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Waring's Problem for Hurwitz Quaternion Integers

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Abstract

Extending recent work of Pollack on Waring's problem for the ring of Lipschitz quaternion integers, we study Waring's problem with respect to the larger ring of Hurwitz quaternion integers.

1 Introduction and statement of the main results

In 1770, Lagrange [7] proved that every positive integer can be written as a sum of four integer squares. The same year Waring [14] claimed that "every positive integer is a sum of nine (integer) cubes, a sum of at most 19 biquadrates, et cetera", however, without giving a proof. In 1909, Hilbert [3] solved the so-called *Waring problem* by showing that *for every* $\ell \in \mathbb{N}$ *there exists some* $g(\ell) \in \mathbb{N}$ *such that every* $n \in \mathbb{N}$ *is a sum of at most* $g(\ell)$ ℓ*-th powers.* For most values of ℓ the true value for $g(\ell)$ is given by $2^\ell + \lfloor (3/2)^\ell \rfloor - 2$; for this and more details on the classical Waring problem we refer to the survey [13] by Vaughan and Wooley.

Ever since Hilbert's proof, Waring's problem has also been studied for other semigroups and rings. Recently, Pollack [12] solved Waring's problem for the set of Lipschitz quaternion integers $\mathcal{L} := \mathbb{Z}[i, j, k]$, introduced by Lipschitz [8] in 1886. Recall that the quaternions

$$\alpha + \beta i + \gamma j + \delta k \qquad \text{with} \quad \alpha, \beta, \gamma, \delta \in \mathbb{R}$$

form a skew field, that is, a non-commutative division ring, where i, j, k are independent square roots of -1 satisfying

$$i^2 = j^2 = k^2 = ijk = -1.$$

Their recent anniversary should not be unmentioned: quaternions were discovered (or invented) by Hamilton a little more than 175 years ago. In 1946, Niven [10] showed that every Lipschitz quaternion integer $\alpha + \beta i + \gamma j + \delta k$ has a representation as a sum of squares provided that the coefficients β, γ, δ are all even; in that case at most three squares are sufficient and $6 + 2i$ is not a sum of two squares. The necessity for those even coefficients follows immediately from

$$(\alpha + \beta i + \gamma j + \delta k)^2 = \alpha^2 - \beta^2 - \gamma^2 - \delta^2 + 2\alpha\beta i + 2\alpha\gamma j + 2\alpha\delta k.$$

Pollack [12] proved that for odd $\ell \in \mathbb{N}$ every Lipschitz quaternion integer is a sum of ℓ-th powers, while for even $\ell = 2^\mu m$ with $\mu \in \mathbb{N}$ and odd m the sums of ℓ-th powers are exactly those Lipschitz quaternion integers $a + bi + cj + dk$ that satisfy

$$b, c, d \equiv 0 \pmod{2^\mu} \qquad \text{and} \qquad b + c + d \equiv 0 \pmod{2^{\mu+1}}.$$

This obstruction shall be seen as natural analogue to the original case where negative integers are excluded for obvious reasons. Pollack's theorem (and our results as well) also include the case of Gaussian integers $\mathbb{Z}[\sqrt{-1}]$ with the usual imaginary unit $\sqrt{-1}$ (via a natural embedding and identifying $\sqrt{-1}$ with i). Pollack's reasoning is influenced by the works of Paley [11] and Cohn [2].

Our approach in this brief note is different and we shall consider Waring's problem for the larger set of Hurwitz quaternion integers $\mathcal{H} := \mathbb{Z}[i, j, k, \rho]$, chosen by Hurwitz [4] in 1896, where

$$\rho := \frac{1}{2}(1 + i + j + k).$$

The additional generator ρ may be considered as the analogue of a sixth root of unity (since ρ satisfies the equation for the sixth cyclotomic polynomial, i.e., $\rho^2 - \rho + 1 = 0$) and plays a central role in our results. Notice that the ring $\mathcal{H} = \mathcal{L}[\rho]$ of Hurwitz quaternion integers has more structure than Lipschitz's ring $\mathcal{L}$; for example, $\mathcal{H}$ is a Euclidean domain while $\mathcal{L}$ is not. For this and further information we refer to Hurwitz's booklet [5] and its upcoming English translation [6] with comments.

