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A Short Proof of the Binomial Identities of Frisch and Klamkin

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Abstract

We present short proofs for Frisch's identity and Klamkin's identity. Furthermore, we deduce variants of Frisch's and Klamkin's identities involving infinite series.

1 Introduction

In their recent paper [4], Gould and Quaintance gave new proofs of Frisch's identity

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{b+k}{c}^{-1} = \frac{c}{n+c} \binom{n+b}{b-c}^{-1} \qquad (b \geq c > 0) \tag{1}$$

and Klamkin's identity

$$\sum_{k=0}^{n} \binom{n}{k} \binom{x}{k+b}^{-1} = \frac{x+1}{x-n+1} \binom{x-n}{b}^{-1} \qquad (x-n \geq b \geq 0)\,. \tag{2}$$

As usual we put $\binom{x}{0} = 1$. For suitable real numbers x, y, the binomial coefficient $\binom{x}{y}$ is to be read in terms of the gamma function, i.e., $\binom{x+y}{y} = \Gamma(x+y+1) / (\Gamma(x+1)\Gamma(y+1))$. The proofs in [4] are based on the well-known formula of Gauss for the hypergeometric function ${}_2F_1(a, b; c; 1)$.

In 1969, Ragnar Frisch (1895–1973) was awarded the first Nobel Prize in Economic Sciences. Identity (1) appeared in Frisch's 1926 dissertation [2]. It was cited and proved in the 2nd edition 1927 of the book [5, pp. 337–338] by Netto. A further proof of Frisch's identity (1), a two-page calculation involving an application of Melzak's formula, can be found in the new book [6, Section 7.2].

In [4] the authors report that identity (2) in its original form [4, Eq. (1)] with $x = n + a$ was stated by Murray S. Klamkin in a letter to Henry W. Gould on May 16, 1966. It is tabulated as Formula (4.2) in Gould's collection [3]. Identity (4.6) in [3] is a special case of this.

The purpose of this note are elementary short proofs without application of hypergeometric functions. They are based on the use of the Euler beta function

$$B\left(x,y\right) = \int_0^1 t^{x-1}\left(1-t\right)^{y-1}dt = \frac{\Gamma\left(x\right)\Gamma\left(y\right)}{\Gamma\left(x+y\right)} \qquad \left(x,y>0\right).$$

Furthermore, we present variants of Frisch's and of Klamkin's identities involving infinite series.

2 Proof of Frisch's and Klamkin's formulas

By direct calculation, we obtain, for $n = 0, 1, 2, \ldots$ and $b \geq c > 0$,

$$\begin{aligned}
\sum_{k=0}^{n}\left(-1\right)^k\binom{n}{k}\binom{b+k}{c}^{-1} &= c\sum_{k=0}^{n}\left(-1\right)^k\binom{n}{k}B\left(b-c+1+k,c\right)\\
&= c\sum_{k=0}^{n}\left(-1\right)^k\binom{n}{k}\int_0^1 t^{b-c+k}\left(1-t\right)^{c-1}dt\\
&= c\cdot B\left(b-c+1,c+n\right) = \frac{c}{n+c}\binom{n+b}{b-c}^{-1},
\end{aligned}$$

which is Frisch's identity (1). For suitable $x \neq -1$, we have

$$\begin{aligned}
\frac{1}{x+1}\sum_{k=0}^{n}\binom{n}{k}\binom{x}{k+b}^{-1} &= \sum_{k=0}^{n}\binom{n}{k}B\left(b+k+1,x-b-k+1\right)\\
&= \sum_{k=0}^{n}\binom{n}{k}\int_0^1 t^b\left(1-t\right)^{x-b}\left(\frac{t}{1-t}\right)^k dt\\
&= B\left(b+1,x-b-n+1\right) = \frac{1}{x-n+1}\binom{x-n}{b}^{-1},
\end{aligned}$$

which proves Klamkin's identity (2).

3 Variants of Frisch's and Klamkin's identities

Since $\binom{n}{k} = 0$, for integers $k > n \geq 0$, the left-hand sides in both identities (1), (2) can be written as infinite sums. Formally replacing n with $-n$ in those equations and using the obvious binomial identity

$$(-1)^k \binom{-n}{k} = \binom{n+k-1}{k}$$

yields

$$\begin{aligned}
\sum_{k=0}^{\infty} \binom{n+k-1}{k} \binom{b+k}{c}^{-1} &= \frac{c}{c-n} \binom{-n+b}{b-c}^{-1}, \\
\sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} \binom{x}{k+b}^{-1} &= \frac{x+1}{x+n+1} \binom{x+n}{b}^{-1}.
\end{aligned}$$

We shall show that both equations are valid also for positive integers n, if the parameters a, b, c are chosen in an appropriate manner.

First we deduce the variant of Frisch's identity.

Theorem 1. *For $b \geq c > n \geq 1$,*

$$\sum_{k=0}^{\infty} \binom{n+k-1}{k} \binom{b+k}{c}^{-1} = \frac{c}{c-n} \binom{b-n}{b-c}^{-1}. \tag{3}$$

The infinite series in Eq. (3) is convergent since

$$\binom{n+k-1}{k} \binom{b+k}{c}^{-1} = \Gamma(n) \frac{\Gamma(n+k)}{\Gamma(1+k)} \Gamma(c+1) \frac{\Gamma(b-c+1+k)}{\Gamma(b+1+k)} \sim \frac{\Gamma(n)\Gamma(c+1)}{k^{1-n+c}}$$

as $k \to \infty$ (see, e.g., [1, (6.1.46)]) and $c > n$.

