



On Two New Classes of B - q -bonacci Polynomials

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Abstract

In this paper we define two new classes of polynomials associated with generalized Fibonacci polynomials. We call them $h(x)$ - B - q -bonacci polynomials and incomplete $h(x)$ - B - q -bonacci polynomials. We present some identities for the two classes of polynomials, and the convolution property of $h(x)$ - B - q -bonacci polynomials and its applications.

1 Introduction

The Fibonacci sequence, polynomials associated with the Fibonacci sequence, and their extended forms produce interesting and fascinating properties. For details see [8, 14]. Arolkar and Valaulikar introduced the B -tribonacci sequence [2] and B -tribonacci polynomials [1].

The B -tribonacci sequence [2] and B -tribonacci polynomials [1] are further extended to q^{th} order recurrence relations in [5] and [6] respectively. Arolkar and Valaulikar extended and studied the $h(x)$ -Fibonacci polynomials [9] to $h(x)$ - B -tribonacci polynomials [3]. Filipponi [7] introduced the incomplete Fibonacci and Lucas numbers. Ramírez [11] studied various identities related to the incomplete k -Fibonacci and k -Lucas numbers. Ramírez [13] introduced interesting classes of polynomials, namely, the incomplete $h(x)$ -Fibonacci and $h(x)$ -Lucas polynomials. Arolkar and Valaulikar [4] extended the incomplete $h(x)$ -Fibonacci and $h(x)$ -Lucas polynomials. Ramírez and Sirvent [12] defined and studied identities related to the incomplete tribonacci numbers and polynomials. Yilmaz and Taskara [10] obtained identities for the incomplete tribonacci-Lucas numbers and polynomials.

The aim of this paper is to extend two classes of polynomials, namely, the $h(x)$ - B -tribonacci polynomials [3] and incomplete $h(x)$ - B -tribonacci polynomials of [4] to the q^{th} order relations. We call them the $h(x)$ - B - q -bonacci polynomials and incomplete $h(x)$ - B - q -bonacci polynomials. We study some properties of these polynomials.

2 $h(x)$ - B - q -bonacci polynomials

We first define the class of $h(x)$ - B - q -bonacci polynomials.

Definition 1. Let $h(x)$ be a polynomial with real coefficients. The $h(x)$ - B - q -bonacci polynomials, denoted by $({}^qB)_{h,n}(x)$, $n \in \mathbb{N} \cup \{0\}$, $q \geq 2$, are defined by

$$({}^qB)_{h,n+q-1}(x) = \sum_{r=0}^{q-1} \frac{(q-1)^r}{r!} h^{q-1-r}(x) ({}^qB)_{h,n+q-2-r}(x), \forall n \geq 1, \quad (1)$$

with $({}^qB)_{h,i}(x) = 0$, $i = 0, 1, 2, 3, \dots, q-2$ and $({}^qB)_{h,q-1}(x) = 1$, where the coefficients of the terms on the right-hand side are the terms of the binomial expansion of $(h(x) + 1)^{q-1}$ and $({}^qB)_{h,n}(x)$ is the n^{th} polynomial.

For simplicity, henceforth we denote $({}^qB)_{h,n}(x)$ by $({}^qB)_{h,n}$ and $h(x)$ by h . We have the following identities for $({}^qB)_{h,n}$.

(1) The n^{th} term $({}^qB)_{h,n}$ of (1) is given by

$$({}^qB)_{h,n} = \sum_{r=0}^{\lfloor \frac{(q-1)(n-(q-1))}{q} \rfloor} \frac{((q-1)(n-(q-1)-r))^r}{r!} h^{(q-1)(n-(q-1)-r)-r}, \quad (2)$$

$n \geq q-1$, where $\lfloor \cdot \rfloor$ denotes the floor function.

Proof. We prove the identity using induction on n .

For $n = q - 1$, (2) implies

$$({}^q B)_{h, q-1} = \sum_{r=0}^0 \frac{((q-1)(-r))^r}{r!} h^{(q-1)(-r)-r} = 1.$$

Hence (2) is true for $n = q - 1$. Assume that (2) is true for $n \leq m$. We divide the result into q cases, namely, $m = qk, qk + 1, qk + 2, \dots, qk + (q - 1)$, for some $k \geq 1$.

