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Log-Concavity and LC-Positivity for Generalized Triangles

Moussa Ahmia
LMAM Laboratory
Department of Mathematics
UMSB
Jijel 18000
Algeria
ahmiamoussa@gmail.com

Hacène Belbachir
USTHB
Faculty of Mathematics
RECITS Laboratory
BP 32
El Alia 16111
Bab Ezzouar
Algiers
Algeria
hacenebelbachir@gmail.com

Abstract

In this paper, we propose the generalized triangles called *s-triangles* for s given positive integer, as a *bi*-indexed sequence of nonnegative numbers $\{a_s(n,k)\}_{0\leq k\leq ns}$ satisfying $a_s(n,k)=0$ for $k<0$. We extend some results of Wang and Yeh, and show that if the *s-triangle* is LC-positive (resp., doubly LC-positive) then it preserves (resp., it doubly preserves) the log-concavity of the sequences. Applications related to bi^{s}nomial coefficients are given.

1 Introduction

A sequence of nonnegative numbers $(x_k)_k$ is *log-concave* (LC for short) if $x_{i-1}x_{i+1} \leq x_i^2$ for all $i > 0$, which is equivalent to $x_{i-1}x_{j+1} \leq x_i x_j$ for all $j \geq i \geq 1$; see [11]. Log-concave sequences arise often in combinatorics, algebra, geometry, analysis, probability and statistics and have been extensively investigated; see Stanley [26] and Brenti [11] for details.

For two polynomials with real coefficients $A(q)$ and $B(q)$, we write $A(q) \geq_q B(q)$ if the difference $A(q) - B(q)$ has only nonnegative coefficients. A polynomial sequence $(A_n(q))_{n\geq 0}$ is called q-log-concave (as introduced by Sagan [23]) if

$$A_{n-1}(q)A_{n+1}(q) \leq_q A_n(q)^2$$

for $n \geq 1$.

It is easy to see that if the sequence $(A_n(q))_{n\geq 0}$ is q-log-concave, then for each fixed nonnegative number q, the sequence $(f_n(q))_{n\geq 0}$ is log-concave. The q-log-concavity of polynomials have been extensively studied; see Butler [14], Krattenthaler [18], Leroux [19] and Sagan [23, 24], for instance.

Let $\{a_s(n,k)\}_{0\leq k\leq ns}$ be a s-triangle of nonnegative numbers with $s \geq 1$. We illustrate a 4-triangle as follows:

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	$\star$																
1	$\star$	$\star$	$\star$	$\star$	$\star$												
2	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$								
3	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$				
4	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$

Table 1. The 4-triangle.

A nice example of such s-triangles is the triangle given by the ordinary multinomials or bi^{s}nomial coefficients [9]: let $s \geq 1$ and $n \geq 0$ be two integers, and $k = 0, 1, \ldots, sn$, the bi^{s}nomial number $\binom{n}{k}_s$ is defined as the k-th coefficient in the expansion

$$(1 + x + x^2 + \cdots + x^s)^n = \sum_{k\geq 0} \binom{n}{k}_s x^k. \tag{1}$$

Below we list some related identities for the bi^{s}nomial coefficients. For more details see [9] and references therein.

- Expression of bi^{s}nomial coefficients in terms of binomial coefficients,

$$\binom{n}{k}_s = \sum_{j_1+j_2+\cdots+j_s=k} \binom{n}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{s-1}}{j_s}. \tag{2}$$

- The symmetry relation

$$\binom{n}{k}_s = \binom{n}{sn-k}_s. \tag{3}$$

- The longitudinal recurrence relation

$$\binom{n}{k}_s = \sum_{j=0}^{s} \binom{n-1}{k-j}_s. \tag{4}$$

These coefficients, as for usual binomial coefficients, are defined as in the Pascal triangle known as the "s-Pascal triangle". One can find the first values of the s-Pascal triangle in the *On-Line Encyclopedia of Integer Sequences* (OEIS) [25] as A027907 for $s = 2$, as A008287 for $s = 3$, and as A035343 for $s = 4$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1												
1	1	1	1										
2	1	2	3	2	1								
3	1	3	6	7	6	3	1						
4	1	4	10	16	19	16	10	4	1				
5	1	5	15	30	45	51	45	30	15	5	1		
6	1	6	21	50	90	126	141	126	90	50	21	6	1
7	1	7	28	77	161	266	357	393	357	266	161	77	...
8	1	8	36	112	266	504	784	1016	1107	1016	784	504	...

Table 2. Triangle of trinomial coefficients: $s = 2$.

