\[
= \sum_{r_1 \geq 0} \left(\frac{t}{1-t}\right)^{r_1} \langle x_1 \rangle^{r_1} \left(\sum_{r_2 \geq 0} \left(\frac{t}{1-t}\right)^{r_2} \langle x_2 \rangle^{r_2} \right) \cdots \left(\sum_{r_k \geq 0} \left(\frac{t}{1-t}\right)^{r_k} \langle x_k \rangle^{r_k} \right)
\]
\[
= \sum_{r_1,r_2,\ldots,r_k \geq 0} \prod_{i=1}^k \frac{1}{r_i!} \left(\frac{t}{1-t}\right)^{r_i} \langle x_1 \rangle^{r_1} \langle x_2 \rangle^{r_2} \cdots \langle x_k \rangle^{r_k}
\]
\[
= \sum_{n \geq 0} \sum_{r_1,r_2,\ldots,r_k} \binom{n}{r_1,r_2,\ldots,r_k} \langle x_1 \rangle^{r_1} \langle x_2 \rangle^{r_2} \cdots \langle x_k \rangle^{r_k} \frac{t^n}{n!}.
\] (12)

The left hand side of the first equality is
\[
\left(1 + \frac{t}{1-t}\right)^{x_1+x_2+\cdots+x_k} = (1-t)^{-(x_1+x_2+\cdots+x_k)}
\]
\[
= \sum_{n \geq 0} (-1)^n \left(-x_1 - x_2 - \cdots - x_k\right) \frac{t^n}{n!}
\]
\[
= \sum_{n \geq 0} \binom{x_1 + x_2 + \cdots + x_k + n - 1}{n} \frac{t^n}{n!}
\]
\[
= \sum_{n \geq 0} \binom{x_1 + x_2 + \cdots + x_k}{n} \frac{t^n}{n!}.
\] (13)

Equating the coefficients of \( \frac{t^n}{n!} \) in (12) and (13) yields (10). Equation (11) follows by substituting \((-x_i)\) for \(i \in [k]\). \(\square\)

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Using Theorem 9, we get, for example

\[(x_1 + x_2)^2 = \langle x_1 \rangle_2 + 2x_1x_2 + \langle x_2 \rangle_2 + 2(x_1 + x_2),\]
\[(x_1 + x_2)^3 = \langle x_1 \rangle_3 + 3\langle x_1 \rangle_2x_2 + 3x_1\langle x_2 \rangle_2 + \langle x_2 \rangle_3 + 6(\langle x_1 \rangle_2 + 2x_1x_2 + \langle x_2 \rangle_2) + 6(x_1 + x_2),\]
\[(x_1 + x_2 + x_3)^2 = \langle x_1 \rangle_2 + \langle x_2 \rangle_2 + \langle x_3 \rangle_2 + 2(x_1x_2 + x_1x_3 + x_2x_3) + 2(x_1 + x_2 + x_3).
\[(x_1 + x_2)^2 = 3\langle x_1 \rangle_2 - 6x_1 - 6x_2 + 6x_1x_2 + 3\langle x_2 \rangle_2.\]

**Theorem 10.** For any \(n, r_1, r_2, \ldots, r_k \geq 1\), we have

\[\sum_{s_1+s_2+\ldots+s_k \leq n} (-1)^{s_i-r_i} \left[ \begin{array}{c} n \\ s_1, s_2, \ldots, s_k \end{array} \right] \left[ \begin{array}{c} s_1 \\ r_1 \\ s_2 \\ r_2 \\ \vdots \\ s_k \\ r_k \end{array} \right] = \left[ \begin{array}{c} n \\ r_1, r_2, \ldots, r_k \end{array} \right]. \quad (14)\]

**Proof.** From Relation (10), we have

\[\langle x_1 + x_2 + \cdots + x_k \rangle_n = \sum_{s_1+s_2+\cdots+s_k \leq n} \left[ \begin{array}{c} n \\ s_1, s_2, \ldots, s_k \end{array} \right] \langle x_1 \rangle_{s_1} \langle x_2 \rangle_{s_2} \cdots \langle x_k \rangle_{s_k} = \sum_{s_1+s_2+\cdots+s_k \leq n} (-1)^{s_i-r_i} \left[ \begin{array}{c} n \\ s_1, s_2, \ldots, s_k \end{array} \right] \left[ \begin{array}{c} s_1 \\ r_1 \\ s_2 \\ r_2 \\ \vdots \\ s_k \\ r_k \end{array} \right] \times \langle x_1 \rangle_{r_1} \langle x_2 \rangle_{r_2} \cdots \langle x_k \rangle_{r_k}.\]