Case (i): Let $m = qk$ and $t = \left\lfloor \frac{(q-1)(qk-s-(q-1))}{q} \right\rfloor$. Then

$$\begin{aligned} & \sum_{s=0}^{q-1} \frac{(q-1)^{\underline{s}}}{s!} h^{q-1-s} ({}^q B)_{h, qk-s} \\ &= \sum_{s=0}^{q-1} \frac{(q-1)^{\underline{s}}}{s!} \sum_{r=0}^t \frac{((q-1)(qk - (q-1) - (r+s)))^r}{r!} h^{(q-1)(qk+1-(q-1)-(r+s))-(r+s)} \\ &= \sum_{s=0}^{q-1} \frac{(q-1)^{\underline{s}}}{s!} \sum_{p=s}^{(q-1)k-(q-2)} \frac{((q-1)(qk - (q-1) - p))^{p-s}}{(p-s)!} h^{(q-1)(qk+1-(q-1)-p)-p} \\ &= \left(\frac{(q-1)^{\underline{0}}}{0!} \sum_{p=0}^{(q-1)k-(q-2)} \frac{((q-1)(qk - (q-1) - p))^p}{p!} \right. \\ & \quad + \frac{(q-1)^{\underline{1}}}{1!} \sum_{p=1}^{(q-1)k-(q-2)} \frac{((q-1)(qk - (q-1) - p))^{p-1}}{(p-1)!} + \dots \\ & \quad \left. + \frac{(q-1)^{\underline{(q-1)}}}{(q-1)!} \sum_{p=q-1}^{(q-1)k-(q-2)} \frac{((q-1)(qk - (q-1) - p))^{p-(q-1)}}{(p-(q-1))!} \right) h^{(q-1)(qk+1-(q-1)-p)-p} \\ &= \left(\frac{((q-1)(qk - (q-1)))^{\underline{0}}}{0!} \right. \\ & \quad \left. + \left(\frac{((q-1)(qk - (q-1) - 1))^{\underline{1}}}{1!} + \frac{(q-1)^{\underline{1}} ((q-1)(qk - (q-1) - 1))^{\underline{0}}}{1! \cdot 0!} \right) + \dots \right. \\ & \quad \left. + \sum_{p=q-1}^{(q-1)k-(q-2)} \sum_{s=0}^{q-1} \frac{(q-1)^{\underline{s}} ((q-1)(qk - (q-1) - p))^{p-s}}{s! (p-s)!} \right) h^{(q-1)(qk+1-(q-1)-p)-p}. \end{aligned}$$

Therefore, using $\sum_{s=0}^{q-1} \frac{(q-1)^s}{s!} \frac{n^{p-s}}{(p-s)!} = \frac{(n+(q-1))^p}{p!}$, we have

$$\begin{aligned} & \sum_{s=0}^{q-1} \frac{(q-1)^s}{s!} h^{q-1-s} ({}^q B)_{h, qk-s} \\ &= \sum_{p=0}^{(q-1)k-(q-2)} \frac{((q-1)(qk+1-(q-1)-p))^p}{p!} h^{(q-1)(qk+1-(q-1)-p)-p} \\ &= ({}^q B)_{h, qk+1}. \end{aligned}$$

Thus, assuming the result for $m = qk$, we have proved it for $m = qk + 1$. Similarly, we can prove the other cases.

We conclude that $\sum_{s=0}^{q-1} \frac{(q-1)^s}{s!} h^{q-1-s} ({}^q B)_{h, m-s} = ({}^q B)_{h, m+1}$.

Hence, by induction, the result follows. \square

(2) The sum of the first $n + 1$ terms of (1) is given by

$$\sum_{r=0}^n ({}^q B)_{h, r} = \frac{({}^q B)_{h, n+1} + \sum_{i=0}^{q-2} \sum_{r=1+i}^{q-1} \frac{(q-1)^r}{r!} h^{q-1-r} ({}^q B)_{h, n-i} - 1}{(h+1)^{q-1} - 1}, \quad (3)$$

provided $\begin{cases} h \neq 0, & \text{if } q \text{ is even;} \\ h \neq 0, -2, & \text{if } q \text{ is odd.} \end{cases}$

Proof. We obtain the result by induction on n . For $n = q - 1$, $\sum_{r=0}^{q-1} ({}^q B)_{h, r} = ({}^q B)_{h, q-1} = 1$. Also,

$$\begin{aligned} & \frac{({}^q B)_{h, q} + \sum_{i=0}^{q-2} \sum_{r=1+i}^{q-1} \frac{(q-1)^r}{r!} h^{q-1-r} ({}^q B)_{h, q-1-i} - 1}{(h+1)^{q-1} - 1} \\ &= \frac{h^{q-1} + \sum_{r=1}^{q-1} \frac{(q-1)^r}{r!} h^{q-1-r} ({}^q B)_{h, q-1} - 1}{(h+1)^{q-1} - 1} = 1. \end{aligned}$$

Thus, (3) is true for $n = q - 1$.

