A Short Proof of the Binomial Identities of Frisch and Klamkin

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Abstract

We present short proofs for Frisch’s identity and Klamkin’s identity. Furthermore, we deduce variants of Frisch’s and Klamkin’s identities involving infinite series.

1 Introduction

In their recent paper [4], Gould and Quaintance gave new proofs of Frisch’s identity

\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{b + k}{c}^{-1} = \frac{c}{n + c} \binom{n + b}{b - c}^{-1} \quad (b \geq c > 0) \]  

and Klamkin’s identity

\[ \sum_{k=0}^{n} \binom{n}{k} \binom{x}{k + b}^{-1} = \frac{x + 1}{x - n + 1} \binom{x - n}{b}^{-1} \quad (x - n \geq b \geq 0). \]  

As usual we put \( \binom{x}{0} = 1 \). For suitable real numbers \( x, y \), the binomial coefficient \( \binom{x}{y} \) is to be read in terms of the gamma function, i.e., \( \binom{x+y}{y} = \Gamma(x+y+1)/\Gamma(x+1)\Gamma(y+1) \). The proofs in [4] are based on the well-known formula of Gauss for the hypergeometric function \( _2F_1(a, b; c; 1) \).
In 1969, Ragnar Frisch (1895–1973) was awarded the first Nobel Prize in Economic Sciences. Identity (1) appeared in Frisch’s 1926 dissertation [2]. It was cited and proved in the 2nd edition 1927 of the book [5, pp. 337–338] by Netto. A further proof of Frisch’s identity (1), a two-page calculation involving an application of Melzak’s formula, can be found in the new book [6, Section 7.2].

In [4] the authors report that identity (2) in its original form [4, Eq. (1)] with \( x = n + a \) was stated by Murray S. Klamkin in a letter to Henry W. Gould on May 16, 1966. It is tabulated as Formula (4.2) in Gould’s collection [3]. Identity (4.6) in [3] is a special case of this.

The purpose of this note are elementary short proofs without application of hypergeometric functions. They are based on the use of the Euler beta function

\[
B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)} \quad (x, y > 0).
\]

Furthermore, we present variants of Frisch’s and of Klamkin’s identities involving infinite series.

## 2 Proof of Frisch’s and Klamkin’s formulas

By direct calculation, we obtain, for \( n = 0, 1, 2, \ldots \) and \( b \geq c > 0 \),

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{b+k}{c}^{-1} = c \sum_{k=0}^{n} (-1)^k \binom{n}{k} B(b - c + 1 + k, c) = c \sum_{k=0}^{n} (-1)^k \binom{n}{k} \int_0^1 t^{b-c+k} (1-t)^{c-1} dt = c \cdot B(b - c + 1, c + n) = \frac{c}{n + c} \left(\frac{x + b}{b - c}\right)^{-1},
\]

which is Frisch’s identity (1). For suitable \( x \neq -1 \), we have

\[
\frac{1}{x+1} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x}{k+b}\right)^{-1} = \sum_{k=0}^{n} \binom{n}{k} B(b + k + 1, x - b - k + 1) = \sum_{k=0}^{n} \binom{n}{k} \int_0^1 t^b (1-t)^{x-b} \left(\frac{t}{1-t}\right)^k dt = B(b + 1, x - b - n + 1) = \frac{1}{x-n+1} \left(\frac{x}{b}\right)^{-1},
\]

which proves Klamkin’s identity (2).
3 Variants of Frisch’s and Klamkin’s identities

Since \( \binom{n}{k} = 0 \), for integers \( k > n \geq 0 \), the left-hand sides in both identities (1), (2) can be written as infinite sums. Formally replacing \( n \) with \( -n \) in those equations and using the obvious binomial identity

\[
(-1)^k \binom{-n}{k} = \binom{n+k-1}{k}
\]

yields

\[
\sum_{k=0}^{\infty} \binom{n+k-1}{k} \left( \frac{b+k}{c} \right)^{-1} = \frac{c}{c-n} \left( \frac{-n+b}{b-c} \right)^{-1},
\]

\[
\sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} \left( \frac{x}{k+b} \right)^{-1} = \frac{x+1}{x+n+1} \left( \frac{x+n}{b} \right)^{-1}.
\]

We shall show that both equations are valid also for positive integers \( n \), if the parameters \( a, b, c \) are chosen in an appropriate manner.

First we deduce the variant of Frisch’s identity.

