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New Congruences for Broken k-Diamond Partitions

Dazhao Tang
College of Mathematics and Statistics
Huxi Campus
Chongqing University
Chongqing—401 331
PR China
dazhaotang@cqu.edu.cn

Abstract

Andrews and Paule introduced a new class of directed graphs, called broken k-diamond partitions. Let $\Delta_k(n)$ denote the number of broken k-diamond partitions of n for a fixed positive integer k. In this paper, we establish new infinite families of congruences modulo 5, 7, 25 and 49 for $\Delta_k(n)$ via a standard q-series technique and modular forms.

1 Introduction

In 2007, Andrews and Paule [1] introduced a new class of directed graphs, called *broken k-diamond partitions*. They proved that the generating function of $\Delta_k(n)$, the number of broken k-diamond partitions of n, is given by

$$\sum_{n=0}^{\infty} \Delta_k(n) q^n = \frac{(q^2;q^2)_\infty (q^{2k+1};q^{2k+1})_\infty}{(q;q)_\infty^3 (q^{2(2k+1)};q^{2(2k+1)})_\infty}.$$

Here and in the sequel, we assume $|q| < 1$ and adopt the following customary notation on q-series and partitions:

$$(a;q)_\infty := \prod_{n=0}^{\infty} (1 - aq^n).$$

The following two congruences modulo 5 were subsequently proved by Chan [4], Radu [9], and Hirschhorn [5]:

$$\Delta_2(25n+14) \equiv 0 \pmod 5,$$
$$\Delta_2(25n+24) \equiv 0 \pmod 5.$$

Moreover, a number of infinite families of congruences modulo 5 satisfied by $\Delta_2(n)$ have been proved. See, for example, Chan [4], Hirschhorn [5], Radu [9], and Xia[12].

On the other hand, Jameson[7] and Xia[11] proved the following congruences modulo 7 enjoyed by $\Delta_3(n)$, which were conjectured by Paule and Radu [8]:

$$\Delta_3(343n+82) \equiv 0 \pmod 7,$$
$$\Delta_3(343n+229) \equiv 0 \pmod 7,$$
$$\Delta_3(343n+278) \equiv 0 \pmod 7,$$
$$\Delta_3(343n+327) \equiv 0 \pmod 7.$$

Quite recently, a variety of infinite families of congruences modulo 7 enjoyed by $\Delta_3(n)$ also have been found. See, for example, Xia [11], Yao, and Wang [13].

In this paper, we establish two infinite families of congruences modulo 5 and 25 for $\Delta_k(n)$ as follows:

Theorem 1. *For all* $n \geq 0$,

$$\Delta_k(25n+24) \equiv 0 \pmod 5, \quad \text{if} \quad k \equiv 12 \pmod{25}. \tag{1}$$

Theorem 2. *For all* $n \geq 0$,

$$\Delta_k(125n+99) \equiv 0 \pmod{25}, \quad \text{if} \quad k \equiv 62 \pmod{125}. \tag{2}$$

Moreover, we obtain the following infinite families of congruences modulo 7 and 49 for $\Delta_k(n)$.

Theorem 3. *For all* $n \geq 0$,

$$\Delta_k(49n+s) \equiv 0 \pmod 7, \quad \text{if} \quad k \equiv 24 \pmod{49}, \tag{3}$$

where $s = 19, 33, 40,$ *and* 47.

Theorem 4. *For all* $n \geq 0$,

$$\Delta_k(343n+t) \equiv 0 \pmod{49}, \quad \text{if} \quad k \equiv 171 \pmod{343}, \tag{4}$$

where $t = 96, 292,$ *and* 341.

2 Proofs of Theorems 1–4

2.1 The case mod 5

To prove (1), we collect some useful identities. Recall that the Ramanujan theta function $f(a,b)$ is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \tag{5}$$

where $|ab| < 1$. The Jacobi triple product identity can be stated as

$$f(a,b) = (-a, -b, ab; ab)_\infty. \tag{6}$$

One specialization of (5) is given by [2]:

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2}.$$

According to (6), we have

$$\psi(q) = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}.$$

The following two identities are given by Hirschhorn [5, 6] and Berndt [3].

Lemma 5.

$$\psi(q) = a + qb + q^3 c, \tag{7}$$
$$\psi^2(q^5) = ab + q^5 c^2. \tag{8}$$

where

$$a = f(q^{10}, q^{15}), \quad b = f(q^5, q^{20}), \quad c = \psi(q^{25}).$$

Eq. (7) comes from [5, Eq. (2.1)], [3, Entry 10(i), p. 262] and Eq. (8) comes from [6, Eq. (34.1.21), p. 313].

