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Riordan Pseudo-Involutions, Continued Fractions and Somos-4 Sequences

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Abstract

We define a three-parameter family of Bell pseudo-involutions in the Riordan group. The defining sequences have generating functions that are expressible as continued fractions. We indicate that the Hankel transforms of the defining sequences, and of the A-sequences of the corresponding Riordan arrays, can be associated with a Somos-4 sequence. We give examples where these sequences can be associated with elliptic curves, and we exhibit instances where elliptic curves can give rise to associated Riordan pseudo-involutions. In the case of a particular one-parameter family of elliptic curves, we show how we can associate a unique Bell pseudo-involution with each such curve.

1 Introduction

The area of Riordan (pseudo) involutions has been the subject of much research in recent years [4, 6, 11, 7, 8, 9, 10]. Recently, a sequence characterization of these involutions has emerged [6]. This sequence is called the Δ-sequence or the B-sequence. In this paper, we study a three-parameter family of Riordan pseudo-involutions defined by a simply described B-sequence. We show that these pseudo-involutions are linked to Catalan defined generating functions, and are linked to Somos sequences and elliptic curves via the Hankel transforms of these generating functions. We show by example that it is possible to start with an appropriate elliptic curve and to derive from its equation an associated Riordan pseudo-involution.

The group of (ordinary) Riordan arrays [1, 13, 14] is the set of lower triangular invertible matrices $(g(x), f(x))$ defined by two power series

$$g(x) = 1 + g_1 x + g_2 x^2 + \cdots ,$$

and

$$f(x) = f_1 x + f_2 x^2 + \cdots ,$$

such that the (n, k)-th element $t_{n,k}$ of the matrix is given by

$$t_{n,k} = [x^n]g(x)f(x)^k,$$

where $[x^n]$ is the functional that extracts the coefficient of x^n from a power series.

The Fundamental Theorem of Riordan Arrays (FTRA) says that we have the law

$$(g(x), f(x)) \cdot h(x) = g(x)h(f(x)).$$

This is equivalent to the matrix represented by $(g(x), f(x))$ operating on the column vector whose elements are the expansion of the generating function $h(x)$. The resulting vector, regarded as a sequence, will have generating function $g(x)h(f(x))$.

The product of two Riordan arrays $(g(x), f(x))$ and $(u(x), v(x))$ is defined by

$$(g(x), f(x)) \cdot (u(x), v(x)) = (g(x)u(f(x)), v(f(x))).$$

The inverse of the Riordan array $(g(x), f(x))$ is given by

$$(g(x), f(x))^{-1} = \left(\frac{1}{g(\bar{f}(x))}, \bar{f}(x) \right),$$

where $\bar{f}(x) = \text{Rev}(f)(x)$ is the compositional inverse of $f(x)$. Thus $\bar{f}(x)$ is the solution $u(x)$ of the equation

$$f(u) = x$$

with $u(0) = 0$.

The identity element is given by $(1, x)$ which as a matrix is the usual identity matrix.

With these operations the set of Riordan arrays form a group, called the Riordan group.

The Bell subgroup of Riordan arrays consists of those lower triangular invertible matrices defined by a power series

$$g(x) = 1 + g_1 x + g_2 x^2 + \cdots ,$$

where the (n, k)-th element $t_{n,k}$ of the matrix is given by

$$t_{n,k} = [x^n]g(x)(xg(x))^k = [x^{n-k}]g(x)^{k+1}.$$

A Bell pseudo-involution is a Bell array $(g(x), xg(x))$ such that the square of the Riordan array $(g(x), -xg(x))$ is the identity matrix. We shall call a generating function $g(x)$ for which this is true an *involutory* generating function.

For a lower triangular invertible matrix A, the matrix $P_A = A^{-1}\bar{A}$ is called the production matrix of A, where $\bar{A}$ is the matrix A with its first row removed. A matrix A is a Riordan array if and only if P_A takes the form

$$\begin{pmatrix} z_0 & a_0 & 0 & 0 & 0 & 0 \\ z_1 & a_1 & a_0 & 0 & 0 & 0 \\ z_2 & a_2 & a_1 & a_0 & 0 & 0 \\ z_3 & a_3 & a_2 & a_1 & a_0 & 0 \\ z_4 & a_4 & a_3 & a_2 & a_1 & a_0 \\ z_5 & a_5 & a_4 & a_3 & a_2 & a_1 \end{pmatrix}.$$

The sequence that begins

$$z_0, z_1, z_2, z_3, \ldots$$

is called the Z-sequence, while the sequence

$$a_0, a_1, a_2, a_3, \ldots$$

is called the A-sequence. For a Riordan array $(g(x), f(x))$, we have

$$A(x) = \frac{x}{\bar{f}(x)} \quad \text{and} \quad Z(x) = \frac{1}{\bar{f}(x)}\left(1 - \frac{1}{g(\bar{f}(x))}\right),$$

where $A(x)$ is the power series $a_0 + a_1 x + a_2 x^2 + \cdots$, and $Z(x)$ is the power series $z_0 + z_1 x + z_2 x + \cdots$.

For a Riordan array $(g(x), f(x))$ to be an element of the Bell subgroup it is necessary and sufficient that

$$A(x) = 1 + xZ(x).$$

If $(g(x), xg(x))$ is a pseudo-involution, then we have that [11]

$$A(x) = \frac{1}{g(-x)}.$$

We have the following result [6].

Proposition 1. *An element $(g(x), xg(x))$ of the Bell subgroup is a pseudo-involution if and only if there exists a sequence*

$$b_0, b_1, b_2, \ldots$$

such that

$$t_{n+1,k} = t_{n,k-1} + \sum_{j \geq 0} b_j \cdot t_{n-j,k+j},$$

where $t_{n,k-1} = 0$ if $k = 0$.

This sequence, when it exists, is called the B-sequence, or the Δ-sequence, of the Riordan pseudo-involution. The relationship between $A(x)$ and $B(x)$, when this latter exists, is given by [11]

$$A(x) = 1 + xB\left(\frac{x^2}{A(x)}\right).$$

Sequences and triangles, where known, will be referenced by their *Annnnnn* number in the On-Line Encyclopedia of Integer Sequences [15, 16]. All number triangles in this note are infinite in extent; where shown, a suitable truncation is used.