On other hand, we have

\[\langle x_1 + x_2 + \cdots + x_k \rangle_n = \sum_{r_1+r_2+\cdots+r_k=n} \left[ \begin{array}{c} n \\ r_1, r_2, \ldots, r_k \end{array} \right] \langle x_1 \rangle_{r_1} \langle x_2 \rangle_{r_2} \cdots \langle x_k \rangle_{r_k}. \quad (15)\]

Equating the coefficients of \(\langle x \rangle_{r_1} \langle x \rangle_{r_2} \cdots \langle x \rangle_{r_k}\) we obtain the result. \(\square\)

**Corollary 11.** The following identity holds

\[\langle x_1 \rangle_{r_1} \langle x_2 \rangle_{r_2} \cdots \langle x_k \rangle_{r_k} = \sum_{s_1,s_2,\ldots,s_k} \left[ \begin{array}{c} r_1 \\ s_1 \\ r_2 \\ s_2 \\ \vdots \\ r_k \\ s_k \end{array} \right] \langle x_1 \rangle_{s_1} \langle x_2 \rangle_{s_2} \cdots \langle x_k \rangle_{s_k}. \quad (16)\]

In the next theorem, we give a convolution identity involving the multivariate Lah numbers.

**Theorem 12.** We have

\[\prod_{i=1}^{k} \left( \frac{r_i + s_i}{r_i} \right) \left[ \begin{array}{c} n \\ r_1 + s_1, r_2 + s_2, \ldots, r_k + s_k \end{array} \right] = \sum_{j=1}^{n} \left[ \begin{array}{c} n \\ j \\ r_1, r_2, \ldots, r_k \end{array} \right] \left[ \begin{array}{c} n-j \\ s_1, s_2, \ldots, s_k \end{array} \right]. \quad (17)\]
Proof. From Theorem 7, we have

$$\sum_{n \geq 0} \prod_{i=1}^{k} \left( \begin{array}{c} r_i + s_i \\ r_i \end{array} \right) \left| r_1 + s_1, \ldots, r_k + s_k \right| \frac{t^n}{n!} = \prod_{i=1}^{k} \left( \begin{array}{c} r_i + s_i \\ r_i \end{array} \right) \frac{1}{(r_i + s_i)!} \left( \frac{t}{1-t} \right)^{(r_i+s_i)}$$

$$= \left( \prod_{i=1}^{k} \frac{1}{(r_i)!} \left( \frac{t}{1-t} \right)^{r_i} \right) \left( \prod_{j=1}^{k} \frac{1}{(s_j)!} \left( \frac{t}{1-t} \right)^{s_j} \right)$$

$$= \left( \sum_{l \geq 0} \left| \begin{array}{c} l \\ r_1, r_2, \ldots, r_k \end{array} \right| \frac{t^l}{l!} \right) \times \left( \sum_{m \geq 0} \left| \begin{array}{c} m \\ s_1, s_2, \ldots, s_k \end{array} \right| \frac{t^m}{m!} \right).$$

Equating the coefficients of $\frac{t^n}{n!}$ in both sides, we get the desired result.

Combinatorial proof of Theorem 12. Let $C = \{1, 2, \ldots, k\}$ be a list of $k$ different colors. The left hand side of the identity counts the number of multipartitions of the set $[n]$ into $(r_1 + s_1, \ldots, r_k + s_k)$-lists such that the elements of $r_i$ lists among $r_i + s_i$ lists get colour $i$, for all $i \in [k]$. In the right hand side, we start by choosing $j$ elements from $n$ and there are $\binom{n}{j}$ ways to do. The $j$ elements have to be partitioned into $(r_1, \ldots, r_k)$-lists such that the elements of the lists $r_i$ get colour $i$ and the remaining $n-j$ elements have to be partitioned into $(s_1, \ldots, s_k)$-lists.

Now we give some recurrence relations satisfied by the multivariate Lah numbers.

Theorem 14. The multivariate Lah numbers satisfy the following recurrence relations

(i) Triangular recurrence relation:

$$\left| \begin{array}{c} n \\ r_1, r_2, \ldots, r_k \end{array} \right| = \sum_{i=1}^{k} \left| \begin{array}{c} n \\ r_1, \ldots, r_i-1, \ldots, r_k \end{array} \right| + (n + r_1 + \cdots + r_k - 1) \left| \begin{array}{c} n-1 \\ r_1, r_2, \ldots, r_k \end{array} \right|,$$

with initial terms $\left| \begin{array}{c} n \\ r_1, r_2, \ldots, r_k \end{array} \right| = 0$ if $n < r_1 + r_2 + \cdots + r_k$. 

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and there are \( (r_1 + \cdots + r_k + j)_{n-j} \sum_{i=1}^{k} \binom{j-1}{r_i-1, \ldots, r_k} \) ways to choose such lists.