Assume that (3) is true for $n \leq m$. Then

$$\begin{aligned} & \sum_{r=0}^{m+1} ({}^q B)_{h, r} = \sum_{r=0}^m ({}^q B)_{h, r} + ({}^q B)_{h, m+1} \\ &= \frac{({}^q B)_{h, m+1} + \sum_{i=0}^{q-2} \sum_{r=1+i}^{q-1} \frac{(q-1)^r}{r!} h^{q-1-r} ({}^q B)_{h, m-i} - 1}{(h+1)^{q-1} - 1} + ({}^q B)_{h, m+1} \\ &= \frac{({}^q B)_{h, m+2} + \sum_{i=0}^{q-2} \sum_{r=1+i}^{q-1} \frac{(q-1)^r}{r!} h^{q-1-r} ({}^q B)_{h, m+1-i} - 1}{(h+1)^{q-1} - 1}. \end{aligned}$$

Thus, the result is true for $n = m + 1$. Hence, by induction, the result follows. \square

(3) The generating function for (1) is given by

$$({}^qG_{(B)})_h(z) = \frac{1}{1 - z(h+z)^{q-1}}, \quad (4)$$

provided $|(z(h+z)^{q-1})| < 1$.

Proof. Let $t = \left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor$. Then

$$\begin{aligned} ({}^qG_{(B)})_h(z) &= \sum_{n=0}^{\infty} ({}^qB)_{h,n} z^{n-(q-1)} \\ &= \sum_{n=0}^{q-2} ({}^qB)_{h,n} z^{n-(q-1)} + \sum_{n=q-1}^{\infty} ({}^qB)_{h,n} z^{n-(q-1)} \\ &= \sum_{n=q-1}^{\infty} ({}^qB)_{h,n} z^{n-(q-1)}, \text{ since } ({}^qB)_{h,i} = 0, \ i = 0, 1, 2, 3, \dots, q-2 \\ &= \sum_{n=q-1}^{\infty} \sum_{r=0}^t \frac{((q-1)(n-(q-1)-r))^r}{r!} h^{(q-1)(n-(q-1)-r)-r} z^{n-(q-1)} \\ &= 1 + h^{q-1}z + \left(h^{2(q-1)} + \frac{(q-1)!}{1!} h^{q-2} \right) z^2 + \dots \\ &= 1 + z(h+z)^{q-1} + z^2(h+z)^{2(q-1)} + \dots \\ &= \frac{1}{1 - z(h+z)^{q-1}}, \text{ provided } |(z(h+z)^{q-1})| < 1. \end{aligned}$$

□

We now obtain the following property.

Theorem 2. (Convolution property for $({}^qB)_{h,n}$)

For all $n \geq q-1$, we have

$$\frac{d}{dx} ({}^qB)_{h,n} = (q-1) \frac{dh}{dx} \left(\sum_{r=0}^{q-2} \frac{(q-2)^r}{r!} h^{(q-2)-r} \sum_{i=0}^{n+q-2-r} ({}^qB)_{h,i} ({}^qB)_{h,n+q-2-r-i} \right). \quad (5)$$

