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Two New Identities Involving the Catalan Numbers and Sign-Reversing Involutions

Jovan Mikić
J.U. SŠC "Jovan Cvijić"
74480 Modriča
Republic of Srpska
jnmikic@gmail.com

Abstract

We give a combinatorial proof of a known sum concerning the product of a binomial coefficient with two central binomial coefficients. The method of description, involution, and exception is used. The same combinatorial argument also proves the "-1 shifted version" of this sum. As a consequence, two new binomial coefficient identities with the Catalan numbers are derived.

1 Introduction

Let n be a non-negative integer. The Catalan numbers are the famous sequence

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

In this paper, we derive two new (to our knowledge) binomial coefficient identities with the Catalan numbers.

Theorem 1. *For non-negative integers n, we have*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}C_k\binom{2n-2k}{n-k} = \binom{n}{\lfloor \frac{n}{2}\rfloor}^2, \tag{1}$$

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}C_kC_{2n-k} = C_n\binom{2n}{n}. \tag{2}$$

In order to prove Theorem 1, we consider a known combinatorial sum

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k}=\begin{cases}0, & \text{if } n \text{ is odd;}\\ \binom{n}{\frac{n}{2}}^2, & \text{if } n \text{ is even.}\end{cases} \tag{3}$$

Eq. (3) appears several times in the literature; see, for example, [2, Eq. (6.12), p. 52], [3, Eq. (6.61), p. 29], and [7, Example 3.6.2, p. 45]. There are two binomial coefficient identities ([3, Eq. (6.10), p. 23; Eq. (7.3), p. 34]) similar to Eq. (3). They are proved combinatorially, for example, in [8]. See [6] and [5, Conclusions] for the connection between these identities and Shapiro's formula [9, Ex. (6.C.18), p. 41] and Segner's recurrence relation [4, Eq. (5.6), p. 117] respectively.

We give a proof of Eq. (3) by using the method of "description, involution, and exception" [1].

By using the same idea, we derive the "-1 shifted version" of Eq. (3). We assert that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{k-1}\binom{2n-2k}{n-k}=\begin{cases}-\binom{n}{\lfloor\frac{n}{2}\rfloor}^2, & \text{if } n \text{ is odd;}\\ 0, & \text{if } n \text{ is even.}\end{cases} \tag{4}$$

Clearly, subtracting Eq. (4) from Eq. (3) and by using the well-known relation $C_k = \binom{2k}{k} - \binom{2k}{k-1}$, we get Eq. (1). Furthermore, it can be shown that Eq. (2) is a consequence of Eq. (1).

Throughout the paper, $[n]$ denotes the set $\{1, 2, \ldots, n\}$, if n is a positive integer; and $[0]$ denotes the empty set $\emptyset$.

Let A and B be sets. Then $|A|$ denotes the cardinality of the set A, and $A\backslash B$ denotes the set difference: $\{x : x \in A, x \notin B\}$.

We end this paper with the generalization of Eqns. (3) and (4).

2 Definitions

Let n be a fixed non-negative integer.

Definition 2. For $A \subset [n]$, we define

$$A^t = \begin{cases}\emptyset, & \text{if } A = \emptyset;\\ \{x+n : x \in A\}, & \text{if } A \neq \emptyset.\end{cases}$$

Obviously, if $A \subset [n]$, then $A^t \subset [2n]\backslash[n]$.

Definition 3. We define the function $\varphi : [2n] \to [2n]$, as follows:

$$\varphi(x) = \begin{cases}x+n, & \text{if } x \in [n];\\ x-n, & \text{if } x \in [2n]\backslash[n].\end{cases}$$

Definition 4. Let $S \subset [2n]$. The set S is *balanced* if $S = (S \cap [n]) \cup (S \cap [n])^t$. Otherwise, S is a *unbalanced* set.

In other words, set S is balanced if $\forall x\,(x \in S \Leftrightarrow \varphi(x) \in S)$. Clearly, if set S is balanced, then $|S|$ must be even.

The union of two balanced sets is a balanced set. Note that the converse does not hold even if two sets are disjoint. For example, unbalanced sets $[n]$ and $[2n]\backslash[n]$ are disjoint, but their union is a balanced set $[2n]$.

However, if S_1 and S_2 are disjoint sets such that $S_1 \cup \varphi(S_1)$ and $S_2 \cup \varphi(S_2)$ are disjoint sets, then $S_1 \cup S_2$ is a balanced set if and only if both sets S_1 and S_2 are balanced.