2 A Bell pseudo-involution defined by continued fractions

In this section, we consider the B-sequence with generating function given by

$$B(x) = \frac{a - cx}{1 + bx}.$$

Proposition 2. *For the Bell pseudo-involution* $(g(x), xg(x))$ *with*

$$B(x) = \frac{a - cx}{1 + bx},$$

we have

$$A(x) = 1 + ax - \frac{x^3(ab+c)}{1 + ax + bx^2}\mathfrak{C}\left(\frac{x^3(ab+c)}{(1+ax+bx^2)^2}\right),$$

and

$$g(x) = \frac{1}{1 - ax - bx^2}\mathfrak{C}\left(\frac{-x^2(b+cx)}{(1-ax-bx^2)^2}\right),$$

where

$$\mathfrak{C}(x) = \frac{1 - \sqrt{1-4x}}{2x}$$

is the generating function of the Catalan numbers A000108 $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Proof. In order to solve for $A(x)$, we must solve the equation

$$u = 1 + x\frac{a - cx^2/u}{bx^2/u + 1}$$

for $u(x)$. We find that the appropriate branch is given by

$$u(x) = A(x) = \frac{1 + ax - bx^2 + \sqrt{1 + 2ax + (a^2+2b)x^2 - 2(ab+2c)x^3 + bx^4}}{2}.$$

Using $\mathfrak{C}(x) = \frac{1-\sqrt{1-4x}}{2x}$ we can put this in the form

$$A(x) = 1 + ax - \frac{x^3(ab+c)}{1+ax+bx^2}\mathfrak{C}\left(\frac{x^3(ab+c)}{(1+ax+bx^2)^2}\right).$$

Now we have

$$g(x) = \frac{1}{A(-x)}.$$

We find that

$$g(x) = \frac{2}{1-ax-bx^2+\sqrt{1-2ax+(a^2+2b)x^2+2(ab+2c)x^3+bx^4}},$$

or

$$g(x) = \frac{-1+ax+bx^2+\sqrt{1-2ax+(a^2+2b)x^2+2(ab+2c)x^3+b^2x^4}}{2x^2(cx+b)}.$$

This last expression can then be expressed as

$$g(x) = \frac{1}{1-ax-bx^2}\mathfrak{C}\left(\frac{-x^2(b+cx)}{(1-ax-bx^2)^2}\right).$$

$\square$

We now recall that $\mathfrak{C}(x)$ can be expressed as the continued fraction [3, 17]

$$\mathfrak{C}(x) = \cfrac{1}{1-\cfrac{x}{1-\cfrac{x}{1-\cdots}}}.$$

Thus we have

Corollary 3. *For the pseudo-involution $(g(x), xg(x))$ with B-sequence given by*

$$B(x) = \frac{a-cx}{1+bx},$$

we have that $g(x)$ can be expressed as the continued fraction

$$g(x) = \cfrac{1}{1-ax-bx^2+\cfrac{x^2(b+cx)}{1-ax-bx^2+\cfrac{x^2(b+cx)}{1-ax-bx^2+\cdots}}}.$$

Corollary 4. *For the pseudo-involution $(g(x), xg(x))$ with B-sequence given by*

$$B(x) = a + dx,$$

we have that $g(x)$ can be expressed as the continued fraction

$$g(x) = \cfrac{1}{1 - ax - \cfrac{dx^3}{1 - ax - \cfrac{dx^3}{1 - ax - \cdots}}}. \tag{1}$$

Corollary 5. *For the pseudo-involution $(g(x), xg(x))$ with B-sequence given by*

$$B(x) = \frac{a}{1 - bx},$$

we have that $g(x)$ can be expressed as the continued fraction

$$g(x) = \cfrac{1}{1 - ax + bx^2 - \cfrac{bx^2}{1 - ax + bx^2 - \cfrac{bx^2}{1 - ax + bx^2 - \cdots}}}.$$

3 Examples

In this section, we examine some examples of the above sequences g_n, where $g(x) = \sum_{n=0}^{\infty} g_n x^n$ is such that $(g(x), xg(x))$ is a pseudo-involution. Thus we have $(g(x), -xg(x))^2 = (1, x)$.

Example 6. We recall that for the pseudo-involution $(g(x), xg(x))$ with B-sequence given by

$$B(x) = a + dx,$$

we have that $g(x)$ can be expressed as the continued fraction of the form given in Eq (1). Thus by Proposition 2, we have

$$g(x) = \frac{1}{1 - ax} \mathfrak{C}\left(\frac{dx^3}{(1 - ax)^2}\right).$$

Using the FTRA, this can be written as

$$g(x) = \left(\frac{1}{1 - ax}, \frac{dx^3}{(1 - ax)^2}\right) \cdot \mathfrak{C}(x).$$

We can determine the (n,k)-th term $t_{n,k}$ of the Riordan array $\left(\frac{1}{1-ax}, \frac{dx^3}{(1-ax)^2}\right)$ as follows.

$$\begin{aligned}
t_{n,k} &= [x^n]\frac{d^k x^{3k}}{(1-ax)^{2k+1}}\\
&= d^k[x^{n-3k}]\sum_{j=0}^{\infty}\binom{-(2k+1)}{j}(-a)^j\\
&= d^k[x^{n-3k}]\sum_{j=0}^{\infty}\binom{2k+1+j-1}{j}a^j\\
&= d^k\binom{2k+n-3k}{n-3}a^{n-3k}\\
&= \binom{n-k}{n-3}d^k a^{n-3k}.
\end{aligned}$$

We then have

$$\begin{aligned}
g_n &= \sum_{k=0}^{\lfloor \frac{n}{3}\rfloor}\binom{n-k}{n-3k}d^k a^{n-3k}C_k\\
&= \sum_{k=0}^{n}\frac{1}{n-k+3}\binom{\frac{2n+k}{3}}{\frac{n+2k}{3}}\binom{\frac{n+2k}{3}}{k}\left(2\cos\left(\frac{2(n-k)\pi}{3}\right)+1\right)a^k d^{(n-k)/3}.
\end{aligned}$$

For instance, we have

$$\begin{pmatrix} g_0\\ g_1\\ g_2\\ g_3\\ g_4\\ g_5\\ g_6\\ g_7 \end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
d & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 3d & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 6d & 0 & 0 & 1 & 0 & 0\\
2d^2 & 0 & 0 & 10d & 0 & 0 & 1 & 0\\
0 & 10d^2 & 0 & 0 & 15d & 0 & 0 & 1
\end{pmatrix}\begin{pmatrix} 1\\ a\\ a^2\\ a^3\\ a^4\\ a^5\\ a^6\\ a^7 \end{pmatrix}.$$

The above matrix (for $d=1$ is an aeration of A060693, which counts the number of Schröder paths from $(0,0)$ to $(2n,0)$ having k peaks.