(iii) Diagonal recurrence relation:

\[
\binom{n}{r_1, r_2, \ldots, r_k} = \sum_{j=0}^{r_1+\cdots+r_k} \sum_{s_1+\cdots+s_k=j} \binom{j}{s_1, \ldots, s_k} \times (n + r_1 + \cdots + r_k - 2j - 1) \binom{n-j-1}{r_1-s_1, \ldots, r_k-s_k}.
\]

Proof. Let us show (19). A multipartition of the set \([n]\) into \((r_1, r_2, \ldots, r_k)\)-lists can be obtained from a multipartition of the set \([n-1]\) into \((r_1, \cdots, r_i-1, \ldots, r_k)\)-lists \((1 \leq i \leq k)\) to which we add a single list \(\{n\}\) of category \(C_i\), or from a multipartition of the set \([n-1]\) into \((r_1, r_2, \ldots, r_k)\)-lists, by adding the element \(\{n\}\) before any existing elements or at the end of any list. Then there are \((n + r_1 + \cdots + r_k - 1)_{r_1, r_2, \ldots, r_k}\) ways.

Next, we show (20). For a given \(r_1 + \cdots + r_k \leq j \leq n\) and \(1 \leq i \leq k\), let us consider the elements of \([j-1]\) which are not in the same list with the element \(\{n\}\). The number of multipartitions of \([j-1]\) into \((r_1, \ldots, r_i-1, \ldots, r_k)\)-lists is \(\binom{j-1}{r_1, \ldots, r_i-1, \ldots, r_k}\), and there are \((r_1 + \cdots + r_k + j)_{n-j}\) ways to add the remaining elements of \([j, n-1]\) into \((r_1, r_2, \ldots, r_k)\)-lists. Summing over all possible \(j\) and \(i\) gives the result.

Finally, we show (21). Let \(j\) \((0 \leq j \leq r_1 + \cdots + r_k)\) be the number of lists which contain exactly one element, then the number of ways to choose a such lists is \(\binom{j}{s_1, \ldots, s_k}\). Now it remains to count the number of multipartitions of \([j+1, n]\) into \((r_1-s_1, \ldots, r_k-s_k)\)-lists. So, the number of multipartitions of \([j+1, n-1]\) into \((r_1-s_1, \ldots, r_k-s_k)\)-lists is \(\binom{n-j-1}{r_1-s_1, \ldots, r_k-s_k}\), and there are \((n + r_1 + \cdots + r_k - 2j - 1)\) ways to add the element \(\{n\}\) in any list. Summing up yields the desired result.

\[\square\]

4 Multivariate Stirling numbers

Definition 15. For any \(n, r_1, r_2, \ldots, r_k \geq 0\), the multivariate Stirling numbers of the first kind, denoted \(\left[ r_1, r_2, \ldots, r_k \right]_n\), are defined as the numbers of multipartitions of the set \([n]\) into \((r_1, r_2, \ldots, r_k)\)-cycles. Analogously, we define the multivariate Stirling numbers of second kind, denoted \(\left\{ r_1, r_2, \ldots, r_k \right\}_n\), as the numbers of multipartitions of \([n]\) into \((r_1, r_2, \ldots, r_k)\)-blocks.

From Definition 15, we deduce that the multivariate Stirling numbers of the first kind satisfy the following recurrence relation

\[
\left[ n \right]_{r_1, r_2, \ldots, r_k} = \sum_{i=1}^{k} \left[ n-1 \right]_{r_1, \ldots, r_i-1, \ldots, r_k} + (n-1) \left[ n-1 \right]_{r_1, r_2, \ldots, r_k},
\]

(22)
with \( \left\{ \begin{array}{c} n \\ r_1, r_2, \ldots, r_k \end{array} \right\} = 0 \) if \( n < r_1 + r_2 + \cdots + r_k \). The multivariate Stirling numbers of the second kind satisfy the following recurrence relation

\[
\left\{ \begin{array}{c} n \\ r_1, r_2, \ldots, r_k \end{array} \right\} = \sum_{i=1}^{k} \left\{ \begin{array}{c} n-1 \\ r_1, \ldots, r_i-1, \ldots, r_k \end{array} \right\} \text{ } + \text{ } (r_1 + r_2 + \cdots + r_k) \left\{ \begin{array}{c} n-1 \\ r_1, r_2, \ldots, r_k \end{array} \right\}. \tag{23}
\]

with \( \left\{ \begin{array}{c} n \\ r_1, r_2, \ldots, r_k \end{array} \right\} = 0 \) if \( n < r_1 + r_2 + \cdots + r_k \).