3 Proof of Eq. (3)

Proof. Let n be a fixed non-negative integer. We use the method of description, involution, and exception, as discussed in [1].

Description:

Let X denote the set

$$\{(A, B, C) : A \subset [n], B \subset A \cup A^t, |A| = |B|, C \subset [2n]\backslash(A \cup A^t), |A| + |C| = n\}.$$

For integers k, where $0 \leq k \leq n$, we define the sets X_k, as follows:

$$X_k = \{(A, B, C) \in X : |A| = k\}.$$

Obviously, $X = \bigcup_{k=0}^{n} X_k$ and $|X_k| = \binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k}$. We have

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k} = \sum_{k=0}^{n}(-1)^k|X_k| = |\mathcal{E}| - |\mathcal{O}|; \tag{5}$$

where

$$\mathcal{E} = \{(A, B, C) \in X : |A| \text{ is even}\} \text{ and } \mathcal{O} = \{(A, B, C) \in X : |A| \text{ is odd}\}.$$

Involution:

Let us define sets D and E, as follows:

$$D = \{(A, B, C) \in X : B \cup C \text{ is an unbalanced set}\} \tag{6}$$
$$E = \{(A, B, C) \in X : B \cup C \text{ is balanced set}\} \tag{7}$$

Obviously, D and E are disjoint sets and $X = D \cup E$.

Let $(A, B, C) \in D$. We let $d_{B,C}$ denote $\min\{x \in B \cup C : \varphi(x) \notin B \cup C\}$. The integer $d_{B,C}$ is well-defined because $B \cup C$ is an unbalanced set.

Let us define the function $\Psi : D \to D$, as follows:

$$\Psi((A,B,C)) = \begin{cases} (A\backslash\{d_{B,C}\}, B\backslash\{d_{B,C}\}, C \cup \{d_{B,C}\}), & \text{if } d_{B,C} \in B \cap [n]; \\ (A\backslash\{d_{B,C} - n\}, B\backslash\{d_{B,C}\}, C \cup \{d_{B,C}\}), & \text{if } d_{B,C} \in B \cap [n]^t; \\ (A \cup \{d_{B,C}\}, B \cup \{d_{B,C}\}, C\backslash\{d_{B,C}\}), & \text{if } d_{B,C} \in C \cap [n]; \\ (A \cup \{d_{B,C} - n\}, B \cup \{d_{B,C}\}, C\backslash\{d_{B,C}\}), & \text{if } d_{B,C} \in C \cap [n]^t. \end{cases} \tag{8}$$

The function Ψ is well-defined and an involution on D. Moreover, if $(A,B,C) \in D \cap \mathcal{E}$, then $\Psi((A,B,C)) \in D \cap \mathcal{O}$; and vice versa. Therefore, we may conclude that $|D \cap \mathcal{E}| = |D \cap \mathcal{O}|$.

We have

$$\begin{aligned} |\mathcal{E}| - |\mathcal{O}| &= |\mathcal{E} \cap X| - |\mathcal{O} \cap X| \\ &= |\mathcal{E} \cap D| + |\mathcal{E} \cap E| - (|\mathcal{O} \cap D| + |\mathcal{O} \cap E|) \qquad (X = D \cup E) \\ &= |\mathcal{E} \cap E| - |\mathcal{O} \cap E| \qquad (\text{because } |\mathcal{E} \cap D| = |\mathcal{O} \cap D|). \end{aligned}$$

Therefore, we obtain

$$|\mathcal{E}| - |\mathcal{O}| = |\mathcal{E} \cap E| - |\mathcal{O} \cap E|. \tag{9}$$

From Eqns. (5) and (9), it follows that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{k} \binom{2n-2k}{n-k} = |\mathcal{E} \cap E| - |\mathcal{O} \cap E|.. \tag{10}$$

Exception:

Let $(A,B,C) \in E$.

By Eq. (7), the set $B \cup C$ is balanced. Sets B and C are disjoint. Moreover, $B \cup \varphi(B)$ and $C \cup \varphi(C)$ are disjoint sets too. Then it follows that both sets B and C must be balanced. Hence integers $|B|$ and $|C|$ are even. Since $|A|$=$|B|$ (by the definition of X), $|A|$ is even and $(A,B,C) \in \mathcal{E}$. Therefore, $E \subset \mathcal{E}$.

Eq. (10) simplifies to

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{k} \binom{2n-2k}{n-k} = |E|. \tag{11}$$

We have two cases:

Case (a): n is odd.

