Journal of Integer Sequences, Vol. 23 (2020),
Article 20.2.3

The Minimal Excludant in Integer Partitions

George E. Andrews
The Pennsylvania State University
University Park, PA 16802
USA
gea1@psu.edu

David Newman
Far Rockaway, New York
USA
DavidSNewman@gmail.com

Abstract

The minimal excludant, or "mex" function, on a set S of positive integers is the least positive integer not in S. In this paper, the mex function is extended to integer partitions generalized by constricting the universal set from all positive integers to those in certain arithmetic progressions. There are numerous surprising partition identities connected with this restricted mex function. This paper provides an account of some of the most conspicuous cases.

1 Introduction

The minimal excludant function (mex-function) appears extensively in combinatorial game theory (e.g., [7]). For each set S of nonnegative integers we define it as follows:

$$\operatorname{mex}(S) = \min(\mathbb{Z}_{\geq 0} \setminus S). \tag{1}$$

The mex function is perhaps best known in its uses in the game of nim and the Sprague-Grundy theory originating in the 1930's.

We shall generalize the mex-function to apply to integer partitions. We define $\text{mex}_{A,a}(\pi)$ to be the smallest integer congruent to a modulo A that is not a part of the integer partition π.

We define $p_{A,a}(n)$ to be the number of partitions π of n, where

$$\text{mex}_{A,a}(\pi) \equiv a \pmod{2A}, \tag{2}$$

and $\overline{p}_{A,a}(n)$ to be the number of partitions π of n, where

$$\text{mex}_{A,a}(\pi) \equiv A + a \pmod{2A}. \tag{3}$$

If $p(n)$ denotes the number of integer partitions of n, then clearly

$$p(n) = p_{A,a}(n) + \overline{p}_{A,a}(n). \tag{4}$$

For example, consider $n = 4$, $A = 2$, and $a = 1$. There are five partitions of 4, so $p(4) = 5$, where $p(n)$ is the number of partitions of n. These partitions along with first missing odd part (m.o.) are 4 (m.0. 1), $3+1$ (m.o. 5), $2+2$ (m.o. 1), $2+1+1$ (m.o. 3) and $1+1+1+1$ (m.o. 3). Thus three partitions have $\text{mex}_{2,1}$ congruent to 1 mod 4, and two have $\text{mex}_{2,1}$ congruent to 3 mod 4. Thus $p_{2,1}(4) = 3$ and $\overline{p}_{2,1}(4) = 2$.

Let us define

$$F_{A,a}(q) = \sum_{n\geq 0} p_{A,a}(n) q^n, \tag{5}$$

and

$$\overline{F}_{A,a}(q) = \sum_{n\geq 0} \overline{p}_{A,a}(n) q^n. \tag{6}$$

Then

$$\begin{aligned} F_{A,a}(q) + \overline{F}_{A,a}(q) &= \sum_{n\geq 0} p(n) q^n \\ &= \prod_{n=1}^{\infty} \frac{1}{1-q^n} := F(q), \text{ by [2, p. 3, Th. 1.1]}. \end{aligned} \tag{7}$$

Thus by (7), every identity for $F_{A,a}(q)$ yields a natural identity for $\overline{F}_{A,a}(q)$ and vice versa.

Our theorems also require $p_e(n)$ (resp., $p_o(n)$), the number of partitions of n into an even (resp., odd) numbers of parts. Hence by the methods of [2, Ch. 1], we get

$$F_e(q) := \sum_{n\geq 0} p_e(n) q^n = \sum_{n=0}^{\infty} \frac{q^{2n}}{(q;q)_{2n}}, \tag{8}$$

and

$$F_o(q) := \sum_{n\geq 0} p_o(n) q^n = \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q;q)_{2n+1}}, \tag{9}$$

where

$$(x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1}). \tag{10}$$

In order to prove our theorems for $p_{1,1}(n)$ and $p_{3,3}(n)$, we must recall two partition statistics, the rank and the crank.

