Journal of Integer Sequences, Vol. 23 (2020),
Article 20.4.5

The Multivariate Lah and Stirling Numbers

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Abstract

An ordered partition of $\{1, 2, \ldots, n\}$ into k blocks $B_1, B_2, \ldots, B_k$ is a partition where the order of blocks is considered. In the present paper, we we consider the case that each block B_i has r_i copies. Using this extension of ordered set partitions, we introduce a new generalization of the Lah and Stirling numbers of both kinds which called multivariate Lah and Stirling numbers, respectively. We study several combinatorial properties such as explicit formulas, recurrence relations, generating functions, and some convolution identities.

1 Introduction

Let $[n] = \{1, \ldots, n\}$. A partition π of $[n]$ is a family of nonempty, pairwise disjoint subsets called *blocks*. A partition of $[n]$ into k blocks is denoted $B_1/B_2/\cdots/B_k$ such that $\min(B_1) < \min(B_2) < \cdots < \min(B_k)$.

For any $n \geq k \geq 0$, let $\genfrac{\{}{\}}{0pt}{}{n}{k}$, $\genfrac{[}{]}{0pt}{}{n}{k}$ and $\genfrac{\lfloor}{\rfloor}{0pt}{}{n}{k}$ be the Stirling numbers of the second kind, first kind and Lah numbers, respectively. The numbers $\genfrac{\{}{\}}{0pt}{}{n}{k}$ count the number of set partitions of $[n]$ into k blocks, $\genfrac{[}{]}{0pt}{}{n}{k}$ count the number of partitions of $[n]$ into k cycles. Similarly, the Lah numbers $\genfrac{\lfloor}{\rfloor}{0pt}{}{n}{k}$ count the number of partitions of set $[n]$ into k nonempty lists.

The falling and rising factorials are defined, respectively by

$$(x)_n = x(x+1)(x+2)\cdots(x+n-1), \qquad (x)_0 = 1,$$

and

$$\langle x\rangle_n = x(x-1)(x-2)\cdots(x-n+1), \qquad \langle x\rangle_0 = 1.$$

The Stirling numbers of second kind appear in the expansion $x^n = \sum_k \left\{ {n \atop k} \right\} \langle x\rangle_k$, and the Stirling numbers of first kind appear in the expansion $(x)_n = \sum_k \left[{n \atop k} \right] x^k$. The Lah numbers are connection coefficients between rising and falling factorials $(x)_n = \sum_k \left\lfloor {n \atop k} \right\rfloor \langle x\rangle_k$.

The Lah numbers can be expressed in terms of Stirling numbers of second and first kinds [6, p. 156], as follows:

$$\left\lfloor {n \atop k} \right\rfloor = \sum_{j=k}^{n} \left[{n \atop j} \right] \left\{ {j \atop k} \right\}.$$

The Lah numbers have the following explicit formula [6, p. 134]:

$$\left\lfloor {n \atop k} \right\rfloor = \frac{n!}{k!}\binom{n-1}{k-1}.$$

An ordered partition π of $[n]$ into k blocks is a partition where the order of blocks is important $\pi_\sigma = B_{\sigma(1)}/B_{\sigma(2)}/\cdots/B_{\sigma(k)}$, where σ is a permutation of $[k]$. The number of ordered set partitions of $[n]$ into k blocks is given by $k!\left\{ {n \atop k} \right\}$, [9, p. 106], as follows:

$$k!\left\{ {n \atop k} \right\} = \sum_{\substack{r_1+r_2+\cdots+r_k=n \\ r_i \geq 1}} \binom{n}{r_1, r_2, \ldots, r_k}, \tag{1}$$

where $\binom{n}{r_1,r_2,\ldots,r_k} = \frac{n!}{r_1!r_2!\cdots r_k!}$ is the multinomial coefficient. The coefficients $\binom{n}{r_1,r_2,\ldots,r_k}$ have the following horizontal generating function

$$(x_1+x_2+\cdots+x_k)^n = \sum_{r_1+r_2+\cdots+r_k=n} \binom{n}{r_1, r_2, \ldots, r_k} x_1^{r_1} x_2^{r_2} \cdots x_k^{r_k}. \tag{2}$$

The relation (2) can be generalized to multivariate falling and rising factorials [7, p. 149]:

$$\langle x_1+x_2+\cdots+x_k\rangle_n = \sum_{r_1+r_2+\cdots+r_k=n} \binom{n}{r_1, r_2, \ldots, r_k} \langle x_1\rangle_{r_1} \langle x_2\rangle_{r_2} \cdots \langle x_k\rangle_{r_k}, \tag{3}$$

and

$$(x_1+x_2+\cdots+x_k)_n = \sum_{r_1+r_2+\cdots+r_k=n} \binom{n}{r_1, r_2, \ldots, r_k} (x_1)_{r_1} (x_2)_{r_2} \cdots (x_k)_{r_k}. \tag{4}$$

Many authors have investigated the Stirling and Lah numbers; see, for instance, [1, 2, 4, 8, 12].