Employing the binomial theorem, we can easily establish the following congruence, which will be frequently used without explicit mention.

Lemma 6. *If p is a prime, α is a positive integer, then*

$$(q^\alpha; q^\alpha)_\infty^p \equiv (q^{p\alpha}; q^{p\alpha}) \pmod p,$$
$$(q;q)_\infty^{p^\alpha} \equiv (q^p;q^p)^{p^{\alpha-1}} \pmod{p^\alpha}.$$

Now, we are ready to state the proof of (1).

Notice that $k \equiv 12 \pmod{25}$, then $2k+1 \equiv 0 \pmod{25}$. Let $2k+1 = 25j$, where j is a positive integer, we get, (all the following congruences are modulo 5)

$$\sum_{n=0}^{\infty} \Delta_k(n) q^n = \frac{(q^2;q^2)_\infty (q^{25j};q^{25j})_\infty}{(q;q)_\infty^3 (q^{50j};q^{50j})_\infty} = \frac{\psi^3(q)}{(q^2;q^2)^5} \frac{(q^{25j};q^{25j})_\infty}{(q^{50j};q^{50j})_\infty}$$
$$\equiv \frac{\psi^3(q)}{(q^{10};q^{10})_\infty} \frac{(q^{25j};q^{25j})_\infty}{(q^{50j};q^{50j})_\infty}.$$

Invoking (7), we obtain

$$\sum_{n=0}^{\infty} \Delta_k(n) q^n \equiv \frac{(q^{25j};q^{25j})_\infty}{(q^{10};q^{10})_\infty (q^{50j};q^{50j})_\infty} \times \Big(a^3 + 3qa^2b + 3q^2ab^2$$
$$+ q^3b^3 + 3q^3a^2c + 6q^4abc + 3q^5b^2c + 3q^6ac^2 + 3q^7bc^2 + q^9c^3 \Big).$$

Extracting those terms involving the powers q^{5n+4} and combining (8), we have

$$\sum_{n=0}^{\infty} \Delta_k(5n+4) q^{5n} \equiv \frac{(abc + q^5c^3)\,(q^{25j};q^{25j})_\infty}{(q^{10};q^{10})_\infty (q^{50j};q^{50j})_\infty} = \frac{c(ab + q^5c^2)(q^{25j};q^{25j})_\infty}{(q^{10};q^{10})_\infty (q^{50j};q^{50j})_\infty}$$
$$= \psi(q^{25})\psi^2(q^5) \frac{(q^{25j};q^{25j})_\infty}{(q^{10};q^{10})_\infty (q^{50j};q^{50j})_\infty},$$

then

$$\sum_{n=0}^{\infty} \Delta_k(5n+4) q^n \equiv \psi(q^5)\psi^2(q) \frac{(q^{5j};q^{5j})_\infty}{(q^2;q^2)_\infty (q^{10j};q^{10j})_\infty} = \psi(q^5) \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty^2} \frac{(q^{5j};q^{5j})_\infty}{(q^{10j};q^{10j})_\infty}$$
$$\equiv \psi(q^5) \frac{(q^{5j};q^{5j})_\infty}{(q^{10j};q^{10j})_\infty} \frac{(q;q)_\infty^3 (q^2;q^2)_\infty^3}{(q^5;q^5)_\infty} := \sum_{n=0}^{\infty} a(n) q^n,$$

say. Thanks to Jacobi's identity [2, Theorem 1.3.9, p. 14]

$$(q;q)_\infty^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2},$$

it follows that

$$\begin{aligned} (q;q)_\infty^3 &= J_0 + J_1 + J_3, \\ (q^2;q^2)_\infty^3 &= J_0^* + J_1^* + J_2^*, \end{aligned} \tag{9}$$

where J_i (resp. J_i^*) consists of those terms in which the power of q is congruent to i modulo 5. Furthermore, we see that $J_3 \equiv 0 \pmod 5$ and $J_1^* \equiv 0 \pmod 5$, so

$$\begin{aligned}(q;q)_\infty^3 &\equiv J_0 + J_1 \pmod 5,\\ (q^2;q^2)_\infty^3 &\equiv J_0^* + J_2^* \pmod 5.\end{aligned}$$

Therefore,

$$(q;q)_\infty^3 (q^2;q^2)_\infty^3 \equiv (J_0 + J_1)(J_0^* + J_2^*) \pmod 5,$$

which contains no terms of the form q^{5n+4}. Hence $a(5n+4) \equiv 0 \pmod 5$, equivalently,

$$\Delta_k(25n+24) \equiv 0 \pmod 5,$$

as desired.

