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A Note on a Permutation Statistic

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Abstract

We study the length of the initial up-down alternating segment of a permutation of $[n]$ selected uniformly at random. It turns out that as n tends to infinity, the expected value and the standard deviation of this statistic converge to small constants.

1 Introduction

Analyzing properties of permutation statistics is a popular subject in combinatorics, cf. Bóna [1]. There has been much recent work on determining and statistically analyzing the length of the longest increasing and alternating subsequences of random permutations, cf. Stanley [4]. A permutation $a_1 a_2 \cdots a_n$ of $[n]$ is up-down alternating if $a_1 < a_2 > a_3 < a_4 > \cdots a_n$. Let A_n denote the number of up-down alternating permutations of $[n] = \{1, 2, \ldots, n\}$. André studied these numbers and established their relations to Euler and tangent numbers, cf. Comtet [3, pp. 258–259]. It is well known that $2A_{n+1} = \sum_{k=0}^{n} \binom{n}{k} A_k A_{n-k}$ for $n \geq 1$ with $A_0 = A_1 = 1$. The first few terms of this sequence are

$$1, 1, 1, 2, 5, 16, 61, 272, 1385, 7936, 50521, 353792, 2702765, \ldots,$$

cf. A000111 in the *On-Line Encyclopedia of Integer Sequences*. For their exponential generating function we have the remarkable exponential generating function identity

$$A(x) = \sum_{k=0}^{\infty} \frac{A_k}{k!} x^k = \sec(x) + \tan(x), \tag{1}$$

cf. Comtet [3]. The convergence radius easily follows as well as

$$A_n/n! \sim 2\left(\frac{2}{\pi}\right)^{n+1} \tag{2}$$

as $n \to \infty$, cf. Borwein et al. [2].

We are interested in the *initial* up-down alternating segments of the permutations. Let X_n denote the length of the initial up-down alternating sequence of a permutation of $[n]$ selected uniformly at random, i.e., we set $X_n = n$ for an up-down alternating permutation, and otherwise, $X_n = 2k$ or $X_n = 2k+1$ if $a_1 < a_2 > a_3 < \cdots > a_{2k-1} < a_{2k} < a_{2k+1}$ or $a_1 < a_2 > a_3 < \cdots < a_{2k} > a_{2k+1} > a_{2k+2}$, respectively. Clearly, $1 \le X_n \le n$ and $X_n = n$ exactly if the permutation is up-down alternating. The random variable X_n has some surprising properties. We will see in Theorems 1 and 3 that the expected value and standard deviation of X_n depends only very slightly on n and the limits of these moments are small constants. Theorem 4 shows that for a large n the probability $P(X_n = k)$ decreases exponentially as k grows while Corollary 2 confirms that $P(X_n = k)$ becomes constant for $n > k$.

2 Main results

Theorem 1. *For $1 \le k \le n-1$ we have that*

$$P(X_n = k) = P(X_n \ge k) - P(X_n \ge k+1) = \frac{A_k}{k!} - \frac{A_{k+1}}{(k+1)!}, \tag{3}$$

$$EX_n = \sum_{k=1}^{n} \frac{A_k}{k!},$$

and $\lim_{n\to\infty} EX_n = \sec(1) + \tan(1) - 1 \approx 2.40822$.

Proof. We use the fact that for any positive integer valued random variable X_n

$$EX_n = \sum_{k=1}^{\infty} P(X_n \ge k). \tag{4}$$

We observe that $P(X_n = n) = A_n/n!$. On the other hand, determining $P(X_n = k)$ for other values of k seems complicated, however, as it turns out, calculating $P(X_n \ge k)$ for $1 \le k \le n-1$ is fairly simple:

$$P(X_n \ge k) = \frac{\binom{n}{k} A_k (n-k)!}{n!} = \frac{A_k}{k!}, \tag{5}$$

since we can pick k elements for the initial segment in $\binom{n}{k}$ ways and place them in up-down alternating order in A_k ways, while the other $n-k$ elements can be put in arbitrary order. Identity (5) also holds for $k = n$. Now (5) and (4) imply identities (3) and

$$EX_n = \sum_{k=1}^{n} \frac{A_k}{k!}.$$

The limit follows by the exponential generating function (1). □

Theorem 1 implies the somewhat surprising

Corollary 2. *The probability $P(X_n = k)$ does not depend on n as long as $n > k$.*

The following theorem is straightforward.

Theorem 3. *We set $B(x) = (A(x) - 1)/x$. For the standard deviation of X_n we have*

$$\sigma(X_n) = \left(\sum_{k=1}^{n-1} k^2 \left(\frac{A_k}{k!} - \frac{A_{k+1}}{(k+1)!} \right) + n^2 \frac{A_n}{n!} - (EX_n)^2 \right)^{1/2},$$

and

$$\begin{aligned}\lim_{n\to\infty} \sigma(X_n) &= \left((A''(x) + A'(x)) - (B''(x) + B'(x)) \mid_{x=1} -(EX_n)^2 \right)^{1/2} \\ &\approx 2.09958.\end{aligned}$$

The next theorem quantifies $P(X_n = k)$ for any $k \geq 4$ and sufficiently large n.

Theorem 4. *For a large n and $k < n$, the probability $P(X_n = k)$ decreases exponentially as k grows. More precisely, for $k \geq 4$ and a large enough n, we have that*

$$P(X_n = k) \approx 2 \left(\frac{2}{\pi} \right)^{k+1} \left(1 - \frac{2}{\pi} \right).$$

Proof. By the asymptotic result (2), the exact distribution (3) and some numerical calculation show that the absolute difference between $P(X_n = k)$ and its approximated value is less than 10^{-3} for $k \geq 4$. □

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(Concerned with sequence A000111.)

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