The present paper is organized as follows: Section 2 introduces the multivariate Lah numbers *multipartitions* set which generalize the set partitions and ordered set partitions. Section 3 presents several properties of the multivariate Lah numbers by algebraic and combinatorial arguments. In the last section, we define the multivariate Stirling numbers of the first and second kinds and we provide an expression for the multivariate Lah numbers in terms of the multivariate Stirling numbers and multinomial Stirling numbers were introduced by Moak [10].

2 Combinatorial definition of multivariate Lah numbers

Let $\mathbf{r}_k := (r_1, r_2, \ldots, r_k)$ be a sequence of nonnegative integers. Now suppose that we have k categories of lists $(C_1, C_2, \ldots, C_k)$ such that $|C_i| = r_i$. Let $\mathcal{OP}_n^{\mathbf{r}_k}$ be the set partitions of $[n]$ into $(r_1 + r_2 + \cdots + r_k)$-lists. A partition $\pi \in \mathcal{OP}_n^{\mathbf{r}_k}$ is of the form $\pi = B_1^{r_1}/B_2^{r_2}/\cdots/B_k^{r_k}$ where $B_i^{r_i} = \underbrace{B_i/B_i/\cdots/B_i}_{r_i \text{ times}}$.

Definition 1. Let $\pi = B_1^{r_1}/B_2^{r_2}/\cdots/B_k^{r_k}$ be a partition of the set $\mathcal{OP}_n^{\mathbf{r}_k}$. A *multipartition* is a permutation of the multiset $\{B_1^{r_1}, B_2^{r_2}, \ldots, B_k^{r_k}\}$. We let $\mathfrak{S}_{n,\mathbf{r}_k}(\pi)$ denote the set of all multipartitions of the multiset $\{B_1^{r_1}, B_2^{r_2}, \ldots, B_k^{r_k}\}$.

Example 2. Let $\pi = \underbrace{1,2}_{B_1}/\underbrace{3}_{B_1}/\underbrace{4,5}_{B_2}/\underbrace{6}_{B_3}$ be a partition of the set $[6]$ into $(2,1,1)$-lists. The set $\mathfrak{S}_{6,\mathbf{r}_3}(\pi)$ of multipartitions associated with the partition π is

$$\underbrace{1,2}_{B_1}/\underbrace{3}_{B_1}/\underbrace{4,5}_{B_2}/\underbrace{6}_{B_3} \quad \underbrace{1,2}_{B_1}/\underbrace{4,5}_{B_2}/\underbrace{3}_{B_1}/\underbrace{6}_{B_3}; \quad \underbrace{1,2}_{B_1}/\underbrace{4,5}_{B_2}/\underbrace{6}_{B_3}/\underbrace{3}_{B_1};$$
$$\underbrace{4,5}_{B_2}/\underbrace{1,2}_{B_1}/\underbrace{6}_{B_3}/\underbrace{3}_{B_1}; \quad \underbrace{4,5}_{B_2}/\underbrace{6}_{B_3}/\underbrace{1,2}_{B_1}/\underbrace{3}_{B_1}; \quad \underbrace{4,5}_{B_2}/\underbrace{1,2}_{B_1}/\underbrace{3}_{B_1}/\underbrace{6}_{B_3}.$$

Definition 3. For any n, $r_1, r_2, \ldots, r_k \geq 1$, the multivariate Lah number, which we denote by $\left\lfloor {n \atop r_1, r_2, \ldots, r_k} \right\rfloor$, is the number of multipartitions of the set $[n]$ into nonempty $\mathbf{r}_k$-lists.

$$\left\lfloor {n \atop r_1, r_2, \ldots, r_k} \right\rfloor = \sum_{\pi \in \mathcal{OP}_n^{\mathbf{r}_k}} |\mathfrak{S}_{n,\mathbf{r}_k}(\pi)|. \tag{5}$$

Example 4. There are 3 multipartitions of the set $[3]$ into $(1,2)$-lists

$$1/2/3\ ; \quad 2/1/3\ ; \quad 2/3/1.$$

We have $\left\lfloor {3 \atop 1,1} \right\rfloor = 12$, so the corresponding multipartitions are

$$1/2,3\ ; \quad 2,3/1\ ; \quad 1/3,2\ ; \quad 3,2/1\ ; \quad 1,2/3\ ; \quad 3/1,2\ ;$$
$$2,1/3\ ; \quad 3/2,1\ ; \quad 1,3/2\ ; \quad 2/1,3\ ; \quad 3,1/2\ ; \quad 2/3,1.$$

In the following theorem we provide an explicit formula for the multivariate Lah numbers.