Since $|B \cup C| = n$, it follows that the set $B \cup C$ is unbalanced and $E = \emptyset$. By Eq. (11), it follows that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{k} \binom{2n-2k}{n-k} = 0,$$

as desired.

Case (b): n is even.

We use the Chu-Vandermonde convolution formula:

$$\sum_{k=0}^{c} \binom{a}{k} \binom{b}{c-k} = \binom{a+b}{c}, \tag{12}$$

where a, b, and c are non-negative integers.

Let us count the number of elements of the set E. The set E is equal to the set

$$\{(A, B_1 \cup B_1^t, C_1 \cup C_1^t) : A \subset [n], |A| \text{ even}, B_1 \subset A, |B_1| = \frac{|A|}{2}, C_1 \subset [n] \backslash A, |C_1| + |B_1| = \frac{n}{2}\}.$$

Obviously, it follows that $|E|$ is equal to

$$|\{(A, B_1, C_1) : A \subset [n], |A| \text{ even}, B_1 \subset A, |B_1| = \frac{|A|}{2}, C_1 \subset [n] \backslash A, |C_1| + |B_1| = \frac{n}{2}\}|.$$

Clearly, there is a one-to-one correspondence between (A, B_1, C_1) and $(B_1 \cup C_1, B_1, A \backslash B_1)$. Therefore, $|E|$ is equal to

$$|\{(B_2, B_1, A_1) : B_2 \subset [n], |B_2| = \frac{n}{2}, B_1 \subset B_2, A_1 \subset [n] \backslash B_2, |A_1| = |B_1|\}|. \tag{13}$$

Let $k = |B_1|$. By Eq. (13), it follows that

$$\begin{aligned} |E| &= \binom{n}{\frac{n}{2}} \sum_{k=0}^{\frac{n}{2}} \binom{\frac{n}{2}}{k} \binom{\frac{n}{2}}{k} \\ &= \binom{n}{\frac{n}{2}} \sum_{k=0}^{\frac{n}{2}} \binom{\frac{n}{2}}{k} \binom{\frac{n}{2}}{\frac{n}{2} - k} && \text{(by symmetry)} \\ &= \binom{n}{\frac{n}{2}}^2 && \text{(by Eq. (12))}. \end{aligned}$$

We obtain

$$|E| = \binom{n}{\frac{n}{2}}^2.$$

By Eq. (11), it follows that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{k} \binom{2n-2k}{n-k} = \binom{n}{\frac{n}{2}}^2,$$

as desired. This completes the proof of Eq. (3). □

4 Proof of Eq. (4)

Proof. The proof of Eq. (4) is similar to the proof of Eq. (3).

Description:

Let X denote the set

$$\{(A, B, C) : A \subset [n], B \subset A \cup A^t, |B| = |A| - 1, C \subset [2n] \backslash (A \cup A^t), |A| + |C| = n\}.$$

For integers k, where $0 \leq k \leq n$, we define the following sets X_k, as follows:

$$X_k = \{(A, B, C) \in X : |A| = k\}.$$

Obviously, $X = \bigcup_{k=0}^{n} X_k$ and $|X_k| = \binom{n}{k}\binom{2k}{k-1}\binom{2n-2k}{n-k}$. We have

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{k-1} \binom{2n-2k}{n-k} = \sum_{k=0}^{n} (-1)^k |X_k| = |\mathcal{E}| - |\mathcal{O}|; \tag{14}$$

where

$$\mathcal{E} = \{(A, B, C) \in X : |A| \text{ is even}\} \text{ and } \mathcal{O} = \{(A, B, C) \in X : |A| \text{ is odd}\}.$$

Involution:

Same as in Eq. (3). Let D, E, and Ψ be same as in Eqns. (6),(7), and (8) respectively. It is readily verified that the function Ψ is well-defined and an involution on D. Hence the equation

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{k-1} \binom{2n-2k}{n-k} = |\mathcal{E} \cap E| - |\mathcal{O} \cap E| \tag{15}$$

holds.

Exception:

Let $(A, B, C) \in E$. By Eq. (7), the set $B \cup C$ is balanced. As before, both sets B and C must be balanced. Thus, integers $|B|$ and $|C|$ are even. Since $|A|$=$|B| + 1$ (by the new definition of X), $|A|$ is odd and $(A, B, C) \in \mathcal{O}$. Therefore, $E \subset \mathcal{O}$. Eq. (15) simplifies to

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{k-1} \binom{2n-2k}{n-k} = -|E|. \tag{16}$$

We have two cases:

Case (a): n is even.

