Two New Identities Involving the Catalan Numbers and Sign-Reversing Involutions

Jovan Mikić
J.U. SŠC “Jovan Cvijić”
74480 Modriča
Republic of Srpska
jnmikic@gmail.com

Abstract

We give a combinatorial proof of a known sum concerning the product of a binomial coefficient with two central binomial coefficients. The method of description, involution, and exception is used. The same combinatorial argument also proves the “−1 shifted version” of this sum. As a consequence, two new binomial coefficient identities with the Catalan numbers are derived.

1 Introduction

Let n be a non-negative integer. The Catalan numbers are the famous sequence

\[ C_n = \frac{1}{n + 1} \binom{2n}{n}. \]

In this paper, we derive two new (to our knowledge) binomial coefficient identities with the Catalan numbers.

Theorem 1. For non-negative integers n, we have

\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} C_k \binom{2n - 2k}{n - k} = \left( \frac{n}{\lfloor \frac{n}{2} \rfloor} \right)^2, \]  \hspace{1cm} (1)

\[ \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} C_k C_{2n-k} = C_n \binom{2n}{n}. \]  \hspace{1cm} (2)
In order to prove Theorem 1, we consider a known combinatorial sum

\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{k} \binom{2n-2k}{n-k} = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ \left(\frac{n}{2}\right)^2, & \text{if } n \text{ is even.} \end{cases} \] (3)

Eq. (3) appears several times in the literature; see, for example, [2, Eq. (6.12), p. 52], [3, Eq. (6.61), p. 29], and [7, Example 3.6.2, p. 45]. There are two binomial coefficient identities ([3, Eq. (6.10), p. 23; Eq. (7.3), p. 34]) similar to Eq. (3). They are proved combinatorially, for example, in [8]. See [6] and [5, Conclusions] for the connection between these identities and Shapiro’s formula [9, Ex. (6.C.18), p. 41] and Segner’s recurrence relation [4, Eq. (5.6), p. 117] respectively.

We give a proof of Eq. (3) by using the method of “description, involution, and exception” [1].

By using the same idea, we derive the “−1 shifted version” of Eq. (3). We assert that

\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{k-1} \binom{2n-2k}{n-k} = \begin{cases} -(\frac{n}{2})^2, & \text{if } n \text{ is odd;} \\ 0, & \text{if } n \text{ is even.} \end{cases} \] (4)

Clearly, subtracting Eq. (4) from Eq. (3) and by using the well-known relation \( C_k = \binom{2k}{k} - \binom{2k}{k-1} \), we get Eq. (1). Furthermore, it can be shown that Eq. (2) is a consequence of Eq. (1).

Throughout the paper, \([n]\) denotes the set \{1, 2, \ldots, n\}, if \(n\) is a positive integer; and \([0]\) denotes the empty set \(\emptyset\).

Let \(A\) and \(B\) be sets. Then \(|A|\) denotes the cardinality of the set \(A\), and \(A \setminus B\) denotes the set difference: \(\{x : x \in A, x \notin B\}\).

We end this paper with the generalization of Eqns. (3) and (4).

2 Definitions

Let \(n\) be a fixed non-negative integer.

Definition 2. For \(A \subset [n]\), we define

\[ A^t = \begin{cases} \emptyset, & \text{if } A = \emptyset; \\ x + n : x \in A, & \text{if } A \neq \emptyset. \end{cases} \]

Obviously, if \(A \subset [n]\), then \(A^t \subset [2n]\setminus[n]\).

Definition 3. We define the function \(\varphi : [2n] \to [2n]\), as follows:

\[ \varphi(x) = \begin{cases} x + n, & \text{if } x \in [n]; \\ x - n, & \text{if } x \in [2n]\setminus[n]. \end{cases} \]
Definition 4. Let \( S \subseteq [2n] \). The set \( S \) is balanced if \( S = (S \cap [n]) \cup (S \cap [n])^t \). Otherwise, \( S \) is an unbalanced set.

In other words, set \( S \) is balanced if \( \forall x (x \in S \Leftrightarrow \varphi(x) \in S) \). Clearly, if set \( S \) is balanced, then \(|S|\) must be even.

The union of two balanced sets is a balanced set. Note that the converse does not hold even if two sets are disjoint. For example, unbalanced sets \([n] \) and \([2n] \setminus [n] \) are disjoint, but their union is a balanced set \([2n] \).