Theorem 5. *For any n, $r_1, r_2, \ldots, r_k \geq 1$, we have*

$$\left\lfloor {n \atop r_1, r_2, \ldots, r_k} \right\rfloor = \frac{(n-1)!}{(r_1 + r_2 + \cdots + r_k - 1)!} \binom{n}{r_1, r_2, \ldots, r_k, n - \sum_{j=1}^{k} r_j}. \tag{6}$$

Proof. To construct a multipartition of $[n]$ into $\mathbf{r}_k$-lists we can do the following. First, we select r_1 elements from $[n]$, each element corresponding to the beginning of one list of category C_1. There are $\binom{n}{r_1}$ possibilities. Then, we choose r_2 elements from the remaining $n-r_1$ elements, which we place at the beginning of the lists of category C_2 with $\binom{n-r_1}{r_2}$ possible ways, and so on. We choose r_k of the remaining $n-r_1-\cdots-r_{k-1}$ elements, which we place at the start of the lists of category C_k. There are $\binom{n-r_1-\cdots-r_{k-1}}{r_k}$ possibilities. The remaining $n-r_1-\cdots-r_k$ elements can be added with $(r_1+\cdots+r_k)(r_1+\cdots+r_k+1)\cdots(n-1)$ possibilities. This gives us,

$$\begin{aligned}\binom{n}{r_1}\binom{n-r_1}{r_2} \quad \cdots \quad \binom{n-r_1-\cdots-r_{k-1}}{r_k} \prod_{i=0}^{n-1-r_1-\cdots-r_k)} (n-1-i) \\ = \quad \frac{(n-1)!}{(r_1+r_2+\cdots+r_k-1)!}\binom{n}{r_1,r_2,\ldots,r_k,n-\sum_{j=1}^{k} r_j},\end{aligned}$$

which completes the proof. □

As particular cases of the multivariate Lah numbers, when $k=1$ we obtain the classical Lah numbers and for $r_1+r_2+\cdots+r_k=n$ we get the multinomial coefficient

$$\left\lfloor \begin{matrix} n \\ r_1,r_2,\ldots,r_k \end{matrix} \right\rfloor = \binom{n}{r_1,r_2,\ldots,r_k}.$$

Also, when $r_i=1$ for all $i\in[k]$ we obtain the ordered Lah numbers,

$$\left\lfloor \begin{matrix} n \\ \underbrace{1,1,\ldots,1}_{k-\text{times}} \end{matrix} \right\rfloor = n!\binom{n-1}{k-1}.$$

Let $\sigma(1),\sigma(2),\ldots,\sigma(k)$ be a permutation of $[k]$. Then

$$\left\lfloor \begin{matrix} n \\ r_1,r_2,\ldots,r_k \end{matrix} \right\rfloor = \left\lfloor \begin{matrix} n \\ r_{\sigma(1)},r_{\sigma(2)},\ldots,r_{\sigma(k)} \end{matrix} \right\rfloor.$$

From Relation (6), we deduce an expression for the multivariate Lah numbers in terms of classical Lah numbers.

Corollary 6. *For any $n,r_1,\ldots,r_k\geq 1$, we have*

$$\left\lfloor \begin{matrix} n \\ r_1,r_2,\ldots,r_k \end{matrix} \right\rfloor = \binom{r_1+r_2+\cdots+r_k}{r_1,r_2,\ldots,r_k}\left\lfloor \begin{matrix} n \\ r_1+r_2+\cdots+r_k \end{matrix} \right\rfloor. \tag{7}$$

3 Fundamental properties of multivariate Lah numbers

In this section, we provide some fundamental properties of the multivariate Lah numbers. We start by given the exponential generating function.

Theorem 7. *The exponential generating function of the multivariate Lah numbers is*

$$\sum_{n\geq 0} \left\lfloor {n \atop r_1, r_2, \ldots, r_k} \right\rfloor \frac{t^n}{n!} = \prod_{i=1}^{k} \frac{1}{r_i!} \left(\frac{t}{1-t} \right)^{r_i}. \tag{8}$$