Since $|B\cup C| = n-1$, $|B\cup C|$ is odd. It follows that the set $B\cup C$ is unbalanced and $E=\emptyset$. By Eq. (16), it follows that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{k-1}\binom{2n-2k}{n-k}=0,$$

as desired.

Case (b): n is odd.

Again, we use Eq. (12). Let us count the number of elements of the set E.

The set E is equal to the set

$$\{(A, B_1\cup B_1^t, C_1\cup C_1^t): \\ A\subset[n], |A| \text{ odd}, B_1\subset A, |B_1|=\frac{|A|-1}{2}, C_1\subset[n]\backslash A, |C_1|+|B_1|=\frac{n-1}{2}\}.$$

Obviously, $|E|$ is equal to

$$|\{(A,B_1,C_1): A\subset[n], |A| \text{ odd}, B_1\subset A, |B_1|=\frac{|A|-1}{2}, C_1\subset[n]\backslash A, |C_1|+|B_1|=\frac{n-1}{2}\}|.$$

Clearly, there is one-to-one correspondence between (A,B_1,C_1) and $(B_1\cup C_1, B_1, A\backslash B_1)$.

Therefore, $|E|$ is equal to

$$|\{(B_2,B_1,A_1): B_2\subset[n], |B_2|=\frac{n-1}{2}, B_1\subset B_2, A_1\subset[n]\backslash B_2, |A_1|=|B_1|+1\}|. \tag{17}$$

Let $k=|B_1|$. By Eq. (17), it follows that

$$\begin{aligned}|E| &= \binom{n}{\frac{n-1}{2}}\sum_{k=0}^{\frac{n-1}{2}}\binom{\frac{n-1}{2}}{k}\binom{\frac{n+1}{2}}{k+1}\\ &= \binom{n}{\frac{n-1}{2}}\sum_{k=0}^{\frac{n-1}{2}}\binom{\frac{n-1}{2}}{k}\binom{\frac{n+1}{2}}{\frac{n-1}{2}-k} && \text{(by symmetry)}\\ &= \binom{n}{\frac{n-1}{2}}^2 && \text{(by Eq. (12)).}\end{aligned}$$

Thus we have shown

$$|E|=\binom{n}{\frac{n-1}{2}}^2.$$

By Eq. (16), it follows that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{k-1}\binom{2n-2k}{n-k}=-\binom{n}{\frac{n-1}{2}}^2,$$

as desired. This completes the proof of Eq. (4). □

5 Proof of Theorem 1

Proof. Eq. (1) directly follows from Eqns. (3),(4), and from the relation [4, p. 106] $C_k = \binom{2k}{k} - \binom{2k}{k-1}$.

Let us prove Eq. (2). By Eq. (1), it follows that

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}C_k\binom{4n-2k}{2n-k} = \binom{2n}{n}^2.$$

Changing k to $2n-k$, we obtain that

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}C_k\binom{4n-2k}{2n-k} = \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}C_{2n-k}\binom{2k}{k}.$$

Now we use a lesser known identity on the Catalan numbers:

$$C_k\binom{4n-2k}{2n-k} + \binom{2k}{k}C_{2n-k} = 2(n+1)C_kC_{2n-k}. \tag{18}$$

Eq. (18) is a special case [5, p. 8] of the following identity:

$$C_k\binom{2n-2k}{n-k} + \binom{2k}{k}C_{n-k} = (n+2)C_kC_{n-k}.$$

We have

$$\begin{aligned}
\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}C_k\binom{4n-2k}{2n-k} + \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}C_{2n-k}\binom{2k}{k} &= 2\binom{2n}{n}^2,\\
2(n+1)\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}C_kC_{2n-k} &= 2\binom{2n}{n}^2 \quad \text{(by Eq. (18))}\\
\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}C_kC_{2n-k} &= \frac{1}{n+1}\binom{2n}{n}^2.
\end{aligned}$$

The last equation above proves Eq. (2). This completes the proof of Theorem 1. □

6 Conclusion

Let n, α, β be non-negative integers. By using the same idea from proofs of Eqns. (3) and (4), we can conclude that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{k-\alpha}\binom{2n-2k}{n-k-\beta} = \begin{cases} 0, & \text{if } n \text{ and } \alpha+\beta \text{ are of opposite parity;}\\ (-1)^\alpha\binom{n}{\frac{n-\alpha-\beta}{2}}\binom{n}{\frac{n-\alpha+\beta}{2}}, & \text{otherwise.}\end{cases} \tag{19}$$

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