We note that the Hankel transform $h_n = |g_{i+j}|_{0\le i,j\le n}$ begins

$$1, 0, -d^2, -d^4, 0, d^{10}, d^{14}, 0, -d^{24}, -d^{30}, 0, \ldots.$$

For $a=d=1$, we get the sequence that begins

$$1, 1, 1, 2, 4, 7, 13, 26, 52, 104, 212, \ldots$$

This is A023431, which counts Motzkin paths of length n with no UD's and no UU's. For $a=1, d=2$ we get the sequence A091565, that begins

$$1, 1, 1, 3, 7, 13, 29, 71, 163, 377, 913, \ldots.$$

For $a=2$, $d=1$, we get the sequence A091561 that begins

$$1, 2, 4, 9, 22, 56, 146, 388, 1048, 2869, 7942, \ldots.$$

The related sequence A152225 that begins

$$1, 1, 2, 4, 9, 22, 56, 146, 388, 1048, 2869, 7942, \ldots$$

counts the number of Dyck paths of semi-length n with no peaks at height 0 (mod 3) and no valleys at height 2 (mod 3).

Example 7. When $c=0$ we obtain that for the pseudo-involution $(g(x), xg(x))$ with B-sequence given by

$$B(x) = \frac{a}{1+bx},$$

we have that $g(x)$ can be expressed as the continued fraction

$$g(x) = \cfrac{1}{1-ax-bx^2+\cfrac{bx^2}{1-ax-bx^2+\cfrac{bx^2}{1-ax-bx^2+\cdots}}}.$$

In this case we have

$$g(x) = \left(\frac{1}{1-ax-bx^2}, \frac{-bx^2}{(1-ax-bx)^2}\right) \cdot \mathfrak{C}(x).$$

We find that

$$\begin{aligned} g_n &= \sum_{k=0}^{n} (\sum_{j=0}^{n-2k} \binom{2k+j}{j}\binom{j}{n-2k-j} b^{n-2k-j} a^{2j+2k-n})(-b)^k C_k \\ &= \sum_{k=0}^{n} \sum_{j=0}^{n-2k} \binom{2k+j}{j}\binom{j}{n-2k-j} b^{n-k-j} a^{2j+2k-n} (-1)^k C_k \\ &= \sum_{k=0}^{n} (\sum_{i=0}^{n-2k} \binom{n-i}{2k}\binom{n-2k-i}{i} b^i a^{n-2k-2i})(-b)^k C_k. \end{aligned}$$

We have the following characterization of these sequences [2, 5].

Proposition 8. *For the pseudo-involution $(g(x), xg(x))$ with B-sequence given by*

$$B(x) = \frac{a}{1+bx},$$

we have that g_n satisfies the recurrence

$$g_n = \begin{cases} 1, & \text{if } n = 0; \\ a, & \text{if } n = 1; \\ ag_{n-1} + bg_{n-2} - b\sum_{k=0}^{n-2} g_k g_{n-2-k}, & \text{if } n > 1. \end{cases}$$

The Hankel transform of g_n begins

$$1, 0, -a^2b^2, -a^4b^4, a^6b^7, 0, -a^{12}b^{15}, a^{16}b^{20}, a^{20}b^{26}, 0, -a^{30}b^{40}, \ldots.$$

For $a = 1$, $b = -1$ we get the RNA sequence A004148 that begins

$$1, 1, 1, 2, 4, 8, 17, 37, 82, \ldots.$$

For $a = 2$, $b = -1$ we get the sequence A187256, which begins

$$1, 2, 4, 10, 28, 82, 248, 770, 2440, \ldots.$$

This counts the number of peakless Motzkin paths of length n, assuming that the $(1, 0)$ steps come in 2 colors (Emeric Deutsch).

Example 9. In the general case, we have that

$$g(x) = \frac{1}{1-ax-bx^2}\mathfrak{c}\left(\frac{-x^2(b+cx)}{(1-ax-bx^2)^2}\right).$$

One expansion of this gives us

$$g_n = \sum_{k=0}^{n}(\sum_{i=0}^{n}\binom{k}{i}c^i b^{k-i}\sum_{m=0}^{n-2k-i}\binom{n-i-m}{n-2k-i-m}\binom{n-2k-i-m}{m}b^m a^{n-2k-2m-i})(-1)^k C_k.$$

For $a = 2$, $b = -1$ and $c = 1$, we obtain the sequence $g_n(2, -1, 1)$ A105633 that begins

$$1, 2, 4, 9, 22, 57, 154, 429, 1223, 3550, 10455, \ldots.$$

This sequence counts the number of Dyck paths of semi-length $n + 1$ avoiding $UUDU$ [12]. We note that the related sequence (we prepend a 1 to the previous sequence) that begins

$$1, 1, 2, 4, 9, 22, 57, 154, 429, 1223, 3550, 10455, \ldots$$

has generating function given by

$$\cfrac{1}{1-x-\cfrac{x^2}{1-\cfrac{x}{1-x-\cfrac{x^2}{1-\cfrac{x}{1-x-\cdots}}}}}.$$

The general term of this sequence is given by

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{n-k} \binom{n-k}{j} N_{j,k}$$

where $(N_{n,k})$ is the Narayana triangle A090181 with

$$N_{n,k} = \frac{1}{n-k+1}\binom{n}{k}\binom{n-1}{n-k},$$

which begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 6 & 6 & 1 & 0 & 0 \\ 0 & 1 & 10 & 20 & 10 & 1 & 0 \\ 0 & 1 & 15 & 50 & 50 & 15 & 1 \end{pmatrix}.$$

Equivalently, this sequence is equal to the diagonal sums of the matrix product

$$\left(\binom{n}{k}\right) \cdot (N_{n,k}),$$

where this product matrix A130749 begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 7 & 6 & 1 & 0 & 0 & 0 \\ 1 & 15 & 24 & 10 & 1 & 0 & 0 \\ 1 & 31 & 80 & 60 & 15 & 1 & 0 \\ 1 & 63 & 240 & 280 & 125 & 21 & 1 \end{pmatrix}.$$

The inverse binomial transform of $g_n(2, -1, 1)$ is the sequence A007477 that begins

$$1, 1, 1, 2, 3, 6, 11, 22, 44, 90, 187, 392, \ldots.$$

This sequence counts the number of Dyck $(n+1)$-paths containing no UDUs and no subpaths of the form $UUPDD$ where P is a nonempty Dyck path (observation by David Callan, [15]).