Proof. From Theorem 5 and relation (7), we have

$$\begin{aligned}
\sum_{n\geq 0} \left\lfloor {n \atop r_1, \ldots, r_k} \right\rfloor \frac{t^n}{n!} &= \sum_{n\geq 0} \frac{(n-1)!}{(r_1+\cdots+r_k-1)!} \binom{n}{r_1, \ldots, r_k, n-\sum_{j=1}^{k} r_j} \frac{t^n}{n!} \\
&= \frac{1}{r_1!\cdots r_k!} \sum_{n\geq 0} \binom{n-1}{r_1+r_2+\cdots+r_k-1} t^n \\
&= \frac{t^{r_1+r_2+\cdots+r_k}}{r_1!\cdots r_k!} \sum_{n\geq 0} \binom{n-1}{n-r_1-r_2-\cdots-r_k} t^n \\
&= \frac{t^{r_1+r_2+\cdots+r_k}}{r_1!\cdots r_k!} \sum_{n\geq 0} \binom{n+r_1+r_2+\cdots+r_k-1}{n} t^n \\
&= \frac{t^{r_1+r_2+\cdots+r_k}}{r_1!\cdots r_k!} \sum_{n\geq 0} \binom{-r_1-r_2-\cdots-r_k}{n} (-t)^n \\
&= \frac{1}{r_1!\cdots r_k!} \prod_{i=1}^{k} \left(\frac{t}{1-t} \right)^{r_i},
\end{aligned}$$

which completes the proof. □

In the following theorem, we give the multivariate exponential generating function for the multivariate Lah numbers.

Theorem 8. *We have*

$$\sum_{n\geq 0} \sum_{r_1\geq 0} \sum_{r_2\geq 0} \cdots \sum_{r_k\geq 0} \left\lfloor {n \atop r_1, r_2, \ldots, r_k} \right\rfloor \frac{t^n}{n!} z_1^{r_1} z_2^{r_2} \cdots z_k^{r_k} = \exp\left(\frac{t}{1-t}(z_1+z_2+\cdots+z_k) \right). \tag{9}$$

Proof. The result follows immediately from Theorem (7). □

An expression for the multivariate falling factorial in terms of the classical rising factorial is given by the following theorem.

Theorem 9. *For any $n \geq 1$, we have*

$$(x_1 + x_2 + \cdots + x_k)_n = \sum_{r_1+r_2+\cdots+r_k \leq n} \left\lfloor \begin{matrix} n \\ r_1, r_2, \ldots, r_k \end{matrix} \right\rfloor \langle x_1 \rangle_{r_1} \langle x_2 \rangle_{r_2} \cdots \langle x_k \rangle_{r_k}, \tag{10}$$

and

$$\langle x_1 + x_2 + \cdots + x_k \rangle_n = \sum_{r_1+r_2+\cdots+r_k \leq n} (-1)^{n-r_1-\cdots-r_k} \left\lfloor \begin{matrix} n \\ r_1, r_2, \ldots, r_k \end{matrix} \right\rfloor (x_1)_{r_1} (x_2)_{r_2} \cdots (x_k)_{r_k}. \tag{11}$$

Proof. We have

$$\begin{aligned}
\left(1 + \frac{t}{1-t}\right)^{x_1+x_2+\cdots+x_k} &= \left(1 + \frac{t}{1-t}\right)^{x_1} \left(1 + \frac{t}{1-t}\right)^{x_2} \cdots \left(1 + \frac{t}{1-t}\right)^{x_k} \\
&= \left(\sum_{r_1 \geq 0} \left(\frac{t}{1-t}\right)^{r_1} \frac{\langle x_1 \rangle_{r_1}}{r_1!} \right) \left(\sum_{r_2 \geq 0} \left(\frac{t}{1-t}\right)^{r_2} \frac{\langle x_2 \rangle_{r_2}}{r_2!} \right) \\
&\quad \times \cdots \times \left(\sum_{r_k \geq 0} \left(\frac{t}{1-t}\right)^{r_k} \frac{\langle r_k \rangle_{r_k}}{r_k!} \right) \\
&= \sum_{r_1, r_2, \ldots, r_k \geq 0} \prod_{i=1}^{k} \frac{1}{r_i!} \left(\frac{t}{1-t}\right)^{r_i} \langle x_1 \rangle_{r_1} \langle x_2 \rangle_{r_2} \cdots \langle x_k \rangle_{r_k} \\
&= \sum_{r_1, r_2, \ldots, r_k} \left(\sum_{n \geq 0} \left\lfloor \begin{matrix} n \\ r_1, r_2, \ldots, r_k \end{matrix} \right\rfloor \frac{t^n}{n!} \right) \langle x_1 \rangle_{r_1} \langle x_2 \rangle_{r_2} \cdots \langle x_k \rangle_{r_k} \\
&= \sum_{n \geq 0} \sum_{r_1, r_2, \ldots, r_k} \left\lfloor \begin{matrix} n \\ r_1, r_2, \ldots, r_k \end{matrix} \right\rfloor \langle x_1 \rangle_{r_1} \langle x_2 \rangle_{r_2} \cdots \langle x_k \rangle_{r_k} \frac{t^n}{n!}. \qquad (12)
\end{aligned}$$