Proof. Equation (4) implies

$$\sum_{n=0}^{\infty} ({}^qB)_{h,n} z^{n-(q-1)} = \frac{1}{1 - z(h+z)^{q-1}}.$$

Differentiating both sides with respect to x , we get

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{d}{dx} (({}^qB)_{h,n}) z^{n-(q-1)} = z(q-1)(h+z)^{q-2} \frac{1}{(1-z(h+z)^{q-1})^2} \frac{dh}{dx} \\
& = \left((q-1) \sum_{r=0}^{q-2} \frac{(q-2)^r}{r!} h^{(q-2)-r} z^{r+1} \left(\sum_{n=0}^{\infty} ({}^qB)_{h,n} z^{n-(q-1)} \right)^2 \right) \frac{dh}{dx} \\
& = \left((q-1) \sum_{r=0}^{q-2} \frac{(q-2)^r}{r!} h^{(q-2)-r} z^{-2(q-1)+r+1} \left(\sum_{n=0}^{\infty} ({}^qB)_{h,n} z^n \right)^2 \right) \frac{dh}{dx} \\
& = (q-1) \frac{dh}{dx} \sum_{r=0}^{q-2} \frac{(q-2)^r}{r!} h^{(q-2)-r} \sum_{n=0}^{\infty} \left(\sum_{i=0}^n ({}^qB)_{h,i} ({}^qB)_{h,n-i} z^{n-2(q-1)+r+1} \right).
\end{aligned}$$

Comparing the coefficients of $z^{n-(q-1)}$, we get

$$\frac{d}{dx} (({}^qB)_{h,n}) = (q-1) \frac{dh}{dx} \left(\sum_{r=0}^{q-2} \frac{(q-2)^r}{r!} h^{(q-2)-r} \sum_{i=0}^{n+q-2-r} ({}^qB)_{h,i} ({}^qB)_{h,n+q-2-r-i} \right).$$

□

We now give an application of the convolution property. It is required to prove an identity in the next section.

Theorem 3. For $n \geq q-1$,

$$\begin{aligned}
& \left[\frac{(q-1)(n-(q-1))}{q} \right] \sum_{r=0}^q r \frac{((q-1)(n-(q-1)-r))^r}{r!} h^{(q-1)(n-(q-1))-qr} \\
& = \frac{(q-1)(n-(q-1))}{q} ({}^qB)_{h,n} \\
& - \frac{h}{q} (q-1) \left(\sum_{r=0}^{q-2} \frac{(q-2)^r}{r!} h^{(q-2)-r} \sum_{i=0}^{n+q-2-r} ({}^qB)_{h,i} ({}^qB)_{h,n+q-2-r-i} \right). \tag{6}
\end{aligned}$$

Proof. Equation (2) implies

$$({}^qB)_{h,n} = \sum_{r=0}^{\left[\frac{(q-1)(n-(q-1))}{q} \right]} \frac{((q-1)(n-(q-1)-r))^r}{r!} h^{(q-1)(n-(q-1))-qr}.$$

Differentiating both sides with respect to x and simplifying, we get

$$\begin{aligned} \frac{d}{dx} \left(({}^q B)_{h,n} \right) h &= ((q-1)(n-(q-1))) ({}^q B)_{h,n} \frac{dh}{dx} \\ - q \frac{dh}{dx} &\sum_{r=0}^{\lfloor \frac{(q-1)(n-(q-1))}{q} \rfloor} r \frac{((q-1)(n-(q-1)-r))^r}{r!} h^{(q-1)(n-(q-1))-qr}. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{dh}{dx} &\sum_{r=0}^{\lfloor \frac{(q-1)(n-(q-1))}{q} \rfloor} r \frac{((q-1)(n-(q-1)-r))^r}{r!} h^{(q-1)(n-(q-1))-qr} \\ &= \frac{(q-1)(n-(q-1))}{q} ({}^q B)_{h,n} \frac{dh}{dx} - \frac{h}{q} \frac{d}{dx} \left(({}^q B)_{h,n} \right). \end{aligned}$$

Hence (5) implies

$$\begin{aligned} &\sum_{r=0}^{\lfloor \frac{(q-1)(n-(q-1))}{q} \rfloor} r \frac{((q-1)(n-(q-1)-r))^r}{r!} h^{(q-1)(n-(q-1))-qr} \\ &= \frac{(q-1)(n-(q-1))}{q} ({}^q B)_{h,n} \\ &- \frac{h}{q} (q-1) \left(\sum_{r=0}^{q-2} \frac{(q-2)^r}{r!} h^{(q-2)-r} \sum_{i=0}^{n+q-2-r} ({}^q B)_{h,i} ({}^q B)_{h,n+q-2-r-i} \right). \end{aligned}$$

□

3 Incomplete $h(x)$ - B - q -bonacci Polynomials

In this section we define the class of incomplete $h(x)$ - B - q -bonacci polynomials and discuss some of its properties.