To prove the remaining congruences (2)–(4), we need to use a result of Radu and Sellers [10, Lemma 2.4]. Before introducing the result of Radu and Sellers, we will briefly interpret some definitions and notation.

For a positive integer M, let $R(M)$ be the set of integer sequences $\{r : r = (r_{\delta_1}, \ldots, r_{\delta_k})\}$ indexed by the positive divisors $1 = \delta_1 < \cdots < \delta_k = M$ of M. For some $r \in R(M)$, define

$$f_r(q) := \prod_{\delta | M} (q^\delta; q^\delta)_\infty^{r_\delta} = \sum_{n=0}^{\infty} c_r(n) q^n.$$

Given a positive integer m. Let $\mathbb{Z}_m^*$ be the set of all invertible elements in $\mathbb{Z}_m$, and $\mathbb{S}_m$ be the set of all squares in $\mathbb{Z}_m^*$. We also define the set

$$P_{m,r}(t) := \left\{ t' \;\middle|\; t' \equiv ts + \frac{s-1}{24} \sum_{\delta|M} \delta r_\delta \pmod m, \quad 0 \le t' \le m-1, \quad [s]_{24m} \in \mathbb{S}_{24m} \right\},$$

where $t \in \{0, \ldots, m-1\}$ and $[s]_m = s + m\mathbb{Z}$.

Let $\Gamma := SL_2(\mathbb{Z})$ and $\Gamma_\infty := \left\{ \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \;\middle|\; h \in \mathbb{Z} \right\}$. For a positive integer N, we define the *congruence subgroup* of level N as

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \;\middle|\; c \equiv 0 \pmod N \right\}.$$

The index of $\Gamma_0(N)$ in Γ is given by

$$[\Gamma : \Gamma_0(N)] = N \prod_{p|N} \left(1 + \frac{1}{p}\right),$$

where the product runs through the distinct primes dividing N.

Let $\kappa = \kappa(m) = \gcd(m^2 - 1, 24)$ and denote Δ^* by the set of tuples $(m, M, N, t, r = (r_\delta))$ satisfying conditions given in [10, p. 2255], we set

$$p_{m,r}(\gamma) = \min_{\lambda \in \{0,\ldots,m-1\}} \frac{1}{24} \sum_{\delta | M} r_\delta \frac{\gcd^2(\delta(a + \kappa\lambda c), mc)}{\delta m}$$

and

$$p^*_{r'}(\gamma) = \frac{1}{24} \sum_{\delta | N} \frac{r'_\delta \gcd^2(\delta, c)}{\delta},$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $r \in R(M)$, and $r' \in R(N)$.

The lemma of Radu and Sellers is stated as follows:

Lemma 7. *Let u be a positive integer, $(m, M, N, t, r = (r_\delta)) \in \Delta^*$, $r' = (r'_\delta) \in R(N)$, n be the number of double cosets in $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$ and $\{\gamma_1, \ldots, \gamma_n\} \subset \Gamma$ be a complete set of representatives of the double coset $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$. Assume that $p_{m,r}(\gamma_i) + p^*_{r'}(\gamma_i) \geq 0$ for all $i = 1, \ldots, n$. Let $t_{\min} := \min_{t' \in P_{m,r}(t)} t'$ and*

$$v := \frac{1}{24}\left(\left(\sum_{\delta|M} r_\delta + \sum_{\delta|N} r'_\delta\right)[\Gamma : \Gamma_0(N)] - \sum_{\delta|N} \delta r'_\delta\right) - \frac{1}{24m}\sum_{\delta|M} \delta r_\delta - \frac{t_{\min}}{m}.$$

Then if

$$\sum_{n=0}^{\lfloor v \rfloor} c_r(mn + t')q^n \equiv 0 \pmod{u}$$

for all $t' \in P_{m,r}(t)$, then

$$\sum_{n=0}^{\infty} c_r(mn + t')q^n \equiv 0 \pmod{u}$$

for all $t' \in P_{m,r}(t)$.

2.2 The cases mod 25

Let

$$\sum_{n=0}^{\infty} b(n)q^n = \frac{(q^2; q^2)_\infty}{(q; q)^3_\infty}.$$

By Lemma 6, we obtain

$$\sum_{n=0}^{\infty} b(n)q^n = \frac{(q^2;q^2)_\infty}{(q;q)_\infty^3} \equiv \frac{(q;q)_\infty^{22}(q^2;q^2)_\infty}{(q^5;q^5)_\infty^5} \pmod{25}. \tag{10}$$

In this case, we may take

$$(m, M, N, t, r = (r_1, r_2, r_5, r_{10})) = (125, 10, 10, 99, (22, 1, -5, 0)) \in \Delta^*.$$

By the definition of $P_{m,r}(t)$, we have $P_{m,r}(t) = \{99\}$. Now we can choose

$$r' = (r'_1, r'_2, r'_5, r'_{10}) = (13, 0, 0, 0).$$

Let

$$\gamma_\delta = \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix}.$$

Radu and Sellers [10, Lemma 2.6] also proved that $\{\gamma_\delta : \delta \mid N\}$ contains a complete set of representatives of the double coset $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$.