The left hand side of the first equality is

$$\begin{aligned}
\left(1 + \frac{t}{1-t}\right)^{x_1+x_2+\cdots+x_k} &= (1-t)^{-(x_1+x_2+\cdots+x_k)} \\
&= \sum_{n \geq 0} (-1)^n \binom{-x_1 - x_2 - \cdots - x_k}{n} t^n \\
&= \sum_{n \geq 0} \binom{x_1 + x_2 + \cdots + x_k + n - 1}{n} t^n \\
&= \sum_{n \geq 0} (x_1 + x_2 + \cdots + x_k)_n \frac{t^n}{n!}. \qquad (13)
\end{aligned}$$

Equating the coefficients of $\frac{t^n}{n!}$ in (12) and (13) yields (10). Equation (11) follows by substituting $(-x_i)$ for $i \in [k]$. □

Using Theorem 9, we get, for example

$$
\begin{aligned}
(x_1+x_2)_2 &= \langle x_1\rangle_2 + 2x_1x_2 + \langle x_2\rangle_2 + 2(x_1+x_2),\\
(x_1+x_2)_3 &= \langle x_1\rangle_3 + 3\langle x_1\rangle_2 x_2 + 3x_1\langle x_2\rangle_2 + \langle x_2\rangle_3 + 6(\langle x_1\rangle_2 + 2x_1x_2 + \langle x_2\rangle_2)\\
&\quad + 6(x_1+x_2),\\
(x_1+x_2+x_3)_2 &= \langle x_1\rangle_2 + \langle x_2\rangle_2 + \langle x_3\rangle_2 + 2(x_1x_2+x_1x_3+x_2x_3) + 2(x_1+x_2+x_3).\\
\langle x_1+x_2\rangle_2 &= 3(x_1)_2 - 6x_1 - 6x_2 + 6x_1x_2 + 3(x_2)_2.
\end{aligned}
$$

Theorem 10. *For any $n, r_1, r_2, \ldots, r_k \geq 1$, we have*

$$
\sum_{s_1+s_2+\ldots+s_k\leq n} (-1)^{\sum_i s_i - r_i} \left\lfloor {n \atop s_1, s_2, \ldots, s_k} \right\rfloor \left\lfloor {s_1 \atop r_1} \right\rfloor \left\lfloor {s_2 \atop r_2} \right\rfloor \cdots \left\lfloor {s_k \atop r_k} \right\rfloor = \binom{n}{r_1, r_2, \ldots, r_k}. \tag{14}
$$

Proof. From Relation (10), we have

$$
\begin{aligned}
(x_1+x_2+\cdots+x_k)_n &= \sum_{s_1+s_2+\cdots+s_k\leq n} \left\lfloor {n \atop s_1, s_2, \ldots, s_k} \right\rfloor \langle x_1\rangle_{s_1}\langle x_2\rangle_{s_2}\cdots\langle x_k\rangle_{s_k}\\
&= \sum_{s_1+s_2+\cdots+s_k\leq n} \left\lfloor {n \atop s_1, s_2, \ldots, s_k} \right\rfloor \sum_{r_1,r_2,\ldots,r_k} (-1)^{\sum_i s_i - r_i} \left\lfloor {s_1 \atop r_1} \right\rfloor \left\lfloor {s_2 \atop r_2} \right\rfloor \cdots \left\lfloor {s_k \atop r_k} \right\rfloor\\
&\qquad \times (x_1)_{r_1}(x_2)_{r_2}\cdots(x_k)_{r_k}.
\end{aligned}
$$

On other hand, we have

$$
(x_1+x_2+\cdots+x_k)_n = \sum_{r_1+\cdots+r_k=n} \binom{n}{r_1, r_2, \ldots, r_k} (x)_{r_1}(x)_{r_2}\cdots(x)_{r_k}. \tag{15}
$$

Equating the coefficients of $(x)_{r_1}(x)_{r_2}\cdots(x)_{r_k}$ we obtain the result. □

Corollary 11. *The following identity holds*

$$
(x_1)_{r_1}(x_2)_{r_2}\cdots(x_k)_{r_k} = \sum_{s_1,s_2,\ldots,s_k} \left\lfloor {r_1 \atop s_1} \right\rfloor \left\lfloor {r_2 \atop s_2} \right\rfloor \cdots \left\lfloor {r_k \atop s_k} \right\rfloor \langle x_1\rangle_{s_1}\langle x_2\rangle_{s_2}\cdots\langle x_k\rangle_{s_k}. \tag{16}
$$

In the next theorem, we give a convolution identity involving the multivariate Lah numbers.