Brondarenko [12] gives a combinatorial interpretation of the bi^{s}nomial coefficient $\binom{n}{k}_s$ as *the number of different ways of distributing "k" balls among "n" cells where each cell contains at most "s" balls.* Using this combinatorial argument, one can easily establish the following relation

$$\binom{n}{k}_s = \sum_{n_1+2n_2+\cdots+sn_s=k} \binom{n}{n_0, n_1, \ldots, n_{s-1}}.$$

These coefficients are also naturally linked to generalized Fibonacci sequence: the "multi-bonacci" sequence, given for $s \geq 1$, by

$$\begin{cases} \Phi_0 = \Phi_1 = \cdots = \Phi_{s-1} = 0, \quad \Phi_s = 1, \\ \Phi_n = \Phi_{n-1} + \Phi_{n-2} + \cdots + \Phi_{n-s-1} \ (n \geq 1). \end{cases}$$

We have the following identity [9]

$$\Phi_{n+1} = \sum_k \binom{n-k}{k}_s.$$

The case $s = 1$ provides a nice identity for Fibonacci numbers (sequence A000045):

$$F_{n+1} = \sum_{k} \binom{n-k}{k}.$$

One of the extensions of binomial coefficients are q-binomial coefficients. Several works and applications were done in this area. For Fibonacci sequences, see Carlitz [15] and Cigler [16]. For Lucas sequences, see Belbachir and Benmezai [7]. For a variant of q-bi^{s}nomials, see Belbachir and Benmezai [6] or our paper [5], and for a recent application to the determinant, see Arikan and Kiliç [4].

Let us consider the following two *linear transformations* of sequences:

$$t_n = \sum_{k=0}^{ns} a_s(n,k)x_k, \quad (n \geq 0), \tag{5}$$

$$z_n = \sum_{k=0}^{ns} a_s(n,k)x_k y_{sn-k}, \quad (n \geq 0). \tag{6}$$

We say that the linear transformation (5) (resp., (6)) has the PLC (resp., double PLC) property if it preserves log-concavity of sequences, i.e., the log-concavity of (x_n) (resp., (x_n) and (y_n)) implies that of (t_n) (resp., (z_n)). The corresponding s-triangle $\{a_s(n,k)\}$ is also called PLC (resp., double PLC).

This is a good way to obtain log-concavity by linear transformations or some operators. For instance, Menon [21] demonstrated that log-concavity is preserved under the ordinary convolution. Walkup in [27], and later, Wang and Yeh [28] also proved that log-concavity is preserved under the binomial convolution. It is also established that the q-binomial convolution preserves log-concavity; see [30]. In [1, 2, 3], we established the preserving log-convexity and log-concavity properties, respectively, for the bi^{s}nomial coefficients and the p,q-binomial coefficients.

In this paper, we generalize the aforementioned results for the generalized triangles like the s-Pascal triangle. In § 2, we give the necessary conditions to establish the PLC (resp., double PLC) property of the generalized triangles $\{a_s(n,k)\}$. In § 3, some examples of the both properties are given include the s-Pascal triangle.

2 LC-positivity and preservation of log-concavity

In this section, we give a relation between LC-positivity (resp., double LC-positivity) and the PLC property (resp., the double PLC property) for generalized triangles. We start with the concept of LC-positivity introduced by Wang and Yeh [28].

Definition 1. Let $s \geq 1$ and $n \geq 0$ be two integers. For $0 \leq r \leq sn$, define the polynomial

$$\mathcal{A}_{s,r}(n;q) := \sum_{k=r}^{ns} a_s(n,k)q^k.$$

We say that the s-triangle $\{a_s(n,k)\}$ has the LC-positive property if for each $r \geq 0$, the sequence of polynomials $(\mathcal{A}_{s,r}(n;q))_{n\geq r}$ is q-log-concave in n.

Definition 2. Let $s \geq 1$ and $n \geq 0$ be two integers. For $0 \leq k \leq sn$, define the reciprocal triangle $\{a_s^*(n,k)\}$ of $\{a_s(n,k)\}$ by

$$a_s^*(n,k) = a_s(n, sn-k)$$

and for $0 \leq r \leq sn$, the polynomial

$$\mathcal{A}_{s,r}^*(n;q) := \sum_{k=r}^{ns} a_s^*(n,k)q^k.$$

We say that the s-triangle $\{a_s(n,k)\}$ has the double LC-positive property if for each $r \geq 0$, the sequence of polynomials $(\mathcal{A}_{s,r}(n;q))_{n\geq r}$ and $\left(\mathcal{A}_{s,r}^*(n;q)\right)_{n\geq r}$ are q-log-concave in n.

We shall need the following lemma due to Wang and Yeh [28].

Lemma 3. *Let $h \in \mathbb{N}$. Suppose that two sequences $a_0, \ldots, a_h$ and $X_0, \ldots, X_h$ of real numbers satisfy the following two conditions:*

1 $\sum_{k=r}^{h} a_k \geq 0 \quad (0 \leq r \leq h)$;

2 $0 \leq X_0 \leq \cdots \leq X_h$.

Then

$$\sum_{k=0}^{h} a_k X_k \geq X_0 \sum_{k=0}^{h} a_k \geq 0.$$

Let $\{a_s(n,k)\}_{0\leq k\leq ns}$ be a s-triangle of nonnegative numbers and $(x_k)_{k\geq 0}$ be a log-concave sequence. Let $(z_n)_{n\geq 0}$ be the sequence defined by (5) and let us consider the difference

$$\triangle_n := \left(\sum_{k=0}^{ns} a_s(n,k)x_k\right)^2 - \left(\sum_{k=0}^{ns-s} a_s(n-1,k)x_k\right)\left(\sum_{k=0}^{ns+s} a_s(n+1,k)x_k\right). \tag{7}$$

Then $\triangle_n$ is a quadratic form in $ns+s+1$ variables $x_0, x_1, \ldots, x_{ns+s}$.