4 Hankel transforms and Somos-4 sequences

The sequence $g_n(a,b,c)$ begins

$$1, a, a^2, a^3-ab-c, a^4-3a^2b-3ac, a^5-6a^3b-6a^2c+ab^2+bc, \ldots.$$

The Hankel transform of $g_n(a,b,c)$ begins

$$1, 0, -a^2b^2-2abc-c^2, -a^4b^4-4a^3b^3c-6a^2b^2c^2-4abc^3-c^4, \ldots.$$

Proceeding numerically, we can conjecture that the sequence t_n that begins

$$-a^2b^2-2abc-c^2, -a^4b^4-4a^3b^3c-6a^2b^2c^2-4abc^3-c^4, \ldots$$

is a $((ab+c)^2, b(ab+c)^2)$ Somos-4 sequence. By this we mean that

$$t_n = \frac{(ab+c)^2 t_{n-1}t_{n-3} + b(ab+c)^2 t_{n-2}^2}{t_{n-4}}.$$

Example 10. We take $a=2$, $b=-2$ and $c=3$. We have

$$g(x) = \frac{1}{1-2x+2x^2}\mathfrak{C}\left(\frac{x^2(2-3x)}{(1-2x+2x^2)^2}\right) = \frac{1-2x+2x^2-\sqrt{1-4x+4x^3+4x^4}}{2x^2(2-3x)}.$$

This expands to give the sequence $g_n(2,-2,3)$ that begins

$$1, 2, 4, 9, 22, 58, 162, 472, 1418, 4357, 13618, \ldots.$$

The Hankel transform of $g_n(3,-2,3)$ begins

$$1, 0, -1, -1, -2, -3, 5, 28, 67, 411, 506, \ldots.$$

Now the sequence

$$-1, -1, -2, -3, 5, 28, 67, 411, 506, \ldots$$

is a $(1,-2)$ Somos-4 sequence, associated with the elliptic curve $y^2+y=x^3+3x^2+x$.

Example 11. The sequence $g_n(-1,2,1)$ has generating function

$$g(x) = \frac{1}{1+x-2x^2}\mathfrak{C}\left(\frac{-x^2(2+x)}{(1+x-2x^2)^2}\right) = \frac{\sqrt{1+2x+5x^2+4x^4}-2x^2+1}{2x^2(x+2)}.$$

The sequence $g_n(-1,2,1)$ begins

$$1, -1, 1, 0, -2, 3, 1, -12, 20, 4, -84, \ldots.$$

It has a Hankel transform that begins

$$1, 0, -1, -1, 2, -1, -9, 16, 73, 145, -1442, \ldots.$$

Here, the sequence $-1, -1, 2, -1, -9, 16, 73, 145, -1442, \ldots$ is a $(1, 2)$ Somos-4 sequence. It is related to A178075, which is the $(1, 2)$ Somos-4 sequence that begins

$$1, 1, -2, 1, 9, -16, -73, -145, 1442, \ldots.$$

Example 12. The sequence $g_n(-1,-2,-1)$ has generating function

$$g(x) = \frac{1}{1+x+2x^2}\mathfrak{C}\left(\frac{x^2(2+x)}{(1+x+2x^2)^2}\right) = \frac{1+x+2x^2-\sqrt{1+2x-3x^2+4x^4}}{2x^2(x+2)}.$$

The sequence begins

$$1, -1, 1, -2, 4, -9, 21, -50, 122, -302, 758, \ldots$$

and its Hankel transform begins

$$1, 0, -1, -1, -2, -1, 7, 16, 57, 113, -670, \ldots.$$

The sequence

$$-1, -1, -2, -1, 7, 16, 57, 113, -670, \ldots$$

is a $(1, -2)$ Somos-4 sequence. Apart from signs, this is A178622, which is associated with the elliptic curve $y^2 - 3xy - y = x^3 - x$. In fact, we can show that the $(1, -2)$ Somos-4 sequence $1, 1, 2, 1, -7, -16, -57, \ldots$ can be described as the Hankel transform of the sequence that begins

$$1, 0, 1, -1, 4, -10, 30, -84, 237, -653, 1771, -4699, 12173, -30625, \ldots$$

with generating function

$$f(x) = \frac{2x}{\sqrt{1+6x+9x^2-4x^3-8x^4}-x-1}.$$

Note that many other sequences can have the same Hankel transform.

The relationship between $g(x)$ and $f(x)$ is given by

$$g(x) = \left(\frac{1}{1+2x}, \frac{-x}{1+2x}\right) \cdot \frac{f(x)(1+2x)-1}{f(x)x(3x+2)}.$$

5 From elliptic curve to Riordan pseudo-involution

In this section, we reprise the last example to make explicit the steps that lead from the elliptic curve given by

$$y^2 - 3xy - y = x^3 - x$$

to the Riordan pseudo-involution defined by $g_n(-1,-2,-1)$.

The first step is to solve the quadratic equation

$$y^2 - 3xy - y = x^3 - x$$

for y. The branch that we require is given by

$$\frac{1+3x-\sqrt{1+2x+9x^2+4x^3}}{2}.$$

This expands to give a sequence that begins

$$0, 1, -2, 1, 3, -7, \ldots.$$

Note that the other branch expands to give the sequence

$$1, 2, 2, -1, -3, 7, \ldots.$$

Apart from the signs, the two sequences agree after the first two terms. We must discard these first two terms, to get the generating function

$$\left(\frac{1+3x-\sqrt{1+2x+9x^2+4x^3}}{2} - x\right)/x^2 = \frac{1+x-\sqrt{1+2x+9x^2+4x^3}}{2x^2}.$$

We now form the fraction

$$\frac{1}{1-x-x^2\left(\frac{1+x-\sqrt{1+2x+9x^2+4x^3}}{2x^2}\right)} = \frac{2}{1-3x+\sqrt{1+2x+9x^2+4x^3}}.$$