Theorem 12. *We have*

$$
\prod_{i=1}^{k} \binom{r_i+s_i}{r_i} \left\lfloor {n \atop r_1+s_1, r_2+s_2, \ldots, r_k+s_k} \right\rfloor = \sum_{j=1}^{n} \binom{n}{j} \left\lfloor {j \atop r_1, r_2, \ldots, r_k} \right\rfloor \left\lfloor {n-j \atop s_1, s_2, \ldots, s_k} \right\rfloor. \tag{17}
$$

Proof. From Theorem 7, we have

$$\begin{aligned}\sum_{n\geq 0}\prod_{i=1}^{k}\binom{r_i+s_i}{r_i}\left\lfloor {n \atop r_1+s_1,\ldots,r_k+s_k}\right\rfloor \frac{t^n}{n!} &= \prod_{i=1}^{k}\binom{r_i+s_i}{r_i}\frac{1}{(r_i+s_i)!}\left(\frac{t}{1-t}\right)^{(r_i+s_i)}\\ &= \left(\prod_{i=1}^{k}\frac{1}{(r_i)!}\left(\frac{t}{1-t}\right)^{r_i}\right)\left(\prod_{j=1}^{k}\frac{1}{(s_j)!}\left(\frac{t}{1-t}\right)^{s_j}\right)\\ &= \left(\sum_{l\geq 0}\left\lfloor {l \atop r_1,r_2,\ldots,r_k}\right\rfloor \frac{t^l}{l!}\right)\\ &\qquad \times\left(\sum_{m\geq 0}\left\lfloor {m \atop s_1,s_2,\ldots,s_k}\right\rfloor \frac{t^m}{m!}\right).\end{aligned}$$

Equating the coefficients of $\frac{t^n}{n!}$ in both sides, we get the desired result. □

Combinatorial proof of Theorem 12. Let $C = \{1, 2, \ldots, k\}$ be a list of k different colors. The left hand side of the identity counts the number of multipartitions of the set $[n]$ into $(r_1+s_1, \ldots, r_k+s_k)$-lists such that the elements of r_i lists among r_i+s_i lists get colour i, for all $i \in [k]$. In the right hand side, we start by choosing j elements from n and there are $\binom{n}{j}$ ways to do. The j elements have to be partitioned into $(r_1, \ldots, r_k)$-lists such that the elements of the lists r_i get colour i and the remaining $n-j$ elements have to be partitioned into $(s_1, \ldots, s_k)$-lists. □

In the following theorem we give a generalization of the formula (17).

Theorem 13. *We have*

$$\prod_{i=1}^{k}\binom{r_i}{r_{1,i},\ldots,r_{t,i}}\left\lfloor {n \atop r_1,r_2,\ldots,r_k}\right\rfloor = \sum_{j_1,\ldots,j_t}\binom{n}{j_1,\ldots,j_t}\prod_{i=1}^{t}\left\lfloor {j_i \atop r_{1,i},r_{2,i},\ldots,r_{t,i}}\right\rfloor. \tag{18}$$

with $r_{1,i}+r_{2,i}+\cdots+r_{t,i} = r_i$ *for* $i \in [t]$.

Now we give some recurrence relations satisfied by the multivariate Lah numbers.

Theorem 14. *The multivariate Lah numbers satisfy the following recurrence relations*

(i) Triangular recurrence relation:

$$\begin{aligned}\left\lfloor {n \atop r_1,r_2,\ldots,r_k}\right\rfloor &= \sum_{i=1}^{k}\left\lfloor {n-1 \atop r_1,\ldots,r_i-1,\ldots,r_k}\right\rfloor\\ &\qquad + (n+r_1+\cdots+r_k-1)\left\lfloor {n-1 \atop r_1,r_2,\ldots,r_k}\right\rfloor,\end{aligned} \tag{19}$$

with initial terms $\left\lfloor {n \atop r_1,r_2,\ldots,r_k}\right\rfloor = 0$ *if* $n < r_1+r_2+\cdots+r_k$.