Let S_t be the sum of terms $x_k x_{t-k}$ in $\triangle_n$. For $0 \leq k \leq \lfloor t/2 \rfloor$ with $0 \leq t \leq 2ns$, let $a_{s,k}(n,t)$ be the coefficient of the term $x_k x_{t-k}$ in $\triangle_n$. Then

$$\triangle_n = \sum_{t=0}^{2ns} S_t \quad \text{with } S_t = \sum_{k=0}^{\lfloor t/2 \rfloor} a_{s,k}(n,t)x_k x_{t-k}. \tag{8}$$

Thus, it suffices to show that $S_t \geq 0$ $(0 \leq t \leq 2ns)$. We have the following inequalities $x_0x_t \leq x_1x_{t-1} \leq x_2x_{t-2} \leq \cdots$. Hence by Lemma 3, it suffices to establish that

$$A_{s,r}(n,t) := \sum_{k=r}^{\lfloor t/2 \rfloor} a_{s,k}(n,t) \geq 0, \quad (0 \leq r \leq \lfloor t/2 \rfloor). \tag{9}$$

Using relation (7), for $k < t/2$, we obtain

$$a_{s,k}(n,t) = 2a_s(n,k)a_s(n,t-k) - a_s(n-1,k)a_s(n+1,t-k) \\ - a_s(n+1,k)a_s(n-1,t-k), \tag{10}$$

and for t even and $k = t/2$, we have

$$a_{s,k}(n,t) = a_s(n,k)^2 - a_s(n-1,k)a_s(n+1,k). \tag{11}$$

Let us remark that $A_{s,r}(n,t)$ is precisely the coefficient of q^t in the polynomial $\mathcal{A}^2_{s,r}(n;q) - \mathcal{A}_{s,r}(n-1;q)\mathcal{A}_{s,r}(n+1;q)$, i.e.,

$$\mathcal{A}^2_{s,r}(n;q) - \mathcal{A}_{s,r}(n-1;q)\mathcal{A}_{s,r}(n+1;q) = \sum_{t=2r}^{2ns} A_{s,r}(n,t)q^t. \tag{12}$$

Hence, the following characterization of positivity holds:

Lemma 4. *The s-triangle $\{a_s(n,k)\}_{0\le k\le ns}$ is LC-positive if and only if $A_{s,r}(n,t) \ge 0$ for all $2r \le t \le 2ns$.*

Now, from the discussion above, we obtain the following:

Theorem 5. *The LC-positive s-triangles are PLC.*

The relation between double LC-positivity and the double PLC property is given by the following proposition.

Proposition 6. *Given a s-triangle $\{a_s(n,k)\}_{0\le k\le ns}$ of nonnegative numbers and two log-concave sequences $(x_k)_{k\ge0}$ and $(y_k)_{k\ge0}$.*

Define three s-triangles $\{b_s(n,k)\}$, $\{c_s(n,k)\}$ and $\{d_s(n,k)\}$ by

$$b_s(n,k) = a_s(n,k)x_k, \quad c_s(n,k) = a_s(n,k)y_{ns-k}, \quad d_s(n,k) = a_s(n,k)x_k y_{ns-k}.$$

For $2r \le t \le 2ns$, define $B_{s,r}(n,t)$, $C_{s,r}(n,t)$ and $D_{s,r}(n,t)$ similar to $A_{s,r}(n,t)$ in (12).

1. *If the s-triangle $\{a_s(n,k)\}$ is LC-positive, then the s-triangle $\{b_s(n,k)\}$ is LC-positive and $B_{s,r}(n,t) \ge A_{s,r}(n,t)x_r x_{t-r}$.*

2. *If the s-triangle $\{a_s(n,k)\}$ is double LC-positive, then the s-triangle $\{c_s(n,k)\}$ is LC-positive and $C_{s,r}(n,t) \ge A_{s,r}(n,t)y_{ns-t+r}y_{ns-r}$ for $t \le ns + r$.*

3. *If the s-triangle $\{a_s(n,k)\}$ is double LC-positive, then the s-triangle $\{d_s(n,k)\}$ is LC-positive and $D_{s,r}(n,t) \ge A_{s,r}(n,t)x_r x_{t-r} y_{ns-t+r} y_{ns-r}$ for $t \le ns + r$.*

Proof.

1. Let $0 \leq t \leq 2ns$. It is easy to see by definition that $b_{s,k}(n,t) = a_{s,k}(n,t)x_k x_{t-k}$ for $0 \leq k \leq \lfloor t/2 \rfloor$. Hence for $0 \leq r \leq \lfloor t/2 \rfloor$

$$B_{s,r}(n,t) := \sum_{k=r}^{\lfloor t/2 \rfloor} b_{s,k}(n,t) = \sum_{k=r}^{\lfloor t/2 \rfloor} a_{s,k}(n,t)x_k x_{t-k},$$

Now $\{a_s(n,k)\}$ is LC-positive and $x_0 x_t \leq x_1 x_{t-1} \leq \cdots$ by the log-concavity of (x_k). From Lemma 3 it follows that

$$B_{s,r}(n,t) \geq x_r x_{t-r} \sum_{k=r}^{\lfloor t/2 \rfloor} a_{s,k}(n,t) = A_{s,r}(n,t)x_r x_{t-r} \geq 0,$$

So the s-triangle $\{b_s(n,k)\}$ is LC-positive.