The *rank* of a partition is the largest part minus the number of parts. The *crank* of a partition is the largest part of the partition if there are no ones as parts, and otherwise is the number of parts larger than the number of ones minus the numbers of ones.

Our first theorem (which is actually a lemma for treating $p_{1,1}(n)$) is originally due to Uncu [10]. Our proof differs from his and is included for completeness. Also see Somos [9].

Theorem 1. *The generating function for partitions with non-negative crank is*

$$\frac{1}{(q;q)_\infty}\sum_{n\geq 0}(-1)^n q^{\binom{n+1}{2}}. \tag{11}$$

This leads directly to the following three results:

Theorem 2. $p_{1,1}(n)$ *equals the number of partitions of* n *with non-negative crank.*

Theorem 3. $p_{3,3}(n)$ *equals the number of partitions of* n *with rank* ≥ -1.

Theorem 4. $p_{2,1}(n) = p_e(n)$.

Our last two theorems are the real surprises in this study. The theory of partitions is replete with uncanny partition identities, of which the first Rogers-Ramanujan identity is the most famous example:

Theorem 5 (Rogers-Ramanujan)**.** *[2, Ch. 7] The number of partitions of* n *in which the difference between parts is at least 2 equals the number of partitions of* n *in which all parts are congruent to 1 or 4 modulo 5.*

It turns out that $p_{4,2}(n)$ and $p_{6,3}(n)$ appear in identities of this nature.

Theorem 6. $p_{4,2}(n) - p_o(n)$ *equals the number of partitions of* n *into parts congruent to* $\pm 4, \pm 6, \pm 8, \pm 10 \pmod{32}$.

Theorem 7. $p_{6,3}(n) - p_o(n)$ *equals the number of partitions of* n *into parts congruent to* $\pm 2, \pm 4, \pm 5, \pm 6, \pm 7, \pm 8 \pmod{24}$.

Table 1 gives the first few values of these sequences.

n	$p_{11}(n)$	$p_{33}(n)$	$p_{21}(n)$	$p_{42}(n)$	$p_{63}(n)$
0	1	1	1	1	1
1	0	1	0	1	1
2	1	2	1	1	2
3	2	2	1	2	2
4	3	4	3	3	4
5	4	5	3	4	5
6	6	8	6	6	8
7	8	10	7	8	10
8	12	15	12	12	15
9	16	20	14	16	19
10	23	28	22	22	27
11	30	36	27	29	34
12	42	50	40	40	48
13	54	64	49	52	60
14	73	86	69	69	81
15	94	110	86	90	102
16	124	145	118	118	135
17	158	184	146	151	169
18	206	238	195	195	220
19	260	300	242	248	224
20	334	384	317	317	352
21	419	481	392	400	437
22	531	608	505	505	554
23	662	756	623	632	684
24	832	948	793	793	860
25	1029	1172	973	985	1057

Table 1: First few values of $p_{1,1}$, $p_{33}(n)$, $p_{21}(n)$, $p_{42}(n)$, and $p_{63}(n)$

We note that $p_{1,1}(n)$ is sequence A064428 in the *On-Line Encyclopedia of Integer Sequeces* (OEIS), $p_{3,3}(n)$ is A260894, $p_{2,1}(n)$ is A027187, and $p_{4,2}(n)$ is A046682. We also note that none of the references to these sequences in the OEIS refer to our results.

We note that the proof of Theorem 3 relies on the study of Garden of Eden partitions given by Hopkins and Sellers [8], while Theorem 6 and 7 rely on identities given by Blecksmith, Brillhart and Gerst [5].