One readily verifies that all assumptions of Lemma 7 are satisfied. Furthermore we obtain the upper bound $\lfloor v \rfloor = 21$.

For all congruences in (11), we check that they hold for n from 0 to their corresponding upper bound $\lfloor v \rfloor$ via *Mathematica*. It follows by Lemma 7 and (10) that

$$b(125n + 99) \equiv 0 \pmod{25} \tag{11}$$

holds for all $n \geq 0$. When $k \equiv 62 \pmod{125}$, i.e., $2k + 1 \equiv 0 \pmod{125}$, (2) is a direct consequence of (11).

2.3 The cases mod 7

Firstly, we have

$$\sum_{n=0}^{\infty} b(n)q^n = \frac{(q^2;q^2)_\infty}{(q;q)_\infty^3} \equiv \frac{(q;q)_\infty^{4}(q^2;q^2)_\infty}{(q^7;q^7)_\infty} \pmod{7}.$$

To prove (3), it suffices to show

$$b(49n + s) \equiv 0 \pmod{7} \tag{12}$$

for $s = 19, 33, 40$, and 47.

We first show the cases $s = 19, 33$, and 40. Taking

$$(m, M, N, t, r = (r_1, r_2, r_7, r_{14})) = (49, 14, 14, 33, (4, 1, -1, 0)) \in \Delta^*.$$

We compute that $P_{m,r}(t) = \{19, 33, 40\}$. Now we can choose

$$r' = (r'_1, r'_2, r'_7, r'_{14}) = (3, 0, 0, 0).$$

and taking γ as in Subsection 2.2, we verify that all these constants satisfy the assumption of Lemma 7. We thus obtain $\lfloor v \rfloor = 6$. With the help of *Mathematica*, we see that (12) holds up to the bound $\lfloor v \rfloor$ with $t \in \{19, 33, 40\}$, and therefore it holds for all $n \geq 0$ by Lemma 7.

Now we will turn to the case $s = 47$. Again we take

$$(m, M, N, t, r = (r_1, r_2, r_7, r_{14})) = (49, 14, 14, 47, (4, 1, -1, 0)) \in \Delta^*$$

and

$$r' = (r'_1, r'_2, r'_7, r'_{14}) = (3, 0, 0, 0).$$

In this case we obtain $P_{m,r}(t) = \{47\}$. One readily computes that $\lfloor v \rfloor = 6$. Thus we verify the first 7 terms of (12) via *Mathematica*. It follows by Lemma 7 that it holds for all $n \geq 0$.

2.4 The case mod 49

Similarly, we have

$$\sum_{n=0}^{\infty} b(n)q^n = \frac{(q^2;q^2)_\infty}{(q;q)_\infty^3} \equiv \frac{(q;q)_\infty^{46}(q^2;q^2)_\infty}{(q^7;q^7)_\infty^7} \pmod{49}.$$

To prove (4), it needs to show

$$b(343n + t) \equiv 0 \pmod{49} \tag{13}$$

for $t = 96$, 292, and 341.

In these cases, we may set

$$(m, M, N, t, r = (r_1, r_2, r_7, r_{14})) = (343, 14, 14, 96, (46, 1, -7, 0)) \in \Delta^*.$$

We compute that $P_{m,r}(t) = \{96, 292, 341\}$. Now we can choose

$$r' = (r'_1, r'_2, r'_7, r'_{14}) = (18, 0, 0, 0).$$

We thus obtain $\lfloor v \rfloor = 56$. Similarly we can verify the first 57 terms of (13) through *Mathematica*. Therefore,

$$\Delta_k(343n + 96) \equiv \Delta_k(343n + 292) \equiv \Delta_k(343n + 341) \equiv 0 \pmod{49}$$

holds for all $n \geq 0$ if $k \equiv 171 \pmod{343}$.

3 Final remarks

We would like to emphasize that unlike most congruences of $\Delta_k(n)$ studied in the literature, where k is usually fixed, the congruences (1)–(4) are valid for infinitely many k.

Moreover, despite the universality of Radu and Sellers' lemma, we should point out that their method is not elementary because it highly relies on modular forms. On the other hand, the proofs of (2)–(4) are routine to some extent. It would be interesting to find elementary proofs of these congruences.

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