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A New Class of Refined Eulerian Polynomials

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Abstract

In this note we introduce a new class of refined Eulerian polynomials defined by

$$A_n(p,q) = \sum_{\pi \in \mathfrak{S}_n} p^{\mathrm{odes}\,(\pi)} q^{\mathrm{edes}\,(\pi)},$$

where $\mathrm{odes}\,(\pi)$ and $\mathrm{edes}\,(\pi)$ enumerate the number of descents of permutation π in odd and even positions, respectively. We show that the refined Eulerian polynomials $A_{2k+1}(p,q), k = 0,1,2,\ldots,$ and $(1+q)A_{2k}(p,q), k = 1,2,\ldots,$ have a nice symmetry property.

1 Introduction

Let $f(q) = a_r q^r + \cdots + a_s q^s (r \leq s)$, with $a_r \neq 0$ and $a_s \neq 0$, be a real polynomial. The polynomial $f(q)$ is *palindromic* if $a_{r+i} = a_{s-i}$ for any i. Following Zeilberger [7], define the *darga* of $f(q)$ to be $r+s$. The set of all palindromic polynomials of darga n is a vector space [6] with gamma basis

$$\Gamma_n := \{q^i(1+q)^{n-2i} \mid 0 \leq i \leq \lfloor n/2 \rfloor\}.$$

Let $f(p,q)$ be a nonzero bivariate polynomial. The polynomial $f(p,q)$ is *palindromic of darga n* if it satisfies the following two equations:

$$f(p,q) = f(q,p),$$
$$f(p,q) = (pq)^n f(1/p, 1/q).$$

See Adin et al. [1] for details. It is known [4] that the set of all palindromic bivariate polynomials of darga n is a vector space with gamma basis

$$\mathcal{B}_n := \{(pq)^i (p+q)^j (1+pq)^{n-2i-j} \mid i,j \geq 0, 2i+j \leq n\}.$$

Let $\mathfrak{S}_n$ denote the set of all permutations of the set $[n] := \{1,2,\ldots,n\}$. For a permutation $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$, an index $i \in [n-1]$ is a *descent* of π if $\pi_i > \pi_{i+1}$, and $\operatorname{des}(\pi)$ denotes the number of descents of π. The classic Eulerian polynomial is defined as the generating polynomial for the statistic des over the set $\mathfrak{S}_n$, i.e.,

$$A_n(q) = \sum_{\pi \in \mathfrak{S}_n} q^{\operatorname{des}(\pi)}.$$

Foata and Schützenberger [3] proved that the Eulerian polynomial $A_n(q)$ can be expressed in terms of the gamma basis Γ_n with nonnegative integer coefficients. A polynomial with nonnegative coefficients under the gamma basis Γ_n is palindromic and unimodal [5].

Ehrenborg and Readdy [2] studied the number of ascents in odd position on 0, 1-words. We define similar statistics on permutations. For a permutation $\pi \in \mathfrak{S}_n$, an index $i \in [n-1]$ is an *odd descent* of π if $\pi_i > \pi_{i+1}$ and i is odd, an *even descent* of π if $\pi_i > \pi_{i+1}$ and i is even, an *odd ascent* of π if $\pi_i < \pi_{i+1}$ and i is odd, an *even ascent* of π if $\pi_i < \pi_{i+1}$ and i is even. Let $\operatorname{Odes}(\pi)$, $\operatorname{Edes}(\pi)$, $\operatorname{Oasc}(\pi)$ and $\operatorname{Easc}(\pi)$ denote the set of all odd descents, even descents, odd ascents and even ascents of π, respectively. The corresponding cardinalities are $\operatorname{odes}(\pi)$, $\operatorname{edes}(\pi)$, $\operatorname{oasc}(\pi)$ and $\operatorname{easc}(\pi)$, respectively. Note that we can also define the above four statistics on words of length n. The joint distribution of odd and even descents on $\mathfrak{S}_n$ is denoted by $A_n(p,q)$, i.e.,

$$A_n(p,q) = \sum_{\pi \in \mathfrak{S}_n} p^{\operatorname{odes}(\pi)} q^{\operatorname{edes}(\pi)}.$$

The polynomial $A_n(p,q)$ is a bivariate polynomial of degree $n-1$. The monomial with degree $n-1$ is $p^{\lfloor n/2 \rfloor} q^{\lfloor (n-1)/2 \rfloor}$ only. If $p = q$, then $A_n(q,q) = A_n(q)$ is the classic Eulerian polynomial. Thus $A_n(p,q)$, $n = 1,2,\ldots$, can be seen as a class of refined Eulerian polynomials. For example, we have

$$\begin{aligned}
A_1(p,q) &= 1,\\
A_2(p,q) &= 1+p,\\
A_3(p,q) &= 1+2p+2q+pq,\\
A_4(p,q) &= 1+6p+5q+5p^2+6pq+p^2q,\\
A_5(p,q) &= 1+13p+13q+16p^2+34pq+16q^2+13p^2q+13pq^2+p^2q^2,\\
A_6(p,q) &= 1+29p+28q+89p^2+152pq+61q^2+61p^3+152p^2q\\
&\quad +89pq^2+28p^3q+29p^2q^2+p^3q^2.
\end{aligned}$$