2 Uncu's theorem

If $M(m,n)$ denotes the number of partitions of n with crank m, then [4]

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} M(m,n)z^m q^n = \frac{(q;q)_\infty}{(zq;q)_\infty(z^{-1}q;q)_\infty}. \tag{12}$$

Now by [2, p. 19, Eq. (2.2.5)] we have

$$\frac{(q;q)_\infty}{(zq;q)_\infty(z^{-1}q;q)_\infty} = (q;q)_\infty \sum_{m=0}^{\infty}\frac{z^m q^m}{(q;q)_m}\sum_{n=0}^{\infty}\frac{z^{-n}q^n}{(q;q)_n}. \tag{13}$$

Thus the generating function for partitions with non-negative crank consists of those terms on the right-hand side of (13), where the exponent on z is non-negative. Hence the required generating function is

$$\begin{aligned}
(q;q)_\infty \sum_{n=0}^{\infty}\sum_{m=n}^{\infty}\frac{q^{m+n}}{(q;q)_m(q;q)_n} &= (q;q)_\infty \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{m+2n}}{(q;q)_{m+n}(q;q)_n} \\
&= (q;q)_\infty \sum_{m=0}^{\infty}\frac{q^m}{(q;q)_m}\sum_{n=0}^{\infty}\frac{q^{2n}}{(q;q)_n(q^{m+1};q)_n} \\
&= (q;q)_\infty \sum_{m=0}^{\infty}\frac{q^m}{(q;q)_m(q^2;q)_\infty(q^{m+1};q)_\infty} \\
&\quad \times \sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n+1}{2}+mn}(q^2;q)_n}{(q;q)_n} \\
&\quad \text{(by [2, p. 19, Cor. 2.3, } a=b=0,\ c=q^{m+1},\ t=q^2\text{])} \\
&= \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{\binom{n+1}{2}}(1-q^{n+1})\sum_{m=0}^{\infty}q^{m(n+1)} \\
&= \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{\binom{n+1}{2}}.
\end{aligned}$$

3 Background lemmas

Lemma 8.

$$F_{2k,k}(q) = \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{kn^2}. \tag{14}$$

Proof.

$$\begin{aligned}\frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{kn^2} &= \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty} q^{4kn^2}(1-q^{4kn+k}) \\ &= \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty} q^{k+3k+5k+\cdots+k(4n-1)}(1-q^{k(4n+1)}) \\ &= \sum_{n=0}^{\infty}\frac{q^{k+3k+5k+\cdots+k(4n-1)}}{\prod_{\substack{m=1\\ m\neq k(4n+1)}}^{\infty}(1-q^m)},\end{aligned}$$

and thus this last expression is clearly the generating function for $p_{2k,k}(n)$. □

Lemma 9.

$$F_{k,k}(q) = \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{k\binom{n+1}{2}}. \tag{15}$$

Proof.

$$\begin{aligned}\frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{k\binom{n+1}{2}} &= \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty} q^{k\binom{2n+1}{2}}(1-q^{k(2n+1)}) \\ &= \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty} q^{k+2k+\cdots+2nk}(1-q^{k(2n+1)}) \\ &= \sum_{n=0}^{\infty}\frac{q^{k+2k+\cdots+2nk}}{\prod_{\substack{m=1\\ m\neq k(2n+1)}}^{\infty}(1-q^m)},\end{aligned}$$

and this last expression is clearly the generating function for $p_{k,k}(n)$. □

4 Proof of Theorem 2

Proof.

$$\sum_{n=0}^{\infty} p_{1,1}(n)q^n = \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{\binom{n+1}{2}} \qquad \text{(by Lemma 9 with } k=1)$$

and by Theorem 1, this is the generating function for partitions with non-negative crank. □

5 Proof of Theorem 3

Proof. In [8], Hopkins and Sellers prove that Garden of Eden partitions are equinumerous with partitions whose rank is ≤ -2. They then quote the work of Dyson [6] to reveal that this generating function is

$$\frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}q^{3\binom{n+1}{2}}.$$

Hence

$$F_{3,3}(q) = \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^{n}q^{3\binom{n+1}{2}} = \frac{1}{(q;q)_\infty}\left(1-\sum_{n=1}^{\infty}(-1)^{n-1}q^{3\binom{n+1}{2}}\right)$$

and this generates all partitions excluding those with rank ≤ -2, i.e., partitions with rank ≥ -1. □