Proof of Theorem 1. By using the well-known power series expansion

$$\sum_{k=0}^{\infty} \binom{n+k-1}{k} z^k = (1-z)^{-n} \qquad (|z| < 1)$$

instead of the binomial formula, it follows in the same manner as in the preceding section that

$$\sum_{k=0}^{\infty} \binom{n+k-1}{k} B(x+k, y) = B(x, y-n) \qquad (x > 0,\ y > n).$$

Rewritten in terms of binomial coefficients the latter identity takes the form

$$x \sum_{k=0}^{\infty} \binom{n+k-1}{k} \binom{x+y+k-1}{y}^{-1} = y \binom{x+y-n-1}{x}^{-1} \qquad (x > 0,\ y > n).$$

Replacing x with $b-c+1$ and y with c, we obtain

$$\sum_{k=0}^{\infty}\binom{n+k-1}{k}\binom{b+k}{c}^{-1}=\frac{c}{b-c+1}\binom{b-n}{b-c+1}^{-1}\qquad(b>c-1,\ c>n)\,.$$

Now Eq. (3) follows since

$$\frac{c}{b-c+1}\binom{b-n}{b-c+1}^{-1}=\frac{c}{c-n}\binom{b-n}{b-c}^{-1}.$$

This completes the proof. □

Remark 2. An alternative approach is the observation

$$\sum_{k=0}^{\infty}\binom{n+k-1}{k}\binom{b+k}{c}^{-1}=\binom{b}{c}^{-1}{}_2F_1\left(1+b-c,n;b+1;1\right).$$

This formula immediately implies Eq. (3) since, by Gauss's formula [1, (15.1.20)],

$${}_2F_1\left(1+b-c,n;b+1;1\right)=\frac{\Gamma\left(c-n\right)\Gamma\left(b+1\right)}{\Gamma\left(c\right)\Gamma\left(b-n+1\right)}\qquad(c>n,\ b>-1)\,.$$

We close with the variant of Klamkin's identity.

Theorem 3. *Let $-a\notin\mathbb{N}$ and $-b\notin\mathbb{N}$. For $-a-1>n\geq 1$ and $a-b+1\notin\mathbb{N}$,*

$$\sum_{k=0}^{\infty}(-1)^k\binom{n+k-1}{k}\binom{a}{b+k}^{-1}=\frac{a+1}{a+n+1}\binom{a+n}{b}^{-1}. \tag{4}$$

The method of proof using the beta integral does not work in the case of Theorem 3 because the arising sum becomes divergent. Therefore, we use the Gauss formula as in the preceding remark.

Proof of Theorem 3. Noting that

$$(-1)^k\binom{a}{b+k}^{-1}=\binom{a}{b}^{-1}\frac{\Gamma\left(b+1+k\right)}{\Gamma\left(b+1\right)}\frac{\Gamma\left(b-a\right)}{\Gamma\left(b-a+k\right)}.$$

we have

$$(-1)^k\binom{n+k-1}{k}\binom{a}{b+k}^{-1}=\gamma\frac{\Gamma\left(n+k\right)}{\Gamma\left(1+k\right)}\frac{\Gamma\left(b+1+k\right)}{\Gamma\left(b-a+k\right)}\sim\gamma k^{n+a}$$

as $k\to\infty$, where $\gamma=\Gamma\left(n\right)\frac{\Gamma(b-a)}{\Gamma(b+1)}\binom{a}{b}^{-1}$. If $a+1+n<0$ we have

$$(-1)^k\binom{n+k-1}{k}\binom{a}{b+k}^{-1}=O\left(k^{-\eta}\right)\qquad(k\to\infty)$$

with a certain constant $\eta > 1$. This shows the convergence of the infinite series in Eq. (4). Finally, we observe

$$\sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} \binom{a}{b+k}^{-1} = \binom{a}{b}^{-1} {}_2F_1\left(n, b+1; b-a; 1\right),$$

for $a+1+n < 0$ and $a-b+1 \notin \mathbb{N}$. Using

$${}_2F_1\left(n, b+1; b-a; 1\right) = \frac{\Gamma\left(-a-n-1\right)\Gamma\left(b-a\right)}{\Gamma\left(b-a-n\right)\Gamma\left(-a-1\right)} = \frac{\Gamma\left(a-b+n+1\right)\Gamma\left(a+2\right)}{\Gamma\left(a-b+1\right)\Gamma\left(a+2+n\right)}$$

the desired formula Eq. (4) follows after a short calculation. □

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References

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Appl. Math. Ser. 55, National Bureau of Standards, 1972.

[2] R. Frisch, Sur les semi-invariants et moments employés dans l'étude des distributions statistiques, *Skrifter utgitt av Det Norske Videnskaps-Akademi i Oslo, II, Historisk-Filosofisk Klasse* **3** (1926), 1–87. Quoted by Th. Skolem, p. 337, in Netto's *Lehrbuch*.

[3] H. W. Gould, *Combinatorial Identities*, Morgantown, West Virginia, 1972.

[4] H. W. Gould and J. Quaintance, On the binomial identities of Frisch and Klamkin, *J. Integer Sequences* **19** (2016) Article 16.7.7.

[5] E. Netto, *Lehrbuch der Kombinatorik*, 2nd edition, Chelsea Publications, 1927.

[6] J. Quaintance and H. W. Gould, *Combinatorial Identities for Stirling Numbers*, World Scientific Press, 2016.

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