Our main results are

Theorem 1. *For every positive integer $\ell \equiv 1, 2 \pmod 3$ there exists some $g_{\mathcal{H}}(\ell) \in \mathbb{N}$ such that every $z \in \mathcal{H}$ can be written as a sum of at most $g_{\mathcal{H}}(\ell)$ ℓ-th powers of quaternions from $\mathcal{H}$.*

The larger set $\mathcal{H}$ allows representations as sums of ℓ-th powers that cannot be realized in $\mathcal{L}$; for example,

$$1 + i + j + k = 2\rho = \rho^2 + \rho^2 + 1^2 + 1^2$$

can be expressed as a sum of squares in $\mathcal{H}$ whereas this is impossible in $\mathcal{L}$. This observation about sums of squares was already made by Niven [10].

For the remaining cases of ℓ, however, the situation is rather different:

Theorem 2. *If $\ell \equiv 0 \pmod 3$ is a positive integer, then no $z \in \mathcal{H} \setminus \mathcal{L}$ is a sum of one or more ℓ-th powers of elements from $\mathcal{H}$.*

Therefore, the extension $\mathcal{H}$ does not lead to further sums of ℓ-th powers for such ℓ. Indeed, if ℓ is a multiple of 3, then, by Pollack's result, every $z \in \mathcal{L}$ is expressible as a sum of ℓ-th powers of quaternions from $\mathcal{L}$ with the exceptions mentioned above in the case that ℓ is even. Unfortunately, the larger ring $\mathcal{H}$ does not help here. The reason for this is that ρ satisfies $\rho^3 = -1$; a similar obstruction appears from $i^2 = -1$. For getting further sums of ℓ-th powers in the case $\ell \equiv 0 \pmod 3$ another extension is needed.

The set $\mathcal{H} = \mathbb{Z}[i, j, k, \rho]$ of Hurwitz quaternion integers may be regarded as the analogue of the ring of integers of a quadratic number field; the quotient structure associated with $\mathcal{H}$ is therefore given by the skew field $\mathbb{Q}[i, j, k]$ (where we may omit ρ). For our next purpose we shall enlarge the field of coefficients by a finite algebraic extension. Thus, we consider quaternions from a skew field $K[i, j, k]$, where K is an appropriate number field. An appropriate choice of K then allows to represent every Hurwitz quaternion integer as a sum of ℓ-th powers.

Theorem 3. *Let $\ell = 2^\mu \cdot 3^\nu \cdot \lambda$ be a positive integer, where $\mu \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \nu, \lambda \in \mathbb{N}$ such that λ is neither divisible by 2 nor by 3. Then there exists an algebraic number ω (depending on ℓ) such that the equations*

$$i = U^{2^\mu} \qquad \text{and} \qquad \rho = V^{3^\nu}$$

have solutions in $\mathcal{W} := \mathbb{Q}(\omega)[i, j, k]$, where $\mathbb{Q}(\omega)$ is the number field generated by ω. Moreover, there exists some $g_{\mathcal{W}}(\ell) \in \mathbb{N}$ such that every $z \in \mathcal{H}$ has a representation as a sum of at most $g_{\mathcal{W}}(\ell)$ ℓ-th powers of quaternions from $\mathcal{W}$.

The proof indicates that the full set $\mathbb{Q}(\omega)$ is not needed to serve for coefficients here. Obviously, $\mathcal{H}$ is contained in $\mathcal{W}$. The idea of an extension of the set of coefficients may be seen as pushing Hurwitz's definition of quaternion integers further. In his case the additional generator ρ replaced the non-euclidean ring of Lipschitz quaternion integers $\mathcal{L}$ with the larger euclidean ring $\mathcal{H}$. In our case as well, roots of unity allow further representations as sums of ℓ-th powers.

In the following section we collect a few useful results from quaternion arithmetic. In the subsequent sections we shall give the proofs of the theorems.