(ii) Horizontal recurrence relation:

$$\left\lfloor {n \atop r_1, r_2, \ldots, r_k} \right\rfloor = \sum_{r_1+\cdots+r_k \le j \le n} (r_1 + \cdots + r_k + j)_{n-j} \sum_{i=1}^{k} \left\lfloor {j-1 \atop r_1, \ldots, r_i - 1, \ldots, r_k} \right\rfloor. \tag{20}$$

(iii) Diagonal recurrence relation:

$$\left\lfloor {n \atop r_1, r_2, \ldots, r_k} \right\rfloor = \sum_{j=0}^{r_1+\cdots+r_k} \sum_{s_1+\cdots+s_k=j} \binom{j}{s_1, \ldots, s_k} \times (n + r_1 + \cdots + r_k - 2j - 1) \left\lfloor {n-j-1 \atop r_1 - s_1, \ldots, r_k - s_k} \right\rfloor. \tag{21}$$

Proof. Let us show (19). A multipartition of the set $[n]$ into $(r_1, r_2, \ldots, r_k)$-lists can be obtained from a multipartition of the set $[n-1]$ into $(r_1, \cdots, r_i - 1, \cdots, r_k)$-lists $(1 \le i \le k)$ to which we add a single list $\{n\}$ of category C_i, or from a multipartition of the set $[n-1]$ into $(r_1, r_2, \ldots, r_k)$-lists, by adding the element $\{n\}$ before any existing elements or at the end of any list. Then there are $(n + r_1 + \cdots + r_k - 1)\left\lfloor {n-1 \atop r_1, r_2, \ldots, r_k} \right\rfloor$ ways.

Next, we show (20). For a given $r_1 + \cdots + r_k \le j \le n$ and $1 \le i \le k$, let us consider the elements of $[j-1]$ which are not in the same list with the element $\{n\}$. The number of multipartitions of $[j-1]$ into $(r_1, \ldots, r_i - 1, \ldots, r_k)$-lists is $\left\lfloor {j-1 \atop r_1, \ldots, r_i - 1, \ldots, r_k} \right\rfloor$, and there are $(r_1 + \cdots + r_k + j)_{n-j}$ ways to add the remaining elements of $[j, n-1]$ into $(r_1, r_2, \ldots, r_k)$-lists. Summing over all possible j and i gives the result.

Finally, we show (21). Let j $(0 \le j \le r_1 + \cdots + r_k)$ be the number of lists which contain exactly one element, then the number of ways to choose a such lists is $\binom{j}{s_1, \ldots, s_k}$. Now it remains to count the number of multipartitions of $[j+1, n]$ into $(r_1 - s_1, \ldots, r_k - s_k)$-lists. So, the number of multipartitions of $[j+1, n-1]$ into $(r_1 - s_1, \ldots, r_k - s_k)$-lists is $\left\lfloor {n-j-1 \atop r_1 - s_1, \ldots, r_k - s_k} \right\rfloor$ and there are $(n + r_1 + \cdots + r_k - 2j - 1)$ ways to add the element $\{n\}$ in any list. Summing up yields the desired result. □

4 Multivariate Stirling numbers

Definition 15. For any n, $r_1, r_2, \ldots, r_k \ge 0$, the *multivariate Stirling* numbers of the first kind, denoted $\left[{n \atop r_1, r_2, \ldots, r_k} \right]$, are defined as the numbers of multipartitions of the set $[n]$ into $(r_1, r_2, \ldots, r_k)$-cycles. Analogously, we define the *multivariate Stirling* numbers of second kind, denoted $\left\{ {n \atop r_1, r_2, \ldots, r_k} \right\}$, as the numbers of multipartitions of $[n]$ into $(r_1, r_2, \ldots, r_k)$-blocks.

From Definition 15, we deduce that the multivariate Stirling numbers of the first kind satisfy the following recurrence relation

$$\left[{n \atop r_1, r_2, \ldots, r_k} \right] = \sum_{i=1}^{k} \left[{n-1 \atop r_1, \ldots, r_i - 1, \ldots, r_k} \right] + (n-1) \left[{n-1 \atop r_1, r_2, \ldots, r_k} \right], \tag{22}$$

with $\left[{n \atop r_1,r_2,\ldots,r_k}\right] = 0$ if $n < r_1 + r_2 + \cdots + r_k$. The multivariate Stirling numbers of the second kind satisfy the following recurrence relation

$$\left\{ {n \atop r_1, r_2, \ldots, r_k} \right\} = \sum_{i=1}^{k} \left\{ {n-1 \atop r_1, \ldots, r_i - 1, \ldots, r_k} \right\} + (r_1 + r_2 + \cdots + r_k) \left\{ {n-1 \atop r_1, r_2, \ldots, r_k} \right\}. \quad (23)$$

with $\left\{{n \atop r_1,r_2,\ldots,r_k}\right\} = 0$ if $n < r_1 + r_2 + \cdots + r_k$.