However, if \( S_1 \) and \( S_2 \) are disjoint sets such that \( S_1 \cup \varphi(S_1) \) and \( S_2 \cup \varphi(S_2) \) are disjoint sets, then \( S_1 \cup S_2 \) is a balanced set if and only if both sets \( S_1 \) and \( S_2 \) are balanced.

3 Proof of Eq. (3)

Proof. Let \( n \) be a fixed non-negative integer. We use the method of description, involution, and exception, as discussed in [1].

Description:

Let \( X \) denote the set

\[
\{(A, B, C) : A \subseteq [n], B \subseteq A \cup A^t, |A| = |B|, C \subseteq [2n] \setminus (A \cup A^t), |A| + |C| = n\}.
\]

For integers \( k \), where \( 0 \leq k \leq n \), we define the sets \( X_k \), as follows:

\[
X_k = \{(A, B, C) \in X : |A| = k\}.
\]

Obviously, \( X = \bigcup_{k=0}^{n} X_k \) and \( |X_k| = \binom{n}{k} \binom{2k}{k} \binom{2n-2k}{n-k} \). We have

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{k} \binom{2n-2k}{n-k} = \sum_{k=0}^{n} (-1)^k |X_k| = |\mathcal{E}| - |\mathcal{O}|; \tag{5}
\]

where

\[
\mathcal{E} = \{(A, B, C) \in X : |A| \text{ is even}\} \text{ and } \mathcal{O} = \{(A, B, C) \in X : |A| \text{ is odd}\}.
\]

Involution:

Let us define sets \( D \) and \( E \), as follows:

\[
D = \{(A, B, C) \in X : B \cup C \text{ is an unbalanced set}\} \tag{6}
\]

\[
E = \{(A, B, C) \in X : B \cup C \text{ is balanced set}\} \tag{7}
\]

Obviously, \( D \) and \( E \) are disjoint sets and \( X = D \cup E \).

Let \( (A, B, C) \in D \). We let \( d_{B,C} \) denote \( \min\{x \in B \cup C : \varphi(x) \notin B \cup C\} \). The integer \( d_{B,C} \) is well-defined because \( B \cup C \) is an unbalanced set.
Let us define the function $\Psi : D \to D$, as follows:

$$
\Psi((A, B, C)) = \begin{cases} 
(A\{d_{B,C}\}, B\{d_{B,C}\}, C \cup \{d_{B,C}\}), & \text{if } d_{B,C} \in B \cap [n]; \\
(A\{d_{B,C} - n\}, B\{d_{B,C}\}, C \cup \{d_{B,C}\}), & \text{if } d_{B,C} \in B \cap [n]^t; \\
(A \cup \{d_{B,C}\}, B \cup \{d_{B,C}\}, C \\setminus \{d_{B,C}\}), & \text{if } d_{B,C} \in C \cap [n]; \\
(A \cup \{d_{B,C} - n\}, B \cup \{d_{B,C}\}, C \\setminus \{d_{B,C}\}), & \text{if } d_{B,C} \in C \cap [n]^t.
\end{cases} \tag{8}
$$

The function $\Psi$ is well-defined and an involution on $D$. Moreover, if $(A, B, C) \in D \cap E$, then $\Psi((A, B, C)) \in D \cap O$; and vice versa. Therefore, we may conclude that $|D \cap E| = |D \cap O|$.

We have

$$
|E| - |O| = |E \cap X| - |O \cap X| \\
= |E \cap D| + |E \cap E| - (|O \cap D| + |O \cap E|) \quad (X = D \cup E) \\
= |E \cap E| - |O \cap E| \quad \text{(because } |E \cap D| = |O \cap D|).$$

Therefore, we obtain

$$
|E| - |O| = |E \cap E| - |O \cap E|. \tag{9}
$$

From Eqns. (5) and (9), it follows that

$$
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{k} \binom{2n-2k}{n-k} = |E \cap E| - |O \cap E|. \tag{10}
$$

**Exception:**

Let $(A, B, C) \in E$.

By Eq. (7), the set $B \cup C$ is balanced. Sets $B$ and $C$ are disjoint. Moreover, $B \cup \phi(B)$ and $C \cup \phi(C)$ are disjoint sets too. Then it follows that both sets $B$ and $C$ must be balanced. Hence integers $|B|$ and $|C|$ are even. Since $|A|=|B|$ (by the definition of $X$), $|A|$ is even and $(A, B, C) \in E$. Therefore, $E \subset E$.