We revert this generating function to obtain the generating function

$$\frac{1+3x-\sqrt{1+6x+9x^2-4x^3-8x^4}}{2x^2}.$$

Finally, we form the generating function

$$\frac{1}{1-x^2\left(\frac{1+3x-\sqrt{1+6x+9x^2-4x^3-8x^4}}{2x^3}\right)}$$

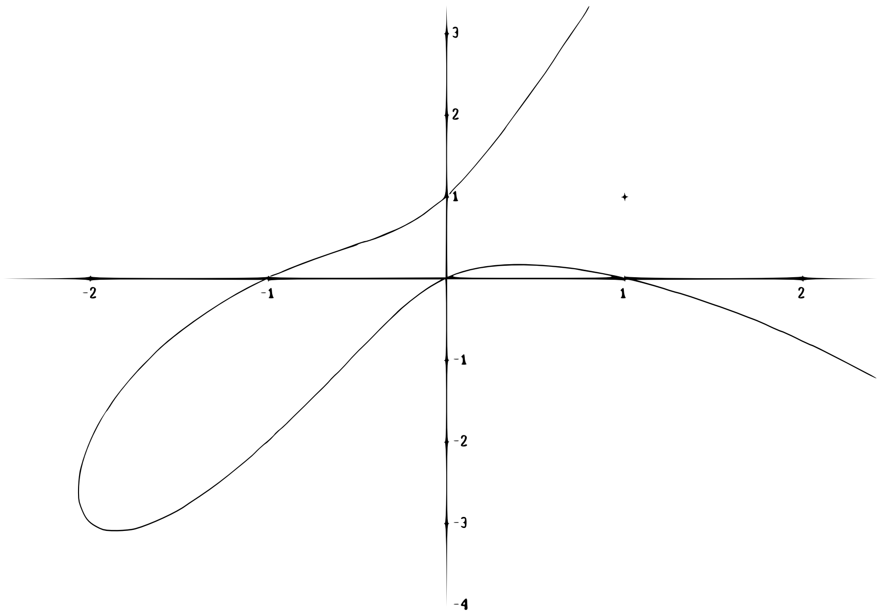

Figure 1: The elliptic curve $y^2 - 3xy - y = x^3 - x$

to arrive at

$$f(x) = \frac{2x}{\sqrt{1+6x+9x^2-4x^3-8x^4} - x - 1}.$$

The sought after involutory generating function is now obtained by

$$g(x) = \left(\frac{1}{1+2x}, \frac{-x}{1+2x}\right) \cdot \frac{f(x)(1+2x)-1}{f(x)x(3x+2)}.$$

Example 13. Inspired by the last section, we now start with the elliptic curve

$$y^2 - 2xy - y = x^3 - x$$

and seek to produce an involutary power series $g(x)$. We begin as before by solving the quadratic in y to get

$$\frac{1+2x-\sqrt{1+4x^2+4x^3}}{2}$$

which expands to give a sequence that begins $0, 1, -1, -1, 1, \ldots$. We now form

$$\left(\frac{1+2x-\sqrt{1+4x^2+4x^3}}{2} - x\right)/x^2 = \frac{1-\sqrt{1+4x^2+4x^3}}{2x^2}.$$

We proceed to form the fraction

$$\frac{1}{1-x-x^2\left(\frac{1-\sqrt{1+4x^2+4x^3}}{2x^2}\right)} = \frac{2}{1-2x+\sqrt{1+4x^2+4x^3}}.$$

We revert this last generating function to obtain the generating function

$$\frac{1+2x-\sqrt{1+4x+4x^2-4x^3-4x^4}}{2x^2}.$$

We finally form the generating function

$$\frac{1}{1-x^2\left(\frac{1+2x-\sqrt{1+4x+4x^2-4x^3-4x^4}}{2x^3}\right)}$$

to get

$$f(x)=\frac{1+\sqrt{1+4x+4x^2-4x^3-4x^4}}{1+x-x^2-x^3}.$$

We now let

$$g(x)=\left(\frac{1}{1+x},\frac{-x}{1+x}\right)\cdot\frac{f(x)(1+x)-1}{f(x)x(2x+1)}.$$

Thus we arrive at

$$g(x)=\frac{1+x^2-\sqrt{1-2x^2+4x^3+x^4}}{2x^2(1-x)}=\frac{1}{1+x^2}\mathfrak{C}\left(\frac{x^2(1-x)}{(1+x^2)^2}\right).$$

This means that the sequence found is the involutory sequence $g_n(0,-1,1)$.

This sequence begins

$$1,0,0,-1,0,-1,2,-1,5,-6,9,-22,28,-57,104,-163,\ldots$$

and its Hankel transform begins

$$1,0,-1,-1,-1,1,2,3,1,-7,-11,\ldots.$$

The sequence

$$1,1,1,-1,-2,-3,-1,7,11,\ldots$$

is the $(1,-1)$ Somos-4 sequence A050512 which is associated with the curve

$$E\,:\,y^2-2xy-y=x^3-x.$$

The association comes about in the following way. We take coordinates of the integer multiples of the point $P=(0,0)$ on E. We use the x-coordinates as the coefficients of x^2 and the ratio of the y and x-coordinates as the coefficients of x in the following continued fraction.

$$\cfrac{1}{1-x-\cfrac{x^2}{1+x-\cfrac{x^2}{1+\cfrac{x^2}{1+3x+\cfrac{2x^2}{1-\frac{x}{2}-\cfrac{(3/4)x^2}{1-\frac{7x}{6}-\cfrac{(2/9)x^2}{1+\cdots}}}}}}},$$

where
$$(0,0),(-1,0),(1,3),(2,-1),(-3/4,-7/8),(-2/9,22/27),\ldots$$
are the coordinates of the multiples of $P=(0,0)$ on the elliptic curve E:$y^2-2xy-y=x^3-x$.

$(nP)_x$	0	-1	2	$-\frac{3}{4}$	$-\frac{2}{9}$	21	$\frac{11}{49}$
$(nP)_y$	0	0	3	-1	$\frac{-7}{8}$	$\frac{22}{27}$	120
$\frac{y}{x}$	1	0	3	$\frac{-1}{2}$	$\frac{7}{6}$	$\frac{-11}{3}$	$\frac{7}{40}$

This generating function expands to give the sequence starting
$$1,1,2,2,5,4,12,10,23,38,17,162,-86,\ldots$$
whose Hankel transform is
$$1,1,1,-1,-2,-3,-1,\ldots.$$
The inverse binomial transform of g_n begins
$$1,-1,1,-2,5,-12,29,-72,182,-466,1207\ldots.$$
This is essentially A025273 or A217333. The sequence g_n is the partial sum sequence of the sequence that begins
$$1,-1,0,-1,1,-1,3,-3,6,-11,15,-31,\ldots.$$
This is the alternating sign version of A025250, whose binomial transform is essentially A025273.