As particular cases, we have

$$\left[{n \atop \underbrace{1, 1, \ldots, 1}_{k-\text{times}}} \right] = k! \left[{n \atop k} \right], \quad \text{for } \ n \geq k,$$

$$\left\{ {n \atop \underbrace{1, 1, \ldots, 1}_{k-\text{times}}} \right\} = k! \left\{ {n \atop k} \right\}, \quad \text{for } \ n \geq k,$$

and

$$\left\{ {n \atop r_1, r_2, \ldots, r_k} \right\} = \binom{n}{r_1, r_2, \ldots, r_k}, \quad \text{for } \ r_1 + r_2 + \cdots + r_k = n.$$

Next, we give an explicit formula for the multivariate Stirling of the second kind.

Theorem 16. *For any $n \geq 0$, we have*

$$\left\{ {n \atop r_1, r_2, \ldots, r_k} \right\} = \frac{1}{r_1! r_2! \cdots r_k!} \sum_{j=0}^{r_1 + \cdots + r_k} (-1)^j \binom{r_1 + \cdots + r_k}{j} (r_1 + \cdots + r_k - j)^n. \quad (24)$$

Proof. The result is obtained by applying the inclusion-exclusion principle. □

Theorem 17. *For any $n \geq 1$, we have*

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{r_1 + r_i + \cdots + r_k \leq n} \left\{ {n \atop r_1, r_2, \ldots, r_k} \right\} \langle x_1 \rangle_{r_1} \langle x_2 \rangle_{r_2} \cdots \langle x_k \rangle_{r_k}. \quad (25)$$

and

$$(x_1 + x_2 + \cdots + x_k)_n = \sum_{r_1 + r_i + \cdots + r_k \leq n} \left[{n \atop r_1, r_2, \ldots, r_k} \right] x_1^{r_1} x_2^{r_2} \cdots x_k^{r_k}. \quad (26)$$

Proof. The result is obtained using induction proof. □

Theorem 18. *The exponential generating function of the multivariate Stirling numbers of the first and second kind*

$$\sum_{n \geq 0} \left[{n \atop r_1, r_2, \ldots, r_k} \right] \frac{t^n}{n!} = \prod_{i=1}^{k} \frac{1}{r_i!} \left(\ln \left(\frac{1}{1-t} \right) \right)^{r_i}, \quad (27)$$

and

$$\sum_{n\geq 0}\left\{ n \atop r_1,r_2,\ldots,r_k \right\}\frac{t^n}{n!}=\prod_{i=1}^{k}\frac{(e^t-1)^{r_i}}{r_i!}. \tag{28}$$

From Theorem 18, we obtain

$$\sum_{n\geq 0}\sum_{r_i\geq 0}\left[n \atop r_1,r_2,\ldots,r_k \right]\frac{t^n}{n!}z_1^{r_1}\cdots z_k^{r_k}=(1-t)^{-z_1-\cdots-z_k}, \tag{29}$$

and

$$\sum_{n\geq 0}\sum_{r_i\geq 0}\left\{ n \atop r_1,r_2,\ldots,r_k \right\}\frac{t^n}{n!}z_1^{r_1}\cdots z_k^{r_k}=\exp\left((e^t-1)(z_1+\cdots+z_k)\right). \tag{30}$$

The connection relation between the multivariate Stirling numbers and classical Stirling numbers is

$$\left\{ n \atop r_1,r_2,\ldots,r_k \right\}=\binom{r_1+r_2+\cdots+r_k}{r_1,r_2,\ldots,r_k}\left\{ n \atop r_1+r_2+\cdots+r_k \right\}, \tag{31}$$

and

$$\left[n \atop r_1,r_2,\ldots,r_k \right]=\binom{r_1+r_2+\cdots+r_k}{r_1,r_2,\ldots,r_k}\left[n \atop r_1+r_2+\cdots+r_k \right]. \tag{32}$$

In the following theorem we express the multivariate Lah numbers in terms of the multivariate Stirling numbers.

Theorem 19. *For any $n\geq 1$, we have*

$$\left\lfloor n \atop r_1,r_2,\ldots,r_k \right\rfloor=\sum_{j_1+\cdots+j_k=r_1+\cdots+r_k}^{n}\left[n \atop j_1,j_2,\ldots,j_k \right]\left\{ j_1 \atop r_1 \right\}\left\{ j_2 \atop r_2 \right\}\cdots\left\{ j_k \atop r_k \right\}. \tag{33}$$

Proof. The result is obtained from (10) and (25). ☐

5 Acknowledgments

The author would like to thank the referee and the editor for helpful comments and suggestions.

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2010 *Mathematics Subject Classification*: Primary 11B37; Secondary 05A19.
Keywords: Lah number, Stirling number, ordered set partition, multinomial coefficient.

(Concerned with sequences A008277, A132393, and A271703.)

Received October 27 2019; revised versions received March 14 2020; April 7 2020. Published in *Journal of Integer Sequences*, April 8 2020.

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