6 Proof of Theorem 4

Proof.

$$\begin{aligned}
F_{2,1}(q) &= \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{n^2} \\
&= \frac{1}{2(q;q)_\infty}\left(\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}+1\right) \\
&= \frac{1}{2(q;q)_\infty}\left(\frac{(q;q)_\infty}{(-q;q)_\infty}+1\right) \\
&= \frac{1}{2}\left(\frac{1}{(-q;q)_\infty}+\frac{1}{(q;q)_\infty}\right) \\
&= \frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{(-1)^n q^n}{(q;q)_n}+\sum_{n=0}^{\infty}\frac{q^n}{(q;q)_n}\right) \\
&= \sum_{n=0}^{\infty}\frac{q^{2n}}{(q;q)_{2n}} \\
&= \sum_{n\geq 0}p_e(n)q^n.
\end{aligned}$$

□

7 Proof of Theorem 6

Proof. In the work of Blecksmith, Brillhart and Gerst [5, Theorem 1 (a)], we find

$$\prod_{\substack{n=1\\ n\not\equiv\pm4,\pm6,\pm8,\pm10 \pmod{32}}}^{\infty}(1-q^n) = 1+\sum_{n=1}^{\infty}(-1)^n\left(q^{n^2}+q^{2n^2}\right). \tag{16}$$

Let us now divide both sides of (16) by $(q;q)_\infty$, and we find

$$\begin{aligned}\prod_{\substack{n=1\\ n\equiv\pm4,\pm6,\pm8,\pm10 \pmod{32}}}^{\infty}\frac{1}{1-q^n} &= F_{2,1}(q)+F_{4,2}(q)-\frac{1}{(q;q)_\infty} \qquad (17)\\ &= F_{4,2}(q)-\overline{F_{2,1}}(q) \qquad \text{(by (7))}\\ &= F_{4,2}(q)-\sum_{n=0}^{\infty}p_o(n)q^n, \qquad \text{(by Theorem 4 and (7))}\end{aligned}$$

and this is equivalent to Theorem 6. □

8 Proof of Theorem 7

Proof. Again, we turn to the paper of Blecksmith, Brillhart and Gerst, where we find [5, Theorem 3 (a)]

$$\prod_{\substack{n=1\\ n\not\equiv\pm2,\pm4,\pm5,\pm6,\pm7,\pm8 \pmod{24}}}^{\infty}(1-q^n) = 1+\sum_{n=1}^{\infty}(-1)^n\left(q^{n^2}+q^{3n^2}\right). \tag{18}$$

We now divide by $(q;q)_\infty$ to obtain

$$\begin{aligned}\prod_{\substack{n=1\\ n\equiv\pm2,\pm4,\pm6,\pm7,\pm8 \pmod{24}}}\frac{1}{1-q^n} &= F_{2,1}(q)+F_{6,3}(q)-\frac{1}{(q;q)_\infty} \qquad (19)\\ &= F_{6,3}(q)-\overline{F_{2,1}}(q) \qquad \text{(by (7))}\\ &= F_{6,3}(q)-\sum_{n=0}^{\infty}p_o(n)q^n, \qquad \text{(by Theorem 4 and (7))}\end{aligned}$$

and this is equivalent to Theorem 7. □

9 Conclusion

We would suggest that this is the tip of the iceberg in the study of $\text{mex}_{A,a}(\pi)$. It is our hope that there are many interesting results awaiting further inquiry.