Eq. (10) simplifies to

$$
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{k} \binom{2n-2k}{n-k} = |E|. \tag{11}
$$

We have two cases:

**Case (a):** $n$ is odd.

Since $|B \cup C| = n$, it follows that the set $B \cup C$ is unbalanced and $E = \emptyset$. By Eq. (11), it follows that

$$
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{k} \binom{2n-2k}{n-k} = 0,
$$

We have two cases:
as desired.

**Case (b):** \( n \) is even.

We use the Chu-Vandermonde convolution formula:

\[
\sum_{k=0}^{c} \binom{a}{k} \binom{b}{c-k} = \binom{a+b}{c},
\]

(12)

where \( a, b, \) and \( c \) are non-negative integers.

Let us count the number of elements of the set \( E \). The set \( E \) is equal to the set

\[
\{(A, B_1 \cup B_1', C_1 \cup C_1') : A \subset [n], |A| \text{ even}, B_1 \subset A, |B_1| = \frac{|A|}{2}, C_1 \subset [n]\setminus A, |C_1| + |B_1| = \frac{n}{2}\}.
\]

Obviously, it follows that \(|E|\) is equal to

\[
|\{(A, B_1, C_1) : A \subset [n], |A| \text{ even}, B_1 \subset A, |B_1| = \frac{|A|}{2}, C_1 \subset [n]\setminus A, |C_1| + |B_1| = \frac{n}{2}\}|.
\]

Clearly, there is a one-to-one correspondence between \((A, B_1, C_1)\) and \((B_1 \cup C_1, B_1, A \setminus B_1)\). Therefore, \(|E|\) is equal to

\[
|\{ (B_2, B_1, A_1) : B_2 \subset [n], |B_2| = \frac{n}{2}, B_1 \subset B_2, A_1 \subset [n]\setminus B_2, |A_1| = |B_1| \}|.
\]

(13)

Let \( k = |B_1| \). By Eq. (13), it follows that

\[
|E| = \left( \frac{n}{2} \right) \sum_{k=0}^{\frac{n}{2}} \binom{\frac{n}{2}}{k} \binom{\frac{n}{2}}{\frac{n}{2} - k}
\]

\[
= \left( \frac{n}{2} \right) \sum_{k=0}^{\frac{n}{2}} \binom{\frac{n}{2}}{k} \binom{\frac{n}{2}}{\frac{n}{2} - k} \quad \text{(by symmetry)}
\]

\[
= \left( \frac{n}{2} \right)^2 \quad \text{(by Eq. (12))}.
\]

We obtain

\[
|E| = \left( \frac{n}{2} \right)^2.
\]

By Eq. (11), it follows that

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{k} \binom{2n-2k}{n-k} = \left( \frac{n}{2} \right)^2,
\]

as desired. This completes the proof of Eq. (3). \( \square \)
4 Proof of Eq. (4)

Proof. The proof of Eq. (4) is similar to the proof of Eq. (3).

Description:

Let $X$ denote the set

$$\{(A, B, C) : A \subset [n], B \subset A \cup A^c, |B| = |A| - 1, C \subset [2n]\setminus(A \cup A^c), |A| + |C| = n\}.$$

For integers $k$, where $0 \leq k \leq n$, we define the following sets $X_k$, as follows:

$$X_k = \{(A, B, C) \in X : |A| = k\}.$$

Obviously, $X = \bigcup_{k=0}^{n} X_k$ and $|X_k| = \binom{n}{k} \binom{2k}{k-1} \binom{2n-2k}{n-k}$. We have

$$\sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \binom{2k}{k-1} \binom{2n-2k}{n-k} = \sum_{k=0}^{n} (-1)^{k} |X_k| = |E| - |O|; \quad (14)$$

where

$$E = \{(A, B, C) \in X : |A| \text{ is even}\} \quad \text{and} \quad O = \{(A, B, C) \in X : |A| \text{ is odd}\}.$$

Involution:

Same as in Eq. (3). Let $D, E,$ and $\Psi$ be same as in Eqns. (6),(7), and (8) respectively. It is readily verified that the function $\Psi$ is well-defined and an involution on $D$. Hence the equation

$$\sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \binom{2k}{k-1} \binom{2n-2k}{n-k} = |E \cap E| - |O \cap E| \quad (15)$$

holds.

Exception:

Let $(A, B, C) \in E$. By Eq. (7), the set $B \cup C$ is balanced. As before, both sets $B$ and $C$ must be balanced. Thus, integers $|B|$ and $|C|$ are even. Since $|A| = |B| + 1$ (by the new definition of $X$), $|A|$ is odd and $(A, B, C) \in O$. Therefore, $E \subset O$. Eq. (15) simplifies to

$$\sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \binom{2k}{k-1} \binom{2n-2k}{n-k} = -|E|. \quad (16)$$

We have two cases:

Case (a): $n$ is even.
Since $|B \cup C| = n - 1$, $|B \cup C|$ is odd. It follows that the set $B \cup C$ is unbalanced and $E = \emptyset$. By Eq. (16), it follows that
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{k-1} \binom{2n-2k}{n-k} = 0,
\]
as desired.