2 Units and a quadratic equation

An important role in our reasoning is played by the units. A quaternion ϵ in $\mathcal{H}$ is called a unit if its norm equals 1; here the norm of a quaternion $z = \alpha + \beta i + \gamma j + \delta k$ is defined by

$$N(z) := zz' = \alpha^2 + \beta^2 + \gamma^2 + \delta^2$$

where

$$z' := \alpha - \beta i - \gamma j - \delta k$$

is the conjugate of z. Thus, $z = 2\alpha - z'$ and

$$z^2 - t(z)z + N(z) = 0, \tag{1}$$

where $t(z) = 2\alpha$ is the trace of z. The name *trace* is chosen with respect to Cayley's matrix representation of quaternions [6] where also the norm equals the determinant; in this context the above quadratic equation turns out to be the corresponding vanishing characteristic polynomial evaluated at its matrix (by the theorem of Cayley-Hamilton) and may have more than two solutions (since multiplication of quaternions is non-commutative in general). The norm of any quaternion integer is a positive rational integer.

There are 24 units in $\mathcal{H}$, namely the eight units $\pm 1, \pm i, \pm j, \pm k$, which lie in $\mathcal{L}$, and the additional 16 units $\varepsilon = \frac{1}{2}(\pm 1 \pm i \pm j \pm k)$ of $\mathcal{H} \setminus \mathcal{L}$ [4, 5, 6]. The quadratic equation (1) above for the Lipschitz units is either $Z^2 + 1 = 0$ or reduces to $Z \mp 1 = 0$, whereas the remaining Hurwitz units ε satisfy

$$Z^2 - \delta Z + 1 = 0$$

with $\varepsilon = \frac{1}{2}(\delta \pm i \pm j \pm k)$ and $\delta \in \{\pm 1\}$. For our later study we shall be very precise about the signs. Therefore, we introduce the representation

$$\varepsilon = \frac{1}{2}(\delta + \delta_i i + \delta_j j + \delta_k k) \qquad \text{with} \quad \delta, \delta_i, \delta_j, \delta_k \in \{\pm 1\}. \tag{2}$$

Then,

$$\varepsilon^2 = \delta\varepsilon - 1 = \frac{1}{2}(-1 + \delta\delta_i i + \delta\delta_j j + \delta\delta_k k). \tag{3}$$

For the third power we compute

$$\varepsilon^3 = \varepsilon \cdot (\delta\varepsilon - 1) = \delta(\delta\varepsilon - 1) - \varepsilon = -\delta, \tag{4}$$

whereas the fourth power is given by

$$\varepsilon^4 = -\delta\varepsilon = \frac{1}{2}(-1 - \delta\delta_i i - \delta\delta_j j - \delta\delta_k k); \tag{5}$$

the fifth power equals

$$\varepsilon^5 = -\varepsilon + \delta = \frac{1}{2}(\delta - \delta_i i - \delta_j j - \delta_k k) = \varepsilon'. \tag{6}$$

Finally, the sixth power is trivial, namely

$$\varepsilon^6 = 1. \tag{7}$$

So every Hurwitz unit $\varepsilon \in \mathcal{H} \setminus \mathcal{L}$ is of order 3 or 6; the Lipschitz units, however, are of order $1, 2$ or 4.

3 Proof of Theorem 1

For $\ell \equiv \pm 1 \pmod 6$ we may assume that $z \in \mathcal{H} \setminus \mathcal{L}$ (since the case of $z \in \mathcal{L}$ is covered by Pollack's theorem). We may step from $\mathcal{H}$ to $\mathcal{L}$ by subtraction of a Hurwitz quaternion integer $e = \frac{1}{2}(-1 + e_i i + e_j j + e_k k)$, where $e_i, e_j, e_k \in \{\pm 1\}$ are arbitrary. Thus, we have

$$z = \omega + e \qquad \text{with some} \quad \omega \in \mathcal{L}.$$

In view of Pollack's theorem [12] it thus suffices to show that e is an ℓ-th power. Since e is a unit, we may solve this task by solving the equation

$$e = \varepsilon^\ell \tag{8}$$

in ε. For this aim we may use the computation of the orbit of the units from the previous section.