Among the more obvious questions are these:

I. Are there bijective proofs of any of these theorems?

II. Is there a theorem for $p_{2,2}(n)$ comparable to Theorem 2 and 3?

To study $p_{2,2}(n)$, we would need to know more about

$$F_{2,2}(q) = \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{n^2+n}.$$

Utilizing the weak form of Bailey's lemma [1, eqs. (3.27) and (3.33) with $a = q$, $\alpha_r = (-1)^n$] we see that

$$F_{2,2}(q) = \sum_{n\geq 0} \frac{q^{n^2+n} B_n(q)}{(q;q)_{2n+1}},$$

where

$$B_n(q) = \sum_{r=0}^{n} (-1)^r \begin{bmatrix} 2n+1 \\ n-r \end{bmatrix}$$

with the q-binomial coefficient given by

$$\begin{bmatrix} m \\ n \end{bmatrix} = \frac{(q;q)_m}{(q;q)_n (q;q)_{m-n}}.$$

It appears that $B_n(q)$

(1) has nonnegative coefficients;

(2) is unimodal; and

(3) enumerates some subset of the partitions into at most n parts each $\leq n+1$.

We are unable to prove any of these assertions.

III. Are there comparable theorems for other $F_{A,a}(q)$?

We would note that in terms of classical theta functions

$$F_{2k,k}(q) = \frac{1}{2(q;q)_\infty} (\theta_4(0, q^k) - 1).$$

Thus one can insert $F_{2k,k}(q)$ into classical modular equations such as [10, p. 289, penultimate equation]

$$\psi(q^2)\theta_4(0, q^5) + q\psi(q^{10})\theta_4(0, q) = (-q;q)_\infty (-q^5; -q^5)_\infty; \tag{20}$$

where [2, p. 23, Eq. (2.2.13)]

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}.$$

However, such complex identities do not appear to yield appealing partition identities comparable to any of Theorems 1 to 7.

IV. Are there other partition-theoretic objects that are related to instances of $p_{A,a}(n)$?

With regard to IV., we note that in [3, Sec. 5] it is proved that $p_{1,1}(n)$ is also the number of concatenated spiral self-avoiding walks with an odd number of turns.

References

[1] George E. Andrews, *q-Series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra*, Vol. 66 of *CBMS Regional Conference Series in Mathematics*, American Mathematical Society, 1986.

[2] George E. Andrews, *The Theory of Partitions*, Cambridge Mathematical Library, Cambridge University Press, 1998.

[3] George E. Andrews, Concave compositions, *Electron. J. Combin.* **18** (2) (2011), Paper 6.

[4] George E. Andrews and F. G. Garvan, Dyson's crank of a partition, *Bull. Amer. Math. Soc.* **18** (1988), 167–171.

[5] Richard Blecksmith, John Brillhart, and Irving Gerst, Parity results for certain partition functions and identities similar to theta function identities, *Math. Comp.* **48** (1987), 29–38.

[6] F. J. Dyson, Some guesses in the theory of partitions, *Eureka* **8** (1944), 10–15.

[7] Aviezri S. Fraenkel and Udi Peled, Harnessing the unwieldy MEX function. In *Games Of No Chance 4*, Vol. 63 of *Math. Sci. Res. Inst. Publ.*, pp. 77–94. Cambridge Univ. Press, 2015.

[8] Brian Hopkins and James A. Sellers, Exact enumeration of Garden of Eden partitions. In B. Landman, M. Nathanson, J. Nešetřil, R. Nowakowski and C. Pomerance, eds., *Combinatorial Number Theory*, pp. 299–303. De Gruyter, Berlin, 2007.

[9] M. Somos, contribution to sequence A064428, in N. J. A. Sloane et al., *On-Line Encyclopedia of Integer Sequences*. Posted January 15 2018.

[10] Ali Kemal Uncu, Weighted Rogers-Ramanujan partitions and Dyson crank, *Ramanujan J.* **46** (2018), 579–591.

2010 *Mathematics Subject Classification*: Primary 11P81; Secondary 11A63, 05A19.
Keywords: minimal excludant, mex function, theta function, partition.

(Concerned with sequences A027187, A046682, A064428, and A260894.)

Received February 24 2019; revised versions received July 24 2019; October 26 2019; October 30 2019; February 16 2020. Published in *Journal of Integer Sequences*, February 17 2020.

Return to Journal of Integer Sequences home page.