2. Let $2r \leq t \leq 2ns$. We need to prove $C_{s,r}(n,t) \geq 0$. For brevity, we do this only for the case t odd since the same technique is still valid for the case where t is even.

   Let $t = 2l + 1$ for $0 \leq k \leq l$. Then we define

$$\begin{aligned} \alpha_k &= a_s(n,k)a_s(n,t-k), \\ \beta_k &= a_s(n-1,k)a_s(n+1,t-k), \\ \gamma_k &= a_s(n+1,k)a_s(n-1,t-k), \\ Y_k &= c_{ns-t+k} y_{ns-k}. \end{aligned}$$

   Then

$$a_{s,k}(n,t) = 2\alpha_k - \beta_k - \gamma_k,$$

   and

$$c_{s,k}(n,t) = 2\alpha_k Y_k - \beta_k Y_{k+s} - \gamma_k Y_{k-s}$$

   by definition. It follows that

$$\begin{aligned} C_{s,r}(n,t) &= \sum_{k=r}^{l} (2\alpha_k Y_k - \beta_k Y_{k+s} - \gamma_k Y_{k-s}) \\ &= \sum_{k=r}^{l} (2\alpha_k - \beta_{k-s} - \gamma_{k+s}) Y_k + \sum_{j=1}^{s} \beta_{r-j} Y_{r+s-j} \\ &\quad - \sum_{j=1}^{s} \gamma_{r+j-1} Y_{r-s+j-1} - \sum_{j=1}^{s} \beta_{l-j+1} Y_{l+s-j+1} + \sum_{j=1}^{s} \gamma_{l+j} Y_{l-s+j}, \end{aligned}$$

where we use the fact that $Y_{l+s-j+1} = Y_{l-s+j}$ and $\beta_{l-j+1} = \gamma_{l+j}$ for $j = \overline{1, s}$. Note that (Y_k) is nondecreasing by the log-concavity of (y_k) and

$$\begin{aligned} 2\alpha_k - \beta_{k-s} - \gamma_{k+s} &= 2a_s^\star(n, ns-k)a_s^\star(n, np-t+k) \\ &\quad - a_s^\star(n-1, ns-k)a^\star(n+1, ns-t+k) \\ &\quad - a_s^\star(n+1, np-k)a_s^\star(n-1, ns-t+k) \\ &= a^\star_{s,ns-t+k}(n, 2ns-t). \end{aligned}$$

Hence by the LC-positivity of $\{a_s^\star(n,k)\}$, we have

$$\begin{aligned} C_{s,r}(n,t) &= \sum_{j=ns-t+r}^{\lfloor (2ns-t)/2 \rfloor} a^\star_{s,j}(n, 2ns-t)Y_{j-ns+t} + \sum_{j=1}^{s} \beta_{r-j}Y_{r+s-j} \\ &\quad - \sum_{j=1}^{s} \gamma_{r+j-1}Y_{r-s+j-1} \\ &\geq Y_r \sum_{j=ns-t+r}^{\lfloor (2ns-t)/2 \rfloor} a^\star_{s,j}(n, 2ns-t) + Y_r \sum_{j=1}^{s} \beta_{r-j} - Y_{r-s} \sum_{j=1}^{s} \gamma_{r+j-1} \\ &= Y_r \sum_{k=r}^{s} (2\alpha_k - \beta_{k-s} - \gamma_{k+s}) + Y_r \sum_{j=1}^{s} \beta_{r-j} - Y_{r-s} \sum_{j=1}^{s} \gamma_{r+j-1} \\ &= Y_r \sum_{k=r}^{s} (2\alpha_k - \beta_k - \gamma_k) + (Y_r - Y_{r-s}) \sum_{j=1}^{s} \gamma_{r+j-1} \\ &= A_{s,r}(n,t)Y_r + (Y_r - Y_{r-s}) \sum_{j=1}^{s} \gamma_{r+j-1}. \end{aligned}$$

Thus $C_{s,r}(n,t) \geq A_{s,r}(n,t)y_{ns-t+r}y_{ns-r}$.

3. We have $d_s(n,k) = a_s(n,k)x_k y_{ns-k} = c_s(n,k)x_k$ and

$$D_{s,r}(n,t) = \sum_{k=r}^{\lfloor t/2 \rfloor} d_{s,k}(n,t) = \sum_{k=r}^{\lfloor t/2 \rfloor} c_{s,k}(n,t)x_k x_{t-k},$$

by 1 and 2, so

$$D_{s,r}(n,t) \geq C_{s,r}(n,t)x_r x_{t-r} \geq A_{s,r}(n,t)x_r x_{t-r} y_{ns-t+r} y_{ns-r}.$$

$\square$

Now we establish the second result.