In fact, if $\ell \equiv 1 \pmod 6$, then let $\varepsilon = e$. Since every Hurwitz unit ϵ is of order 3 or 6, it suffices to consider residues $\ell \pmod 6$. For $\ell \equiv 5 \pmod 6$ we use (6) and choose

$$\varepsilon = \frac{1}{2}(-1 - e_i i - e_j j - e_k k).$$

Now let us assume that $\ell \equiv \pm 2 \pmod 6$. In this case we shall use Hilbert's solution of the original Waring problem. Recall that the ring of Hurwitz quaternion integers is given by $\mathcal{H} = \mathbb{Z}[i, j, k, \rho]$, so an individual $z \in \mathcal{H}$ is of the form

$$z = \pm a\rho \pm bi \pm cj \pm dk \qquad \text{with some} \quad a, b, c, d \in \mathbb{N}_0,$$

and appropriate signs. Notice that the 'real part' of a quaternion z in the ring of Hurwitz quaternion integers is *hidden* in the coefficient $\pm a$ of ρ. By Hilbert's solution of Waring's problem, every coefficient a, b, c, d is a sum of ℓ-th powers,

$$a = \sum_{s=1}^{g(\ell)} a_s^\ell, \quad b = \sum_{t=1}^{g(\ell)} b_t^\ell, \ \ldots,$$

say. If we now can show that also $\pm\rho, \pm i, \pm j$, and $\pm k$ are also sums of ℓ-th powers, then we are done. However, *there is a problem here:* Since multiplication of quaternions in general is not commutative, the product of two ℓ-th powers is not necessarily an ℓ-th power; for example, $(1+i)^2 \cdot (1+j)^2 = 4k$ is not a square in $\mathcal{H}$. If one of the ℓ-th powers is real, a_s^ℓ say, however, then we can merge these powers, e.g.,

$$a_s^\ell \cdot q^\ell = (a_s \cdot q)^\ell,$$

where q may be any quaternion.

First, we consider $\pm\rho$. In view of (3) and (5) we have

$$-\rho = (\rho^2)^2 = \rho^4 \qquad \text{and} \qquad \rho = \rho^2 + 1^2 = \left(\rho'\right)^4 + 1^4,$$

where we have used that $\rho' = \rho^5$. Taking into account that both ρ and its conjugate are of order six, $\pm\rho$ is expressible as a sum of ℓ-th powers for every $\ell \equiv \pm 2 \pmod 6$.

Next we consider $\pm i$; in view of the symmetries $i \leftrightarrow j$ and $i \leftrightarrow k$ the remaining cases $\pm j, \pm k$ can be treated similarly. For $\ell = 2$ we find, by using again (3) and (5),

$$\left(\frac{1}{2}(1 \pm i + j - k)\right)^2 + \left(\frac{1}{2}(1 \pm i - j + k)\right)^2 + 1^2 = \pm i.$$

And if $\ell = 4$, then

$$\left(\frac{1}{2}(1 \pm i + j - k)\right)^4 + \left(\frac{1}{2}(1 \pm i - j + k)\right)^4 + 1^4 = \mp i.$$

Since the involved Hurwitz units are either of order 3 or 6, it follows that $\pm i$ is for $\ell \equiv \pm 2 \pmod 6$ a sum of at most three ℓ-th powers. This finishes the proof of Theorem 1.

The latter reasoning implies for instance the upper bound

$$g_{\mathcal{H}}(4) \leq (2+3+3+3)g(4) = 11 \cdot 19 = 209,$$

where $g(4) = 19$ is a celebrated result due to Balasubramanian, Dress, and Deshouillers [1]. The bound for $g_{\mathcal{H}}(4)$ is very likely rather poor.

4 Proof of Theorem 2

The same reasoning as in the case of odd $\ell \equiv \pm 1 \pmod 6$ in the proof of Theorem 1 above is in general impossible since equation (8) is unsolvable in general. In particular (4) and (7) indicate trouble when ℓ is a multiple of 3. To show that e cannot be written as a sum of ℓ-th powers, however, we have to do a little more.

It follows from (1) that

$$z^3 = (t(z)^2 - N(z))z - t(z)N(z).$$

For $z = \alpha + \beta i + \gamma j + \delta k \in \mathcal{H} \setminus \mathcal{L}$ we have

$$\alpha = a + \frac{1}{2}, \quad \beta = b + \frac{1}{2}, \quad \gamma = c + \frac{1}{2}, \quad \delta = d + \frac{1}{2}$$

for some $a, b, c, d \in \mathbb{Z}$. Hence, $t(z) = 2\alpha = 2a + 1$ and

$$N(z) = a(a+1) + b(b+1) + c(c+1) + d(d+1) + 1 \equiv 1 \pmod 2$$

(since one of two consecutive integers $n, n+1$ is even). It follows that $t(z)^2 - N(z) \equiv 0 \pmod 2$ (which cancels the denominator 2 of the coefficients of z) and therefore $z^3 \in \mathcal{L}$. This proves that no element of $\mathcal{H} \setminus \mathcal{L}$ can be written as a sum of cubes and, of course, the same statement holds for any power with an exponent divisible by 3. This proves Theorem 2.