**Case (b):** $n$ is odd.

Again, we use Eq. (12). Let us count the number of elements of the set $E$.

The set $E$ is equal to the set
\[
\{(A, B_1 \cup B_1', C_1 \cup C_1') : A \subset [n], |A| \text{ odd}, B_1 \subset A, |B_1| = \frac{|A|-1}{2}, C_1 \subset [n] \setminus A, |C_1| + |B_1| = \frac{n-1}{2}\}.
\]

Obviously, $|E|$ is equal to
\[
|\{(A, B_1, C_1) : A \subset [n], |A| \text{ odd}, B_1 \subset A, |B_1| = \frac{|A|-1}{2}, C_1 \subset [n] \setminus A, |C_1| + |B_1| = \frac{n-1}{2}\}|.
\]

Clearly, there is one-to-one correspondence between $(A, B_1, C_1)$ and $(B_1 \cup C_1, B_1, A \setminus B_1)$. Therefore, $|E|$ is equal to
\[
|\{(B_2, B_1, A_1) : B_2 \subset [n], |B_2| = \frac{n-1}{2}, B_1 \subset B_2, A_1 \subset [n \setminus B_2, |A_1| = |B_1| + 1\}|.
\]

Let $k = |B_1|$. By Eq. (17), it follows that
\[
|E| = \left(\frac{n}{n-1}\right)^2 \sum_{k=0}^{n-1} \binom{n-1}{2k} \binom{n}{k+1} \binom{n+1}{2k} \binom{n+1}{k+1} \binom{n+1}{n}.
\]

By symmetry
\[
|E| = \left(\frac{n}{n-1}\right)^2 \sum_{k=0}^{n-1} \binom{n-1}{k} \binom{2n-2k}{n-k} \binom{n}{k-1} \binom{2k}{n-1-k} \binom{2n-2k}{n-k}.
\]

Thus we have shown
\[
|E| = \left(\frac{n}{n-1}\right)^2.
\]

By Eq. (16), it follows that
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{k-1} \binom{2n-2k}{n-k} = -\left(\frac{n}{n-1}\right)^2,
\]
as desired. This completes the proof of Eq. (4). \qed
5 Proof of Theorem 1

Proof. Eq. (1) directly follows from Eqns. (3),(4), and from the relation [4, p. 106] \( C_k = \binom{2k}{k} - \binom{2k}{k-1} \).

Let us prove Eq. (2). By Eq. (1), it follows that

\[
\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} C_k \binom{4n-2k}{2n-k} = \left( \frac{2n}{n} \right)^2.
\]

Changing \( k \) to \( 2n - k \), we obtain that

\[
\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} C_k \binom{4n-2k}{2n-k} = \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} C_{2n-k} \binom{2k}{k}.
\]

Now we use a lesser known identity on the Catalan numbers:

\[
C_k \binom{4n-2k}{2n-k} + \binom{2k}{k} C_{2n-k} = 2(n+1)C_k C_{2n-k}.
\] (18)

Eq. (18) is a special case [5, p. 8] of the following identity:

\[
C_k \binom{2n-2k}{n-k} + \binom{2k}{k} C_{n-k} = (n+2)C_k C_{n-k}.
\]

We have

\[
2(n+1) \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} C_k C_{2n-k} = 2 \left( \frac{2n}{n} \right)^2, \\
\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} C_k C_{2n-k} = \frac{1}{n+1} \left( \frac{2n}{n} \right)^2.
\]

The last equation above proves Eq. (2). This completes the proof of Theorem 1. \( \square \)

6 Conclusion

Let \( n, \alpha, \beta \) be non-negative integers. By using the same idea from proofs of Eqns. (3) and (4), we can conclude that

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \left( \frac{2k}{k-\alpha} \right) \binom{2n-2k}{n-k-\beta} = \begin{cases} 0, & \text{if } n \text{ and } \alpha + \beta \text{ are of opposite parity;} \\ (-1)^\alpha \binom{n}{\frac{n-\alpha-\beta}{2}} \binom{n}{\frac{n+\beta}{2}}, & \text{otherwise.} \end{cases}
\] (19)
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References


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