Theorem 7. *The double LC-positive s-triangles are double PLC.*

Proof. Let the s-triangle $\{a_s(n,k)\}$ be doubly LC-positive. Suppose that both (x_k) and (y_k) are log-concave. Then the s-triangle $\{a_s(n,k)x_k y_{ns-k}\}$ is LC-positive by Proposition 6 (3) and is therefore PLC by Theorem 5. Thus the row-sum sequence

$$z_n = \sum_{k=0}^{ns} a_s(n,k)x_k y_{ns-k}, \qquad n = 0, 1, 2, \dots$$

is log-concave. In other words, the s-triangle $\{a_s(n,k)\}$ is double PLC. □

By Lemma 4, $\{a_s(n,k)\}$ is LC-positive if and only if the inequality $\sum_{k=r}^{\lfloor t/2 \rfloor} a_{s,k}(n,t) \geq 0$ for all $2r \leq t \leq 2ns$, so the following corollary is immediate.

Corollary 8. *Suppose that the following two conditions hold:*

A *There exists an index $m = m(n,t)$ such that $a_k(n,t) < 0$ for $k < m$ and $a_{s,k}(n,t) \geq 0$ for $k \geq m$;*

B *The sequence $(\mathcal{A}_{s,0}(n;q))_{n\geq 0}$ is q-log-concave.*

Then the s-triangle $\{a_s(n,k)\}$ is LC-positive and therefore PLC.

Corollary 9. *Suppose that s-triangle $\{a_s(n,k)\}$ satisfies Conditions (A) and (B) in Corollary 8 and $\{a_s^*(n,k)\}$ satisfies Condition (A). Then $\{a_s(n,k)\}$ is doubly LC-positive and therefore double PLC.*

Proof. It suffices to show that $\left(\mathcal{A}_{s,0}^\star(n;q)\right)$ is q-log-concave. We have

$$\mathcal{A}_{s,0}^\star(n;q) = \sum_{k=0}^{ns} a_s(n, ns-k)q^k = \sum_{k=0}^{ns} a_s(n,k)q^{ns-k} = q^{ns}\mathcal{A}_{s,0}(n;q^{-1})$$

It follows that

$$\mathcal{A}_{s,0}^{\star 2}(n;q) - \mathcal{A}_{s,0}^\star(n-1;q)\mathcal{A}_{s,0}^\star(n+1;q) = q^{2ns}\left(\mathcal{A}_{s,0}^2(n;q^{-1}) - \mathcal{A}_{s,0}(n-1;q^{-1})\mathcal{A}_{s,0}(n+1;q^{-1})\right)$$

which has nonnegative coefficients by the q-log-concavity of $(\mathcal{A}_{s,0}(n;q))$. □

3 Application to linear operators of finite order

In this section, for selected examples of s-triangles we show their LC-positivity leading to the PLC property.

Let $\mathfrak{S}$ denote the set of sequences $(u_k)_{k\in\mathbb{Z}}$ of nonnegative numbers. Given $(s+1)$ nonnegative numbers $\lambda_0, \lambda_1, \dots, \lambda_s$, define the linear operator $L = L[\lambda_0, \lambda_1, \dots, \lambda_s]$, on $\mathfrak{S}$ by

$$L(u_k) = \sum_{j=0}^{s} \lambda_j u_{k-j} \quad (k \in \mathbb{Z}).$$

For $n \geq 2$, define $L^n := L(L^{n-1})$ by induction. It is convenient to view L^0 as the identity operator. Let $(u_k)_{k\in\mathbb{Z}}$ be a log-concave sequence.

Lemma 10. *If the sequence* $(\lambda_0, \lambda_1, \dots, \lambda_s)$ *is log-concave, then so is the sequence* $(L^n(u_k))_{k\in\mathbb{Z}}$.

Proof. In fact

$$\begin{aligned}
&(L(u_k))^2 - L(u_{k-1})L(u_{k+1}) = \left(\sum_{j=0}^{s} \lambda_j u_{k-j}\right)^2 - \sum_{j=0}^{s} \lambda_j u_{k-j-1} \sum_{j=0}^{s} \lambda_j u_{k-j+1} \\
&= \sum_{j=0}^{s} \lambda_j^2 (u_{k-j}^2 - u_{k-j-1}u_{k-j+1}) + \sum_{0\leq l<j\leq s} \lambda_j \lambda_l (u_{k-j}u_{k-l} - u_{k-j-1}u_{k-l+1}) \\
&\quad + \sum_{0\leq l<j\leq s} \lambda_j \lambda_l u_{k-j}u_{k-l} - \sum_{0\leq l<j\leq 0} \lambda_j \lambda_l u_{k-j+1}u_{k-l-1} \\
&= T_1 + T_2 + T_3,
\end{aligned}$$

with

$$\begin{aligned}
T_1 &= \sum_{j=0}^{s} \lambda_j^2 (u_{k-j}^2 - u_{k-j-1}u_{k-j+1}), \\
T_2 &= \sum_{0\leq l<j\leq s} \lambda_j \lambda_l (u_{k-j}u_{k-l} - u_{k-j-1}u_{k-l+1})
\end{aligned}$$

and

$$T_3 = -\sum_{1\leq l+1<j\leq s} \lambda_j \lambda_l (u_{k-j+1}u_{k-l-1} - u_{k-j}u_{k-l}).$$