5 Proof of Theorem 3

We begin with the equation

$$V^{3^\nu} = \rho = \frac{1}{2}(1 + i + j + k) \tag{9}$$

and do the ansatz $V = X + Y\kappa$, where X, Y are real unknowns and $\kappa = i + j + k = 2\rho - 1$. It follows from the binomial theorem in combination with (3)-(7) that

$$\frac{1}{2}(1+\kappa) = \rho = \sum_{0 \leq f \leq 3^\nu} \binom{3^\nu}{f} X^{3^\nu - f} Y^f \kappa^f.$$

Computing κ^f explicitly for even $f = 2r$ as $(-3)^r$, and for odd $f = 2r+1$ as $(-3)^r\kappa$, respectively, and separating the summands according to even and odd indices f, this leads to

$$\frac{1}{2}(1+\kappa) = \sum_{\substack{0\le f\le 3^\nu \\ f\equiv 0 \pmod 2}} \binom{3^\nu}{f}(-3)^{\frac{f}{2}} X^{3^\nu-f}Y^f + \kappa \sum_{\substack{1\le f\le 3^\nu \\ f\equiv 0 \pmod 2}} \binom{3^\nu}{f}(-3)^{\frac{f-1}{2}} X^{3^\nu-f}Y^f.$$

Since $\kappa \notin \mathbb{R}$ we may read the latter equation with quaternion coefficients as two independent equations with real coefficients, namely

$$\frac{1}{2} = \sum_{\substack{0\le f\le 3^\nu \\ f\equiv 0 \pmod 2}} \binom{3^\nu}{f}(-3)^{\frac{f}{2}} X^{3^\nu-f}Y^f, \tag{10}$$

$$\frac{1}{2} = \sum_{\substack{1\le f\le 3^\nu \\ f\equiv 0 \pmod 2}} \binom{3^\nu}{f}(-3)^{\frac{f-1}{2}} X^{3^\nu-f}Y^f. \tag{11}$$

Since

$$1 = N(\rho) = N\big((X+Y\kappa)^{3^\nu}\big) = N(X+Y\kappa)^{3^\nu},$$

it follows that $1 = N(X+Y\kappa) = X^2+3Y^2$. Substituting this in (10) and (11), respectively, yields separate equations in X and Y:

$$\frac{1}{2} = \sum_{\substack{0\le f\le 3^\nu \\ f\equiv 0 \pmod 2}} \binom{3^\nu}{f} X^{3^\nu-f}(X^2-1)^{\frac{f}{2}},$$

$$\frac{1}{2} = \sum_{\substack{1\le f\le 3^\nu \\ f\equiv 0 \pmod 2}} \binom{3^\nu}{f}(-3)^{\frac{f-1}{2}}(3Y^2-1)^{3^\nu-\frac{f}{2}}Y^f.$$

Both equations have rational coefficients and are of odd degree. Hence, there exist real algebraic numbers x and y solving these equations. By the primitive element theorem, there exists a real algebraic number ξ such that $\mathbb{Q}(x,y) = \mathbb{Q}(\xi)$, and Equation (9) is solvable in $\mathbb{Q}(\xi)(i,j,k)$.

We illustrate the latter reasoning in the easiest case $\ell = 3$. Following the reasoning above, we arrive at the equations

$$8X^3 - 6X - 1 = 0 \qquad \text{and} \qquad 24Y^3 - 6Y + 1 = 0.$$

Now let x, y be solutions to these equations and define $v = x + y(i+j+k)$. In view of

$$v^6 = \rho^2 = \rho - 1 = v^3 - 1$$

(as follows from (4) above) v may be considered as a quaternion root of the 18-th cyclotomic polynomial $\Phi_{18} = V^6 - V^3 + 1$ (Mollin's book [9] on algebraic number theory). Hence, $\mathcal{W}$ may be considered as an analogue of the ring of integers in the cyclotomic fields $\mathbb{Q}(\exp(2\pi\sqrt{-1}/18))$.