6 The A-sequence and Somos-4 sequences

We recall that for a Bell pseudo-involution $(g(x),xg(x))$ for which
$$B(x)=\frac{a-cx}{1+bx},$$
we have
$$A(x)=1+ax-\frac{x^3(ab+c)}{1+ax+bx^2}\mathfrak{C}\left(\frac{x^3(ab+c)}{(1+ax+bx^2)^2}\right).$$
The Hankel transform of the expansion of $A(x)$ is not a Somos sequence, so we look at the element given by
$$\frac{1}{1+ax+bx^2}\mathfrak{C}\left(\frac{x^3(ab+c)}{(1+ax+bx^2)^2}\right).$$
We can express this generating function as the continued fraction
$$\cfrac{1}{1+ax+bx^2-\cfrac{(ab+c)x^3}{1+ax+bx^2-\cfrac{(ab+c)x^3}{1+ax+bx^2-\cdots}}}.$$

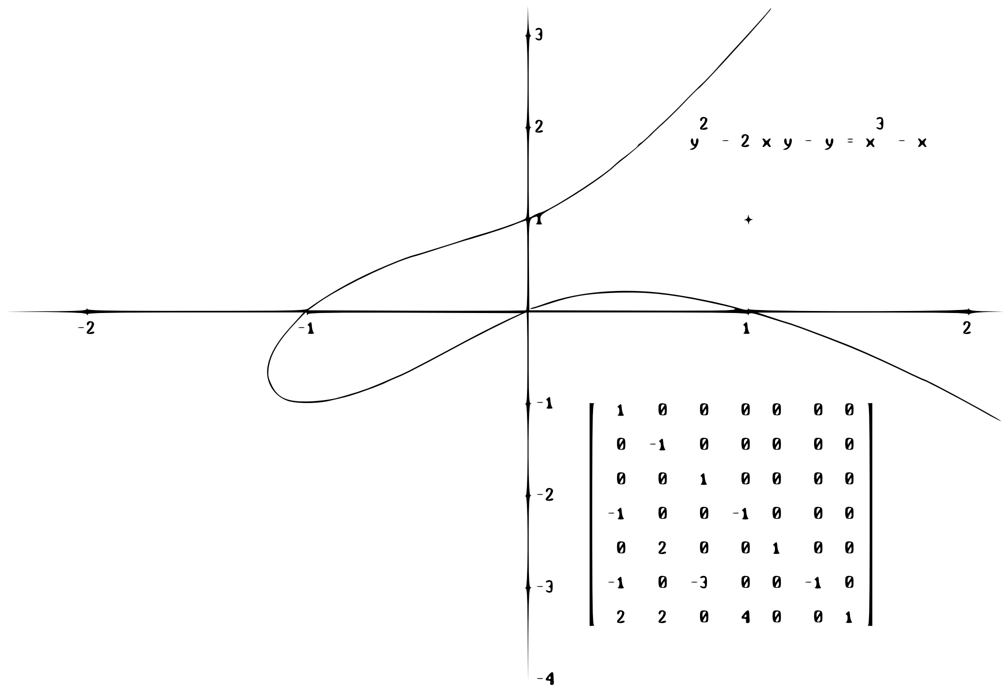

Figure 2: The elliptic curve $y^2 - 2xy - y = x^3 - x$ and its associated Riordan pseudo-involution

This expands to give a sequence that begins

$$1, -a, a^2 - b, -a^3 + 3ab + c, a^4 - 6a^2b - 3ac + b^2, \ldots$$

with a Hankel transform that begins

$$1, -b, -abc - c^2, -a^3b^3c + a^2b^2(b^3 - 3c^2) + abc(2b^3 - 3c^2) + c^2(b^3 - c^2), \ldots.$$

Once again, we can conjecture that this Hankel transform is a $((ab+c)^2, b(ab+c)^2)$ Somos-4 sequence.

Example 14. For $(a, b, c) = (-1, 1, 2)$, we obtain the sequence (essentially A025258 that begins

$$1, 1, 0, 0, 2, 3, 1, 2, 11, 17, 12, \ldots$$

with generating function

$$\frac{1}{1 - x + x^2} \mathfrak{C}\left(\frac{x^3}{(1 - x + x^2)^2}\right) = \frac{1 - x + x^2 - \sqrt{1 - 2x + 3x^2 - 6x^3 + x^4}}{2x^3}.$$

This sequence has a Hankel transform that begins

$$1, -1, -2, -1, 5, 9, -8, -41, -61, 241, \ldots.$$

This is a $(1,1)$ Somos-4 sequence (essentially A178627) associated with the elliptic curve

$$y^2 + xy - y = x^3 - x^2 + x.$$

Example 15. For $(a,b,c) = (-1,-1,-2)$ we obtain the sequence that begins

$$1, 1, 2, 2, 2, -1, -7, -20, -37, -53, -40, 49, 301, \ldots$$

with generating function

$$\frac{1}{1-x-x^2}\mathfrak{C}\left(\frac{-x^3}{(1-x-x^2)^2}\right) = \frac{-1+x+x^2+\sqrt{1-2x-x^2+6x^3+x^4}}{2x^3}.$$

This sequence has a Hankel transform that begins

$$1, 1, -2, -3, -7, 5, 32, 83, 87, -821, \ldots,$$

, which is a $(1,-1)$ Somos-4 sequence.

Example 16. We take $(a,b,c) = (1,2,-1)$. Thus we obtain the sequence that begins

$$1, -1, -1, 4, -4, -5, 23, -28, -28, 164, -232, -166, \ldots$$

with generating function

$$\frac{1}{1+x+2x^2}\mathfrak{C}\left(\frac{x^3}{(1+x+2x^2)^2}\right) = \frac{1+x+2x^2-\sqrt{1+2x+5x^2+4x^4}}{2x^3}.$$

This sequence has a Hankel transform that begins

$$1, -2, 1, 9, -16, -73, -145, 1442, 3951, -49121, \ldots.$$

This is a $(1,2)$ Somos-4 sequence, essentially A178075.

Example 17. We let $(a,b,c) = (-1,-2,-1)$. We obtain the sequence that begins

$$1, 1, 3, 6, 14, 33, 79, 194, 482, 1214, 3090, 7936, 20544, \ldots$$

with generating function

$$\frac{1}{1-x-2x^2}\mathfrak{C}\left(\frac{x^3}{(1-x-2x^2)^2}\right) = \frac{1-x-2x^2-\sqrt{1-2x-3x^2+4x^4}}{2x^3}.$$

This sequence has a Hankel transform that begins

$$1, 2, 1, -7, -16, -57, -113, 670, 3983, 23647, \ldots.$$

This is a $(1,-2)$ Somos-4 sequence, essentially A178622, which is associated with the elliptic curve $y^2 - 3xy - y = x^3 - x$. We note further that the sequence that begins

$$1, 2, 1, 1, 3, 6, 14, 33, 79, 194, 482, 1214, 3090, 7936, 20544, \ldots$$

or A025243 counts the number of Dyck $(n-1)$-paths that contain no $DUDU$'s and no $UUDD$'s for $n \geq 3$.