It follows that

$$
\begin{aligned}
&(L(u_k))^2 - L(u_{k-1})L(u_{k+1}) = \sum_{j=1}^{s-1} \left(\lambda_j^2 - \lambda_{j-1}\lambda_{j+1}\right) \times \left(u_{k-j}^2 - u_{k-j-1}u_{k-j+1}\right) \\
&+ \lambda_0^2 \left(u_k^2 - u_{k-1}u_{k+1}\right) + \lambda_s^2 \left(u_{k-s}^2 - u_{k-s-1}u_{k-s+1}\right) \\
&+ \sum_{2\le l+2<j\le s-1} (\lambda_l\lambda_j - \lambda_{l-1}\lambda_{j+1}) \times (u_{k-j}u_{k-l} - u_{k-j-1}u_{k-l+1}) \\
&+ \sum_{l=0}^{s-1} \lambda_l\lambda_{l+1} \left(u_{k-l-1}u_{k-l} - u_{k-l-2}u_{k-l+1}\right) \\
&+ \sum_{l=0}^{s-2} \lambda_l\lambda_{l+2} \left(u_{k-l-2}u_{k-l} - u_{k-l-3}u_{k-l+1}\right) \\
&\ge 0.
\end{aligned}
$$

By induction, the polynomial sequence $(L^n(u_k))_{k\in\mathbb{Z}}$ is also log-concave for $n \ge 0$. □

This brings us to the following theorem.

Theorem 11. *Given $(s+1)$ nonnegative numbers $\lambda_0, \lambda_1, \ldots, \lambda_s$ and a log-concave sequence $(u_k)_{k\in\mathbb{Z}}$, define*

$$a_s(n,k) = L^n(u_k),\ \ (0 \le k \le ns)\,.$$

If $(\lambda_0, \lambda_1, \ldots, \lambda_s)$ is log-concave. Then the s-triangle $\{a_s(n,k)\}$ is doubly LC-positive and therefore double PLC.

Proof. Denote $a_k = L^{n-1}(u_k)$ for $k \in \mathbb{Z}$ and $\mathcal{A}_{s,r}(n-1;q) = \sum_{k=r}^{ns-s} a_k q^k$. If $(\lambda_0, \lambda_1, \ldots, \lambda_s)$ is log-concave, then by Lemma 10 so is the sequence $(a_k)_{k\in\mathbb{Z}}$. We have

$$
\begin{aligned}
\mathcal{A}_{s,r}(n;q) &= \lambda_0 \sum_{k=r}^{ns} a_k q^k + \lambda_1 \sum_{k=r}^{ns} a_{k-1} q^k + \cdots + \lambda_s \sum_{k=r}^{ns} a_{k-s} q^k \\
&= \mathcal{A}_{s,r}(n-1;q) \sum_{j=0}^{s} \lambda_j q^j + \sum_{j=1}^{s} \lambda_j \sum_{l=1}^{j} a_{r-l} q^{r+j-l} \\
&\quad + \sum_{j=0}^{s-1} \lambda_j \sum_{l=1}^{s-j} a_{ns-s+l} q^{ns-s+l+j},
\end{aligned}
$$

thus

$$\begin{aligned}
&\mathcal{A}_{s,r}(n;q)^2 - \mathcal{A}_{s,r}(n-1;q)\mathcal{A}_{s,r}(n+1;q) = \\
&\sum_{j=1}^{s}\sum_{l=1}^{j}\sum_{f=0}^{s}\lambda_j\lambda_f\left(\sum_{k=r}^{ns}[a_{r-l}a_{k-f} - a_{r-f-l}a_k]q^{k+r+j-l}\right.\\
&\left. + \sum_{k=ns-s+1}^{ns} a_{r-f-l}a_k q^{k+r+j-l}\right)\\
&+\sum_{j=0}^{s-1}\sum_{l=1}^{s-j}\sum_{f=0}^{s}\lambda_j\lambda_f\left(\sum_{k=r}^{ns}\left(a_{ns-s+l}a_{k-f} - a_{ns+l-f}a_{k-s}\right)q^{k+ns-s+l+j}\right.\\
&\left. + \sum_{k=r}^{r+s-1} a_{ns+l-f}a_k q^{k+ns-s+l+j}\right),
\end{aligned}$$

which has nonnegative coefficients by the log-concavity of the sequence (a_k). Hence the s-triangle $\{a_s(n,k)\}_{0\le k\le ns}$ is LC-positive.

On the other hand, let $u_k^\star = u_{-k}$ for $k \in \mathbb{Z}$. Then the sequence $(u_k^\star)_{k\in\mathbb{Z}}$ is log-concave and $a_s^\star(n,k) = L^n[\lambda](u_k^\star)$. Thus the s-triangle $\{a_s^\star(n,k)\}_{0\le k\le ns}$ is also LC-positive, and the s-triangle $\{a_s(n,k)\}_{0\le k\le ns}$ is therefore doubly LC-positive. □

Corollary 12. *Let a and b be two nonnegative integers with $a \ge b$. If the sequences (x_k) and (y_k) are log-concave, then so is the sequence*

$$z_n = \sum_{k=0}^{ns}\binom{a+n}{b+k}_s x_k y_{sn-k}, \quad (n \ge 0).$$

Proof. Using relation (4), we have $\binom{a+n}{b+k}_s = \sum_{j=0}^{s}\binom{a+n-1}{b+k-j}_s$, and taking $u_k = \binom{a}{b+k}_s$ with $\lambda_j = 1$, $(1 \le j \le s)$ in Theorem 11, we obtain the result. □

When $s = 1$, we obtain the result of Y. Wang [29, Corollary 3.4]. Taking $a = b = 0$ in Corollary 12, we obtain the following nice result.