In order to have a representation of an arbitrary Hurwitz quaternion integer as a sum of ℓ-th powers in the case where $\ell \equiv 0 \pmod 3$, recall that $\ell = 2^\mu \cdot 3^\nu \cdot \lambda$ for some positive integer $\lambda \equiv \pm 1 \pmod 6$. As in the proof of Theorem 1 it suffices to show that $\pm\rho, \pm i$ etc. are ℓ-th powers (since then with Hilbert's solution of Waring's problem for $\mathbb{N}$ all integer linear combinations $\alpha\rho + \beta i + \cdots$ can be represented as a sum of ℓ-th powers).

We observe that $2^\mu \equiv 2 \pmod 6$ if μ is odd and $2^\mu \equiv 4 \pmod 6$ if μ is even. Thus, by (3), for odd μ,

$$\rho = 1 + \rho^2 = 1 + \rho^{2^\mu}.$$

For the case of odd λ, let ζ be a root of (9). If $\lambda = 1 + 6s$ for some $s \in \mathbb{N}_0$, then the right hand side equals

$$1 + (\rho^\lambda)^{2^\mu} = 1 + \left(\zeta^{3^\nu}\right)^{2^\mu \cdot \lambda} = 1^\ell + \zeta^\ell;$$

if $\lambda = 5 + 6s$, however, we switch to the conjugates and rewrite the right hand side as

$$1 + \left((\rho')^\lambda\right)^{2^\mu} = 1^\ell + \left(\zeta'\right)^\ell;$$

here ζ' may be computed from $\zeta\zeta' = 1$ as $\zeta' = \zeta^{2\cdot 3^{\nu+1}-1}$. Thus, ρ is a sum of two ℓ-th powers. For the case of $-\rho$ we use the counterpart of (3) for the conjugate of ρ, i.e., $-\rho = \rho' - 1 = (\rho')^2$, and deduce

$$-\rho = \left(\rho'\right)^2 = \left(\rho'\right)^{2^\mu}.$$

If $\lambda = 1 + 6s$, then

$$-\rho = \left(\left(\rho'\right)^\lambda\right)^{2^\mu} = \left(\zeta'\right)^\ell,$$

and replacing ρ' by ρ^5 leads in the case $\lambda = 5 + 6s$ to the result ζ^ℓ. This shows that $\pm\rho$ is a sum of ℓ-th powers in case that μ is odd.

For the case that μ is even, we first notice that, for $\mu = 0$, we have $\pm\rho = (\pm\rho)^\lambda = (\pm\zeta)^\ell$ if $\lambda = 1 + 6s$. Otherwise, when $\lambda = 5 + 6s$, then, by (6),

$$\pm\rho = \pm(1 - \rho^5) = \pm 1 \mp \rho^\lambda = (\pm 1)^\ell + (\mp\zeta)^\ell.$$

For the case of even $\mu \geq 2$ we have in view of (3) that, for $\lambda = 1 + 6s$,

$$\rho = 1 + \rho^2 = 1 + \left(\rho'\right)^{2^\mu} = 1 + \left(\left(\rho'\right)^\lambda\right)^{2^\mu} = 1^\ell + \left(\zeta'\right)^\ell;$$

if $\lambda = 5+6s$, then switching from ζ' to ζ^5 leads to a representation of ρ as a sum of two ℓ-th powers. Moreover, by (5),

$$-\rho = \rho^4 = \rho^{2^\mu} = \left(\rho^\lambda\right)^{2^\mu} = \zeta^\ell$$

for $\lambda = 1+6s$, while in a similar way

$$-\rho = \left(\left(\rho'\right)^\lambda\right)^{2^\mu} = (\zeta')^\ell$$

for $\lambda = 5+6s$. Hence, in all cases $\pm\rho$ is indeed a sum of ℓ-th powers. This proves in particular the statement of the theorem for the case $\ell = 3$ or any odd power of 3.

For even ℓ, however, we need to consider in addition the equation

$$i = U^{2^\mu}. \tag{12}$$

In view of the natural embedding of the complex number field $\mathbb{C} = \mathbb{R}[\sqrt{-1}]$ via mapping $\sqrt{-1}$ to i into the skew field of quaternions, we may first solve this equation in $\mathbb{C}$ (which is possible thanks to the Fundamental Theorem of Algebra) and interpret its solutions as quaternions. Notice that quaternions of the form $\alpha+\beta i$ form a commutative subring. Since a solution $u = \alpha+\beta i$ to equation (12) has real algebraic coefficients, it follows after another application of the primitive element theorem that both equations (9) and (12) are solvable in quaternions with coefficients from an appropriate number field $\mathbb{Q}(\omega)$.