Definition 4. The incomplete $h(x)$ - B - q -bonacci polynomials are defined by

$$({}^q B)_{h,n}^l(x) = \sum_{r=0}^l \frac{((q-1)(n-(q-1)-r))^r}{r!} h^{(q-1)(n-(q-1)-r)-r}(x), \quad (7)$$

$$\forall 0 \leq l \leq \left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor \text{ and } n \geq q-1.$$

Note that $({}^qB)_{h,n}^{\lfloor \frac{(q-1)(n-(q-1))}{q} \rfloor}(x) = ({}^qB)_{h,n}(x)$.

For simplicity, we use $({}^qB)_{h,n}^l(x) = ({}^qB)_{h,n}^l$, $({}^qB)_{h,n}(x) = ({}^qB)_{h,n}$ and $h(x) = h$. We prove identities related to the recurrence relation for $({}^qB)_{h,n}^l$.

Theorem 5. *The recurrence relation for $({}^qB)_{h,n}^l$ is given by*

$$({}^qB)_{h,n+q}^{l+q-1} = \sum_{r=0}^{q-1} \frac{(q-1)^r}{r!} h^{q-1-r} ({}^qB)_{h,n+q-1-r}^{l+q-1-r}, \quad 0 \leq l \leq \left\lfloor \frac{(q-1)(n-q)}{q} \right\rfloor, \quad \forall n \geq q. \quad (8)$$

Proof. Consider,

$$\begin{aligned} & \sum_{r=0}^{q-1} \frac{(q-1)^r}{r!} h^{q-1-r} ({}^qB)_{h,n+q-1-r}^{l+q-1-r} \\ &= \sum_{r=0}^{q-1} \frac{(q-1)^r}{r!} h^{q-1-r} \\ & \sum_{i=0}^{l+q-1-r} \frac{((q-1)(n+q-1-r-(q-1)-i))^i}{i!} h^{(q-1)(n+q-1-r-(q-1)-i)-i} \\ &= \sum_{r=0}^{q-1} \frac{(q-1)^r}{r!} h^{q-1-r} \sum_{i=0}^{l+q-1-r} \frac{((q-1)(n-r-i))^i}{i!} h^{(q-1)(n-r)-qi} \\ &= \sum_{r=0}^{q-1} \frac{(q-1)^r}{r!} \sum_{i=0}^{l+q-1-r} \frac{((q-1)(n-r-i))^i}{i!} h^{(q-1)(n+1)-qr-qi} \\ &= \sum_{r=0}^{q-1} \frac{(q-1)^r}{r!} \sum_{i=0}^{l+q-1-r} \frac{((q-1)(n-(r+i)))^i}{i!} h^{(q-1)(n+1)-q(r+i)}. \end{aligned}$$

Taking $j = i + r$, we get

$$\begin{aligned} & \sum_{r=0}^{q-1} \frac{(q-1)^r}{r!} ({}^qB)_{h,n+q-1-r}^{l+q-1-r} h^{q-1-r} \\ &= \sum_{r=0}^{q-1} \frac{(q-1)^r}{r!} \sum_{j=r}^{l+q-1} \frac{((q-1)(n-j))^{j-r}}{(j-r)!} h^{(q-1)(n+1)-qj} \\ &= \sum_{j=0}^{l+q-1} \frac{((q-1)(n+1-j))^j}{j!} h^{(q-1)(n+1)-qj} \\ &= ({}^qB)_{h,n+q}^{l+q-1}. \end{aligned}$$

□

Theorem 6. For $s \geq 1$,

$$({}^q B)_{h,n+qs}^{l+(q-1)s} = \sum_{i=0}^{(q-1)s} \frac{((q-1)s)^i}{i!} ({}^q B)_{h,n+i}^{l+i} h^i, \quad (9)$$

$$0 \leq l \leq \left\lfloor \frac{(q-1)(n-s-(q-1))}{q} \right\rfloor.$$

Proof. Follows using induction. \square

Theorem 7. For $n \geq \left\lfloor \frac{ql+2(q-1)}{q-1} \right\rfloor$, $({}^q B)_{h,n+(q-1)+s}^{l+(q-1)} - h^{(q-1)s} ({}^q B)_{h,n+q-1}^{l+q-1}$

$$= \sum_{i=0}^{s-1} \sum_{r=1}^{q-1} \frac{(q-1)^r}{r!} h^{(q-1)s-(q-1)i-r} ({}^q B)_{h,n+(q-1)+i-r}^{l+(q-1)-r}. \quad (10)$$

Proof. Follows using induction. \square

The next theorem is related to the sum of incomplete $h(x)$ - B - q -bonacci polynomials $({}^q B)_{h,n}^l$.