7 Elliptic pseudo-involutions

In this section, we consider the methods outlined above, as applied to a particular one-parameter family of elliptic curves. We obtain a result concerning what may be called "elliptic" pseudo-involutions in the Riordan group, as each such pseudo-involution is associated in a unique way with an elliptic curve of the type discussed below. Prior to this, we need to establish the following result.

Proposition 18. *The generating function*

$$g(x) = \frac{1}{1-ax-bx^2}\mathfrak{C}\left(\frac{-x^2(b+cx)}{(1-ax-bx^2)^2}\right)$$

is involutory.

Proof. We must establish that

$$(g(x), -xg(x))^{-1} = (g(x), -xg(x)).$$

Now

$$(g(x), -xg(x))^{-1} = \left(\frac{1}{g\left(\mathrm{Rev}(-xg(x))\right)}, \mathrm{Rev}(-xg(x))\right).$$

Thus a first requirement is to show that

$$\mathrm{Rev}(-xg(x)) = -xg(x).$$

This follows from solving the equation

$$\frac{-u}{1-au-bu^2}\mathfrak{C}\left(\frac{-u^2(b+cu)}{(1-au-bxu^2)^2}\right) = x,$$

where we take the solution that satisfies $u(0) = 0$. We next require that

$$g(x) = \frac{1}{g\left(\mathrm{Rev}(-xg(x))\right)},$$

or that

$$g(x)g(-xg(x)) = 1.$$

Equivalently, we must show that

$$xg(x)g(-xg(x)) = x.$$

Now

$$\begin{aligned} x &= (\mathrm{Rev}(-xg(x)))(-xg(x)) \\ &= (-xg(x))(-xg(x)) \\ &= xg(x)g(-xg(x)). \end{aligned}$$

□

Proposition 19. *The elliptic curve*

$$E : y^2 - axy - y = x^3 - x$$

defines a pseudo-involution $(g(x), xg(x))$ in the Riordan group whose B-sequence is given by

$$B(x) = \frac{2 - a + (1 - 3a + a^2)x}{1 + (1 - a)x}.$$

Proof. We solve the quadratic (in y) given by

$$y^2 - axy - y = x^3 - x$$

to obtain

$$y = \frac{1 + ax - \sqrt{1 + 2(a-2)x + a^2x^2 + 4x^3}}{2}.$$

This expands to a sequence that begins

$$0, 1, 1 - a, 1 - 3a + a^2, \ldots.$$

We remove the first two terms, giving the generating function

$$\left(\frac{1 + ax - \sqrt{1 + 2(a-2)x + a^2x^2 + 4x^3}}{2} - x\right)/x^2$$

$$= \frac{1 - (2-a)x - \sqrt{1 + 2(a-2)x + a^2x^2 + 4x^3}}{2x^2}.$$

We now form the fraction

$$\frac{1}{1 - x - x^2\left(\frac{1-(2-a)x-\sqrt{1+2(a-2)x+a^2x^2+4x^3}}{2x^2}\right)} = \frac{2}{\sqrt{1 + 2(a-2)x + a^2x^2 + 4x^3}}.$$

We revert this generating function $\frac{2x}{\sqrt{1+2(a-2)x+a^2x^2+4x^3}}$ and divide the result by x to get

$$\frac{1 + ax - \sqrt{1 + 2ax + a^2x^2 - 4x^3 + 4(1-a)x^4}}{2x^3}.$$

We let

$$f(x) = \frac{1}{1 - x^2\left(\frac{1+ax-\sqrt{1+2ax+a^2x^2-4x^3+4(1-a)x^4}}{2x^3}\right)},$$

or

$$f(x) = \frac{2x}{\sqrt{1 + 2ax + a^2x^2 - 4x^3 + 4(1-a)x^4} + (2-a)x - 1}.$$

Finally, we form

$$g(x) = \left(\frac{1}{1+(a-1)x}, \frac{-x}{1+(a-1)x}\right) \cdot \frac{f(x)(1+(a-1)x)-1}{xf(x)(ax+a-1)}.$$

This gives us

$$g(x) = \frac{\sqrt{1+2(a-2)x+(a^2-6a+6)x^2+2a(3-a)x^3+(a-1)^2x^4}+(1-a)x^2+(2-a)x-1}{2x^2((a^2-3a+1)x+a-1)}.$$

This can now be put in the form

$$g(x) = \frac{1}{1-(2-a)x-(1-a)x^2}\mathfrak{C}\left(\frac{-x^2((1-a)-(1-3a+a^2)x)}{(1-(2-a)x-(1-a)x^2)^2}\right).$$

Comparing this with

$$\frac{1}{1-\alpha x-\beta x^2}\mathfrak{C}\left(\frac{-x^2(\beta+\gamma x)}{(1-\alpha x-\beta x^2)^2}\right)$$

we see that this shows that $g(x)$ is an involutory generating function associated with the B-sequence given by

$$B(x) = \frac{2-a+(1-3a+a^2)x}{1+(1-a)x}.$$

□

The B sequence with generating function $B(x) = \frac{2-a+(1-3a+a^2)x}{1+(1-a)x}$ begins

$$2-a, -1, -(a-1), -(a-1)^2, -(a-1)^3, -(a-1)^4, -(a-1)^5, -(a-1)^6, -(a-1)^7, -(a-1)^8, \ldots.$$

The sequence g_n begins

$$1, 2-a, a^2-4a+4, -a^3+6a^2-12a+7, a^4-8a^3+24a^2-29a+10, \ldots,$$

and it has a Hankel transform $|g_{i+j}|_{0\le i,j\le n}$ which begins

$$1, 0, -1, -1, 1-a, -a^2+3a-1, \ldots.$$

The sequence

$$1, 1, a-1, a^2-3a+1, -a^3+4a^2-6a+2, \ldots$$

is in fact the Hankel transform of the sequence whose generating function is $f(x)$. This Hankel transform is a $(1, 1-a)$ Somos-4 sequence [2, 5].

a	b_n	$g(x)$
0	$2, -1, 1, -1, 1, -1, \ldots$	$\frac{\sqrt{1-4x+6x^2+x^4}+x^2+2x-1}{2x^2(1-x)}$
1	$1, -1, 0, 0, 0, \ldots$	$\frac{\sqrt{1-2x+x^2+4x^3}+x-1}{2x^3}$
2	$0, -1, -1, -1, \ldots$	$\frac{\sqrt{1-2x^2+4x^3+x^4}-x^2-1}{2x^2(x-1)}$
3	$-1, -1, -2, -4, -8, \ldots$	$\frac{1+x+2x^2-\sqrt{1+2x-3x^2+4x^4}}{2x^2(x+2)}$
4	$-2, -1, -3, -9, -27, \ldots$	$\frac{1+2x+3x^2-\sqrt{1+4x-2x^2-8x^3+9x^4}}{2x^2(5x+3)}$
5	$-3, -1, -4, -16, \ldots$	$\frac{1+3x+4x^2-\sqrt{1+6x+x^2-20x^3+16x^4}}{2x^2(11x+4)}$