**Theorem 1.** For \( b \geq c > n \geq 1 \),

\[
\sum_{k=0}^{\infty} \binom{n+k-1}{k} \left( \frac{b+k}{c} \right)^{-1} = \frac{c}{c-n} \left( \frac{b-n}{b-c} \right)^{-1}.
\]

The infinite series in Eq. (3) is convergent since

\[
\binom{n+k-1}{k} \left( \frac{b+k}{c} \right)^{-1} = \Gamma(n) \frac{\Gamma(n+k)}{\Gamma(1+k)} \frac{\Gamma(b-c+1+k)}{\Gamma(b+1+k)} \frac{\Gamma(b-c+1+k)}{\Gamma(b+1+k)} \sim \frac{\Gamma(n) \Gamma(c+1)}{k^{1-n+c}}
\]

as \( k \to \infty \) (see, e.g., [1, (6.1.46)]) and \( c > n \).

**Proof of Theorem 1.** By using the well-known power series expansion

\[
\sum_{k=0}^{\infty} \binom{n+k-1}{k} z^k = (1-z)^{-n} \quad (|z| < 1)
\]

instead of the binomial formula, it follows in the same manner as in the preceding section that

\[
\sum_{k=0}^{\infty} \binom{n+k-1}{k} B(x+k,y) = B(x,y-n) \quad (x > 0, \ y > n).
\]

Rewritten in terms of binomial coefficients the latter identity takes the form

\[
x \sum_{k=0}^{\infty} \binom{n+k-1}{k} \left( \frac{x+y+k-1}{y} \right)^{-1} = y \left( \frac{x+y-n-1}{x} \right)^{-1} \quad (x > 0, \ y > n).
\]
Replacing $x$ with $b - c + 1$ and $y$ with $c$, we obtain
\[ \sum_{k=0}^{\infty} \binom{n+k-1}{k} \binom{b+k}{c}^{-1} = \frac{c}{b-c+1} \binom{b-n}{b-c+1}^{-1} \quad (b > c - 1, \ c > n). \]

Now Eq. (3) follows since
\[ \frac{c}{b-c+1} \binom{b-n}{b-c+1}^{-1} = \frac{c}{c-n} \binom{b-n}{b-c}^{-1}. \]

This completes the proof. \[ \Box \]

**Remark 2.** An alternative approach is the observation
\[ \sum_{k=0}^{\infty} \binom{n+k-1}{k} \binom{b+k}{c}^{-1} = \binom{b}{c}^{-1} {}_2F_1 \left( 1+b-c, n; b+1; 1 \right). \]

This formula immediately implies Eq. (3) since, by Gauss’s formula \[1, (15.1.20)],
\[ {}_2F_1 \left( 1+b-c, n; b+1; 1 \right) = \frac{\Gamma(c-n) \Gamma(b+1)}{\Gamma(c) \Gamma(b-n+1)} \quad (c > n, \ b > -1). \]

We close with the variant of Klamkin’s identity.

**Theorem 3.** Let $-a \notin \mathbb{N}$ and $-b \notin \mathbb{N}$. For $-a - 1 > n \geq 1$ and $a - b + 1 \notin \mathbb{N},$
\[ \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} \binom{a}{b+k}^{-1} = \frac{a+1}{a+n+1} \binom{a+n}{b}^{-1}. \quad (4) \]

The method of proof using the beta integral does not work in the case of Theorem 3 because the arising sum becomes divergent. Therefore, we use the Gauss formula as in the preceding remark.

**Proof of Theorem 3.** Noting that
\[ (-1)^k \binom{a}{b+k}^{-1} = \binom{a}{b}^{-1} \frac{\Gamma(b+1+k)}{\Gamma(k+1)} \frac{\Gamma(b-a)}{\Gamma(b-a+k)}, \]
we have
\[ (-1)^k \binom{n+k-1}{k} \binom{a}{b+k}^{-1} = \gamma \frac{\Gamma(n+k)}{\Gamma(1+k)} \frac{\Gamma(b+1+k)}{\Gamma(b-a+k)} \sim \gamma a^{a+k} \]
as $k \to \infty$, where $\gamma = \Gamma(n) \frac{\Gamma(b-a)}{\Gamma(b+1)} \binom{a}{b}^{-1}$. If $a + 1 + n < 0$ we have
\[ (-1)^k \binom{n+k-1}{k} \binom{a}{b+k}^{-1} = O(k^{-\eta}) \quad (k \to \infty) \]
with a certain constant \( \eta > 1 \). This shows the convergence of the infinite series in Eq. (4). Finally, we observe

\[
\sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} \frac{a}{b+k} = \left(\frac{a}{b}\right)^{-1} _2F_1 (n, b + 1; b - a; 1),
\]

for \( a + 1 + n < 0 \) and \( a - b + 1 \notin \mathbb{N} \). Using

\[
_2F_1 (n, b + 1; b - a; 1) = \frac{\Gamma (-a - n - 1) \Gamma (b - a)}{\Gamma (b - a - n) \Gamma (-a - 1)} = \frac{\Gamma (a - b + n + 1) \Gamma (a + 2)}{\Gamma (a - b + 1) \Gamma (a + 2 + n)}
\]

the desired formula Eq. (4) follows after a short calculation.

\[\square\]

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References


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