For $\mu \geq 1$ and $3^\nu\lambda = 1+4m$ with some $m \in \mathbb{N}_0$, we find $i = (i)^{3^\nu\lambda} = u^\ell$ with u satisfying (12) and

$$-i = i' = \left(i'\right)^{3^\nu\lambda} = \left(u'\right)^\ell,$$

where $i' = -i$ and $u' = u^{2^{\mu+2}-1}$ (according to $ii' = 1$ and so on). In the case $3^\nu\lambda = 3+4m$ we find similarly $-i = (i)^{3^\nu\lambda} = u^\ell$ and

$$i = -i' = \left(i'\right)^{3^\nu\lambda} = \left(u'\right)^\ell.$$

This shows that $\pm i$ is always a sum of ℓ-th powers.

It remains to consider the units $\pm j$ and $\pm k$. It appears that no further elements have to be adjoined. In fact, conjugation with an appropriate Hurwitz unit ε solves the problem. Define

$$\varepsilon := \frac{1}{2}(1+i+j-k),$$

then

$$j = \varepsilon' \cdot i \cdot \varepsilon = \tilde{u}^{2\mu} \qquad \text{with} \quad \tilde{u} := \varepsilon' \cdot u \cdot \varepsilon$$

and the above results for $\pm i$ may be transfered to $\pm j$ by replacing u by $\tilde{u}$ and their respective conjugates. Hence, also the Lipschitz units $\pm j$ are sums of ℓ-th powers. The same reasoning with another Hurwitz unit implies the same statement for $\pm k$. This proves Theorem 3.

As a final remark, let us mention that still not every quaternion from $\mathcal{W}$ is expressible as a sum of ℓ-th powers for certain integers ℓ. If we aim at a representation for all as a sum of ℓ-th powers with an arbitrary ℓ, we would need to adjoin all (infinitely many) roots of unity.

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References

[1] R. Balasubramanian, F. Dress, and J.-M. Deshouillers, Problème de Waring pour les bicarrés, *C. R. Acad. Sci. Paris Sér. I Math.* **303** (1986), 85–88, 161–163.

[2] J. H. E. Cohn, Waring's problem in quadratic number fields, *Acta Arith.* **20** (1972), 1–16.

[3] D. Hilbert, Beweis für die Darstellbarkeit der ganzen Zahlen durch eine feste Anzahl n-ter Potenzen (Waringsches problem). Dem Andenken an Hermann Minkowski gewidmet, *Math. Ann.* **67** (1909), 281–300.

[4] A. Hurwitz, Ueber die Zahlentheorie der Quaternionen, *Gött. Nachr.* (1896), 314–340.

[5] A. Hurwitz, *Vorlesungen über die Zahlentheorie der Quaternionen*, Springer, 1919.

[6] A. Hurwitz, *Lectures on The Number Theory of Quaternions*, in N. Oswald and J. Steuding, eds., EMS (to appear).

[7] J. L. Lagrange, Démonstration d'un theorème d'arithmétique, *Nouveaux Memoires de L'Académie Royale des Sciences et Belles-Lettres de Berlin* (1770), 123–133.

[8] R. Lipschitz, *Untersuchungen über die Summen von Quadraten*, Cohen & Sohn, 1886.

[9] R. A. Mollin, *Algebraic Number Theory*, Chapman & Hall/CRC Press, 1999.

[10] I. Niven, A note on the number theory of quaternions, *Duke Math. J.* **13** (1946), 397–400.

[11] R. E. A. C. Paley, Theorems on polynomials in a Galois field, *Quart. J. Math.* **4** (1933), 52–63.

[12] P. Pollack, Waring's problem for integral quaternions, *Indagationes Math.* **29** (2018), 1259–1269.

[13] R. C. Vaughan and T. Wooley, Waring's problem, in M. A. Bennett et al., eds., *Number Theory for the Millennium III*, A. K. Peters, 2002, pp. 301–340.

[14] E. Waring, *Meditationes Algebraicae*, 1770. D. Weeks ed., English transl., AMS, 1991.

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