Corollary 13. *If the sequences (x_k) and (y_k) are log-concave, then so is*

$$z_n = \sum_{k=0}^{ns}\binom{n}{k}_s x_k y_{sn-k} \quad (n \ge 0).$$

The following theorem is in a sense dual to Theorem 11.

Theorem 14. *Let $\lambda_0, \lambda_1, \ldots, \lambda_s, (s+1)$ nonnegative numbers and $\{a_s(n,k)\}$ an s-triangle of nonnegative numbers. Suppose that each row of $\{a_s(n,k)\}$ is log-concave and satisfies the following recurrence relation*

$$a_s(n,k) = \sum_{j=0}^{s}\lambda_j a_s(n+1,k+j), \quad (0 \le k \le ns). \tag{13}$$

Then the s-triangle $\{a_s(n,k)\}$ is LC-positive and therefore double PLC.

Proof. Denote $a_s(n+1,k) = v_k \quad (0 \le k \le ns+s)$. Then the sequence (v_k) is log-concave and $\mathcal{A}_{s,r}(n+1;q) = \sum_{k=r}^{ns+s} v_k q^k$. By the recurrence relation (13) we have

$$\mathcal{A}_{s,r}(n;q) = \mathcal{A}_{s,r}(n+1;q)\sum_{j=0}^{s}\lambda_j q^{-j} - \sum_{j=0}^{s-1}\sum_{l=1}^{s-j}\lambda_j v_{ns+j+l}q^{ns+l} - \sum_{j=1}^{s}\sum_{l=0}^{j-1}\lambda_j v_{r+l}q^{r+l-j}.$$

It follows that

$$\begin{aligned}
&\mathcal{A}_{s,r}^2(n;q) - \mathcal{A}_{s,r}(n-1;q)\mathcal{A}_{s,r}(n+1;q)\\
&= \mathcal{A}_{s,r}(n;q)\left(\mathcal{A}_{s,r}(n+1;q)\sum_{j=0}^{s}\lambda_j q^{-j} - \sum_{j=0}^{s-1}\sum_{l=1}^{s-j}\lambda_j v_{ns+j+l}q^{ns+l}\right.\\
&\left.-\sum_{j=1}^{s}\sum_{l=0}^{j-1}\lambda_j v_{r+l}q^{r+l-j}\right)\\
&- \mathcal{A}_{s,r}(n+1;q)\left(\mathcal{A}_{s,r}(n;q)\sum_{j=0}^{s}\lambda_j q^{-j} - \sum_{j=0}^{s-1}\sum_{l=1}^{s-j}\sum_{f=0}^{s}\lambda_j\lambda_f v_{ns-s+j+l+f}q^{ns-s+l}\right.\\
&\left.-\sum_{j=1}^{s}\sum_{l=0}^{j-1}\sum_{f=0}^{s}\lambda_j\lambda_f v_{r+l+f}q^{r+l-j}\right)\\
&= S_1 + S_2 + S_3,
\end{aligned}$$

with

$$S_3 = \sum_{j=0}^{s-1}\sum_{f=0}^{s}\lambda_j\lambda_f\left(\sum_{l=1}^{s-j}\sum_{k=r}^{r+s-1} v_{ns-s+j+l+f}v_k q^{k+ns-s+l} + \sum_{l=0}^{j-1}\sum_{k=ns+1}^{ns+s} v_{r+l+f}v_k q^{k+r+l-j}\right),$$

and

$$\begin{aligned}
S_1 &= \sum_{j=0}^{s-1}\sum_{f=0}^{s}\sum_{l=1}^{s-j}\lambda_j\lambda_f\left(\left(\sum_{k=r+s}^{ns} + \sum_{k=ns+1}^{ns+j} + \sum_{k=ns+j+1}^{ns+s}\right)(v_{ns-s+j+l+f}v_k\right.\\
&\quad\left. - v_{ns+j+l}v_{k+f-s})\, q^{k+ns-s+l}\right)\\
&= \sum_{j=0}^{s-1}\sum_{f=0}^{s}\sum_{l=1}^{s-j} q^{ns-s+l}\lambda_j\lambda_f\left(\sum_{k=r+s}^{ns}(v_{ns-s+j+l+f}v_k - v_{ns+j+l}v_{k+f-s})q^k\right.\\
&\quad\left. + \sum_{k=ns+j}^{ns+s}(v_{ns-s+j+l+f}v_k - v_{k+(ns+j+l-k)}v_{ns-s+j+l+f-(ns+j+l-k)})q^k\right) \qquad (14)
\end{aligned}$$

since

$$\sum_{j=0}^{s-1}\sum_{f=0}^{s}\sum_{l=1}^{s-j}\sum_{k=ns+j+1}^{ns+s}\lambda_j\lambda_f(v_{ns-s+j+l+f}v_k-v_{ns+j+l}v_{k+f-s})q^{k+ns-s+l}$$
$$=\sum_{j=0}^{s-1}\sum_{f=0}^{s}\lambda_j\lambda_f\sum_{l=1}^{s-j}\left(\left(\sum_{k=ns+j+1}^{ns+j+l}+\sum_{k=ns+j+l}^{ns+s}\right)\right.$$
$$\left.(v_{ns-s+j+l+f}v_k-v_{ns+j+l}v_{k+f-s})q^{k+ns-s+l}\right)$$
$$=0,$$