Theorem 8. For all $n \geq q-1$,

$$\begin{aligned} \sum_{l=0}^{\left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor} ({}^q B)_{h,n}^l &= \left(\left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor + \frac{q-(q-1)(n-(q-1))}{q} \right) ({}^q B)_{h,n} \\ &+ \frac{h}{q} (q-1) \left(\sum_{r=0}^{q-2} \frac{(q-2)^r}{r!} h^{(q-2)-r} \sum_{i=0}^{n+q-2-r} ({}^q B)_{h,i} ({}^q B)_{h,n+q-2-r-i} \right). \end{aligned} \quad (11)$$

Proof. $\sum_{l=0}^{\left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor} ({}^q B)_{h,n}^l$

$$\begin{aligned} &= ({}^q B)_{h,n}^0 + ({}^q B)_{h,n}^1 + \cdots + ({}^q B)_{h,n}^r + \cdots + ({}^q B)_{h,n}^{\left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor} \\ &= \frac{((q-1)(n-(q-1)))^0}{0!} h^{(q-1)(n-(q-1))} \\ &+ \left(\frac{((q-1)(n-(q-1)))^0}{0!} h^{(q-1)(n-(q-1))} + \frac{(q-1)(n-(q-1)-1)^1}{1!} h^{(q-1)(n-(q-1))-q} \right) \\ &+ \cdots + \\ &\left(\frac{((q-1)(n-(q-1)))^0}{0!} h^{(q-1)(n-(q-1))} + \cdots + \frac{((q-1)(n-(q-1)-r))^r}{r!} h^{(q-1)(n-(q-1))-qr} \right) \\ &+ \cdots \end{aligned}$$

$$\begin{aligned}
& + \left(\frac{((q-1)(n-(q-1)))^0}{0!} h^{(q-1)(n-(q-1))} + \dots \right. \\
& + \frac{((q-1)(n-(q-1)-r))^r}{r!} h^{(q-1)(n-(q-1))-qr} + \dots \\
& + \frac{\left((q-1) \left(n - (q-1) - \lfloor \frac{(q-1)(n-(q-1))}{q} \rfloor \right) \right)^{\lfloor \frac{(q-1)(n-(q-1))}{q} \rfloor}}{\left(\lfloor \frac{(q-1)(n-(q-1))}{q} \rfloor \right)!} h^{(q-1)(n-(q-1))-q \lfloor \frac{(q-1)(n-(q-1))}{q} \rfloor} \Big) \\
& = \sum_{r=0}^{\lfloor \frac{(q-1)(n-(q-1))}{q} \rfloor} \left(\lfloor \frac{(q-1)(n-(q-1))}{q} \rfloor + 1 - r \right) \frac{((q-1)(n-(q-1)-r))^r}{r!} h^{(q-1)(n-(q-1))-qr} \\
& = \sum_{r=0}^{\lfloor \frac{(q-1)(n-(q-1))}{q} \rfloor} \left(\left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor + 1 \right) \frac{((q-1)(n-(q-1)-r))^r}{r!} h^{(q-1)(n-(q-1))-qr} \\
& - \sum_{r=0}^{\lfloor \frac{(q-1)(n-(q-1))}{q} \rfloor} r \frac{((q-1)(n-(q-1)-r))^r}{r!} h^{(q-1)(n-(q-1))-qr} \\
& = \left(\left\lfloor \frac{(q-1)(n-(q-1))}{q} \right\rfloor + 1 - \frac{(q-1)(n-(q-1))}{q} \right) ({}^q B)_{h,n} \\
& + \frac{h}{q} (q-1) \left(\sum_{r=0}^{q-2} \frac{(q-2)^r}{r!} h^{(q-2)-r} \sum_{i=0}^{n+q-2-r} ({}^q B)_{h,i} ({}^q B)_{h,n+q-2-r-i} \right).
\end{aligned}$$

Using (6) of Theorem 3 in Section 2, the result follows. \square

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