Example 20. We take the case of $a=-3$. Thus we start with the elliptic curve

$$E: y^2+3xy-y=x^3-x.$$

We find that

$$g(x)=\frac{\sqrt{1-10x+33x^2-36x^3+16x^4}+4x^2+5x-1}{2x^2(4-19x)},$$

which expands to give the sequence g_n that begins

$$1, 5, 25, 124, 610, 2979, 14457, 69784, 335330, 1605334, 7662014, \ldots.$$

The corresponding Riordan pseudo-involution then begins

$$\left(\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 25 & 10 & 1 & 0 & 0 & 0 & 0 & 0 \\ 124 & 75 & 15 & 1 & 0 & 0 & 0 & 0 \\ 610 & 498 & 150 & 20 & 1 & 0 & 0 & 0 \\ 2979 & 3085 & 1247 & 250 & 25 & 1 & 0 & 0 \\ 14457 & 18258 & 9300 & 2496 & 375 & 30 & 1 & 0 \\ 69784 & 104580 & 64512 & 21755 & 4370 & 525 & 35 & 1 \end{array}\right),$$

which has a production matrix that begins

$$\left(\begin{array}{ccccccc} 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 5 & 1 & 0 & 0 & 0 \\ 5 & -1 & 0 & 5 & 1 & 0 & 0 \\ -21 & 5 & -1 & 0 & 5 & 1 & 0 \\ 84 & -21 & 5 & -1 & 0 & 5 & 1 \\ -326 & 84 & -21 & 5 & -1 & 0 & 5 \end{array}\right).$$

For this case, we have

$$B(x)=\frac{5+19x}{1+4x}.$$

The Hankel transform of g_n begins

$$1, 0, -1, -1, 4, -19, -83, -1112, 12171, \ldots$$

corresponding to the $(1,4)$ Somos-4 sequence that begins

$$1, 1, -4, 19, 83, 1112, -12171, \ldots.$$

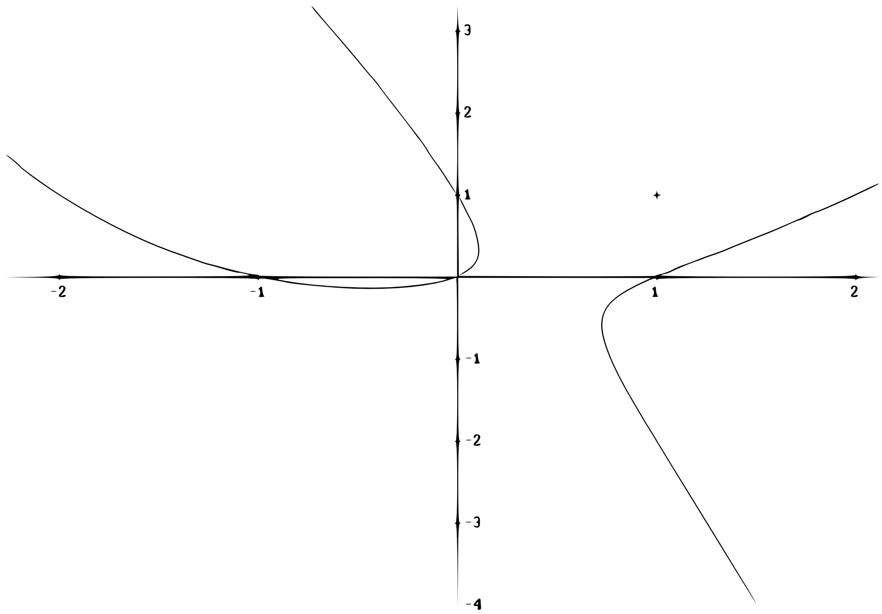

Figure 3: The elliptic curve $y^2 + 3xy - y = x^3 - x$

Example 21. When $a = 0$, we find that

$$g(x) = \left(\frac{1}{1-2x-x^2}, \frac{x^2(x-1)}{(1-2x-x^2)^2} \right) \cdot \mathfrak{C}(x).$$

This expands to give the sequence g_n that begins

$$1, 2, 4, 7, 10, 9, -6, -53, -151, -284, -301, 278, 2482, 7717, \ldots.$$

This sequence has a Hankel transform that begins

$$1, 0, -1, -1, 1, -1, -2, 1, 3, 5, \ldots,$$

where the sequence

$$1, 1, -1, 1, 2, -1, -3, -5, \ldots$$

which is A006769 is the elliptic divisibility sequence [18] associated with the elliptic curve

$$E : y^2 - y = x^3 - x.$$

This is a $(1, 1)$ Somos-4 sequence. It is the Hankel transform of the expansion of

$$f(x) = \frac{2x}{\sqrt{1 - 4x^3 + 4x^4} + 2x - 1}.$$

This expansion begins

$$1, 0, 1, -1, 1, -1, 0, 0, 0, -2, 4, -4, -1, 11, -16, \ldots.$$

We note finally that the generating function

$$g(x)=\frac{1}{1-(2-a)x-(1-a)x^2}\mathfrak{C}\left(\frac{-x^2((1-a)-(1-3a+a^2)x)}{(1-(2-a)x-(1-a)x^2)^2}\right)$$

can be put in the form of the continued fraction [3, 17]

$$\cfrac{1}{1+(a-2)x+(a-1)x^2-\cfrac{x^2(a-1+(1-3a+a^2)x)}{1+(a-2)x+(a-1)x^2-\cdots}}.$$

8 Conclusions

In this note, we have exhibited a three-parameter family of involutory generating functions defined by the B-sequence with generating function

$$B=\frac{a-cx}{1+bx}.$$

A special feature of this family is that, via Hankel transforms, it is closely linked to Somos-4 sequences. In turn, these Somos sequences are linked to elliptic curves. We have shown that it is possible in certain circumstances to start with an elliptic curve, and by a sequence of transformations, arrive at an involutory power series. In particular, we have shown that the one-parameter family of elliptic curves $E: y^2-axy-y=x^3-x$ gives rise to a corresponding family of Bell pseudo-involutions in the Riordan group.

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