by setting, $l'=k-ns-j$ and $k'=l+j+ns$ in the second term. The sum (14) has nonnegative coefficients by log-concavity of $(v_k)_k$, and the first term of (14) gives the following: if $ns-s+j+l+f\le k$, then

$$v_{ns-s+j+l+f}v_k-v_{ns+j+l}v_{k+f-s}=v_{ns-s+j+l+f}v_k-v_{k+(ns+j+l-k)}v_{ns-s+j+l+f-(ns+j+l-k)}\ge 0;$$

and otherwise,

$$v_{ns-s+j+l+f}v_k-v_{ns+j+l}v_{k+f-s}=v_{ns-s+j+l+f}v_k-v_{ns-s+j+f+(s-f)}v_{k-(s-f)}\ge 0.$$

$$S_2=\sum_{f=1}^{s}\lambda_f\left(\sum_{k=r+1}^{ns}\lambda_j(v_{r+f}v_k-v_rv_{k+f})q^{k+r-j}\right.$$
$$+\sum_{j=2}^{s}\lambda_j\left(\sum_{k=r+1}^{r+j-1}(v_{r+f}v_k-v_rv_{k+f})q^{k+r-j}\right.$$
$$+\sum_{k=r+j}^{ns}(v_{r+f}v_k-v_rv_{k+f})q^{k+r-j}+\sum_{l=1}^{j-1}\left((v_{r+l+f}v_r-v_{r+l}v_{r+f})q^{2r+l-j}\right.$$
$$+\sum_{k=r+1}^{r+l}(v_{r+l+f}v_k-v_{r+l}v_{k+f})q^{k+r+l-j}+\sum_{k=r+l}^{r+j-1}(v_{r+l+f}v_k-v_{r+l}v_{k+f})q^{k+r+l-j}$$
$$\left.\left.\left.+\sum_{k=r+j}^{ns}(v_{r+l+f}v_k-v_{r+l}v_{k+f})q^{k+r+l-j}\right)\right)\right)$$
$$=\sum_{f=1}^{s}\lambda_f\left(\sum_{k=r+1}^{ns}\lambda_j(v_{r+f}v_k-v_rv_{k+f})q^{k+r-j}+\sum_{j=2}^{s}\sum_{k=r+j}^{ns}\lambda_j(v_{r+f}v_k-v_rv_{k+f})q^{k+r-j}\right.$$
$$\left.+\sum_{j=2}^{s}\sum_{l=1}^{j-1}\sum_{k=r+j}^{ns}\lambda_j(v_{r+l+f}v_k-v_{r+l}v_{k+f})q^{k+r+l-j}\right)\tag{15}$$

since, by setting $k' = l + r$ in the second term

$$\sum_{j=2}^{s}\sum_{f=1}^{s}\lambda_j\lambda_f\left(\sum_{k=r+1}^{r+j-1}(v_{r+f}v_k - v_r v_{k+f})q^{k+r-j} + \sum_{l=1}^{j-1}(v_{r+l+f}v_r - v_{r+l}v_{r+f})q^{2r+l-j}\right) = 0,$$

also, by setting $k' = l + r$ and $l' = k - r$ in second term

$$\sum_{j=2}^{s}\sum_{f=1}^{s}\sum_{l=1}^{j-1}\lambda_j\lambda_f\left(\left(\sum_{k=r+1}^{r+l} + \sum_{k=r+l}^{r+j-1}\right)(v_{r+l+f}v_k - v_{r+l}v_{k+f})q^{k+r+l-j}\right) = 0.$$

The sum (15) has nonnegative coefficients by the log-concavity of $(v_k)_k$. Hence the polynomial $\mathcal{A}_{s,r}^2(n;q) - \mathcal{A}_{s,r}(n-1;q)\mathcal{A}_{s,r}(n+1;q)$ has nonnegative coefficients. So the triangle $\{a_s(n,k)\}$ is LC-positive.

Clearly, the reciprocal s-triangle $\{a_s^*(n,k)\}$ possesses the same property as $\{a_s(n,k)\}$ does. Hence $\{a_s^*(n,k)\}$ is also LC-positive. Thus the s-triangle $\{a_s(n,k)\}$ is doubly LC-positive and therefore double PLC. □

In Theorem 14, the choice $\lambda_j = 1$ $(1 \leq j \leq s)$ and $a_s(n,k) = \binom{a-n}{b-k}_s$ $(0 \leq k \leq ns)$, leads to the following:

Corollary 15. *Let $a, b \in \mathbb{N}$ with $a \geq b$. If the sequences (x_k) and (y_k) are log-concave, then so is the sequence*

$$z_n = \sum_{k=0}^{ns}\binom{a-n}{b-k}_s x_k y_{sn-k}, \quad (n \geq 0).$$

By setting $s = 1$ in the above result, we obtain the result of Wang [29, Corollary 3.9].

We conclude this paper with the following.

Conjecture 16. The s-triangle $\left(\binom{n}{k}_s\binom{a-n}{b-k}_s\right)_k$ is double PLC.

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