As particular cases, we have

\( \left\{ \begin{array}{c} n \\ 1, 1, \ldots, 1 \end{array} \right\} = k! \left\{ \begin{array}{c} n \\ k \end{array} \right\}, \text{ for } n \geq k, \) \tag{k-times}

and

\( \left\{ \begin{array}{c} n \\ r_1, r_2, \ldots, r_k \end{array} \right\} = \left( \begin{array}{c} n \\ r_1, r_2, \ldots, r_k \end{array} \right), \text{ for } r_1 + r_2 + \cdots + r_k = n. \)

Next, we give an explicit formula for the multivariate Stirling of the second kind.

**Theorem 16.** For any \( n \geq 0 \), we have

\[
\left\{ \begin{array}{c} n \\ r_1, r_2, \ldots, r_k \end{array} \right\} = \frac{1}{r_1! r_2! \cdots r_k!} \sum_{j=0}^{r_1 + \cdots + r_k} (-1)^j \begin{pmatrix} r_1 + \cdots + r_k \\ j \end{pmatrix} (r_1 + \cdots + r_k - j)^n. \tag{24}
\]

**Proof.** The result is obtained by applying the inclusion-exclusion principle. \( \square \)

**Theorem 17.** For any \( n \geq 1 \), we have

\[
(x_1 + x_2 + \cdots + x_k)^n = \sum_{r_1+r_2+\cdots+r_k \leq n} \left\{ \begin{array}{c} n \\ r_1, r_2, \ldots, r_k \end{array} \right\} (x_1)^{r_1} (x_2)^{r_2} \cdots (x_k)^{r_k}. \tag{25}
\]

and

\[
(x_1 + x_2 + \cdots + x_k)^n = \sum_{r_1+r_2+\cdots+r_k \leq n} \left[ \begin{array}{c} n \\ r_1, r_2, \ldots, r_k \end{array} \right] x_1^{r_1} x_2^{r_2} \cdots x_k^{r_k}. \tag{26}
\]

**Proof.** The result is obtained using induction proof. \( \square \)

**Theorem 18.** The exponential generating function of the multivariate Stirling numbers of the first and second kind

\[
\sum_{n \geq 0} \left\{ \begin{array}{c} n \\ r_1, r_2, \ldots, r_k \end{array} \right\} \frac{t^n}{n!} = \prod_{i=1}^{k} \frac{1}{r_i} \left( \ln \left( \frac{1}{1-t} \right) \right)^{r_i}, \tag{27}
\]
and

$$\sum_{n \geq 0} \left\{ \begin{array}{c} n \\ r_1, r_2, \ldots, r_k \end{array} \right\} \frac{t^n}{n!} = \prod_{i=1}^{k} \frac{(e^t - 1)^{r_i}}{r_i!}. \quad (28)$$

From Theorem 18, we obtain

$$\sum_{n \geq 0} \sum_{r_i \geq 0} \left[ \begin{array}{c} n \\ r_1, r_2, \ldots, r_k \end{array} \right] \frac{t^n}{n!} z_1^{r_1} \cdots z_k^{r_k} = (1 - t)^{-z_1 - \cdots - z_k}, \quad (29)$$

and

$$\sum_{n \geq 0} \sum_{r_i \geq 0} \left\{ \begin{array}{c} n \\ r_1, r_2, \ldots, r_k \end{array} \right\} \frac{t^n}{n!} z_1^{r_1} \cdots z_k^{r_k} = \exp \left( (e^t - 1)(z_1 + \cdots + z_k) \right). \quad (30)$$

The connection relation between the multivariate Stirling numbers and classical Stirling numbers is

$$\left\{ \begin{array}{c} n \\ r_1, r_2, \ldots, r_k \end{array} \right\} = \left( r_1 + r_2 + \cdots + r_k \right) \left\{ \begin{array}{c} n \\ r_1, r_2, \ldots, r_k \end{array} \right\}, \quad (31)$$

and

$$\left[ \begin{array}{c} n \\ r_1, r_2, \ldots, r_k \end{array} \right] = \left( r_1 + r_2 + \cdots + r_k \right) \left[ \begin{array}{c} n \\ r_1, r_2, \ldots, r_k \end{array} \right]. \quad (32)$$

In the following theorem we express the multivariate Lah numbers in terms of the multivariate Stirling numbers.

**Theorem 19.** For any \( n \geq 1 \), we have

$$\left[ \begin{array}{c} n \\ r_1, r_2, \ldots, r_k \end{array} \right] = \sum_{j_1 + \cdots + j_k = r_1 + \cdots + r_k} \left[ \begin{array}{c} n \\ j_1, j_2, \ldots, j_k \end{array} \right] \left\{ \begin{array}{c} j_1 \\ r_1 \end{array} \right\} \left\{ \begin{array}{c} j_2 \\ r_2 \end{array} \right\} \cdots \left\{ \begin{array}{c} j_k \\ r_k \end{array} \right\}. \quad (33)$$

**Proof.** The result is obtained from \((10)\) and \((25)\). \qed

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**References**


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