For convenience, we denote

$$\widetilde{A}_n(p,q) = \begin{cases} A_n(p,q), & \text{if } n = 2k+1, \\ (1+q)A_n(p,q), & \text{if } n = 2k. \end{cases}$$

Our main result is the following

Theorem 1. *For any $n = 1, 2, \ldots$, the polynomial $\widetilde{A}_n(p,q)$ is palindromic of darga $\lfloor \frac{n}{2} \rfloor$.*

In the next section we give a proof of Theorem 1. In Section 3 we study the case $q = 1$ and the case $p = 1$, the polynomials $A_n(p,1)$ and $A_n(1,q)$ are the generating functions for the statistics odes and edes over the set $\mathfrak{S}_n$, respectively. In the last section, we propose a conjecture that $\widetilde{A}_n(p,q)$ can be expressed in terms of the gamma basis $\mathcal{B}_{\lfloor \frac{n}{2} \rfloor}$ with nonnegative integer coefficients.

2 The proof of Theorem 1

Let $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$, we define the *reversal* π^r of π to be

$$\pi^r := \pi_n\pi_{n-1}\cdots\pi_1,$$

the *complement* π^c of π to be

$$\pi^c := (n+1-\pi_1)(n+1-\pi_2)\cdots(n+1-\pi_n),$$

and the *reversal-complement* π^{rc} of π to be

$$\pi^{rc} := (\pi^c)^r = (\pi^r)^c.$$

If i is a descent of π, then i is an ascent of π^c and if i is an ascent of π, then i is a descent of π^c. In other words, $\mathrm{odes}\,(\pi) + \mathrm{odes}\,(\pi^c) = \lfloor \frac{n}{2} \rfloor$ and $\mathrm{edes}\,(\pi) + \mathrm{edes}\,(\pi^c) = \lfloor \frac{n-1}{2} \rfloor$. Then

$$\begin{aligned} A_n(p,q) &= \sum_{\pi\in\mathfrak{S}_n} p^{\mathrm{odes}\,(\pi)} q^{\mathrm{edes}\,(\pi)} = \sum_{\pi\in\mathfrak{S}_n} p^{\lfloor \frac{n}{2} \rfloor - \mathrm{odes}\,(\pi^c)} q^{\lfloor \frac{n-1}{2} \rfloor - \mathrm{edes}\,(\pi^c)} \\ &= p^{\lfloor \frac{n}{2} \rfloor} q^{\lfloor \frac{n-1}{2} \rfloor} \sum_{\pi\in\mathfrak{S}_n} \left(\frac{1}{p}\right)^{\mathrm{odes}\,(\pi^c)} \left(\frac{1}{q}\right)^{\mathrm{edes}\,(\pi^c)} = p^{\lfloor \frac{n}{2} \rfloor} q^{\lfloor \frac{n-1}{2} \rfloor} A_n\left(\frac{1}{p}, \frac{1}{q}\right). \end{aligned}$$

Specially, for any $k = 1, 2, \ldots$, we have $A_{2k}(p,q) = p^k q^{k-1} A_{2k}(1/p, 1/q)$ and for any $k = 0, 1, 2, \ldots$, we have $A_{2k+1}(p,q) = (pq)^k A_{2k+1}(1/p, 1/q)$.

It can be derived that i is a descent of π if and only if i is an ascent of π^c. It is also easy to see that i is a descent of π if and only if $n - i$ is an ascent of π^r. Then, given a permutation $\pi = \pi_1\pi_2\cdots\pi_{2k+1} \in \mathfrak{S}_{2k+1}$,

$$i \text{ is a descent of } \pi \text{ if and only if } 2k+1-i \text{ is a descent of } \pi^{rc}.$$

Specially, i is an odd descent of π if and only if $2k+1-i$ is an even descent of π^{rc}, and i is an even descent of π if and only if $2k+1-i$ is an odd descent of π^{rc}. So we have

$$\widetilde{A}_{2k+1}(p,q) = \sum_{\pi\in\mathfrak{S}_{2k+1}} p^{\mathrm{odes}\,(\pi)} q^{\mathrm{edes}\,(\pi)} = \sum_{\pi\in\mathfrak{S}_{2k+1}} p^{\mathrm{edes}\,(\pi^{rc})} q^{\mathrm{odes}\,(\pi^{rc})} = \widetilde{A}_{2k+1}(q,p).$$

Thus for any $k = 1, 2, \ldots$, the polynomial $\widetilde{A}_{2k+1}(p,q)$ is palindromic of darga k.

In addition,

$$\widetilde{A}_{2k}(p,q) = (1+q)p^k q^{k-1} A_{2k}\left(\frac{1}{p}, \frac{1}{q}\right) = \left(1+\frac{1}{q}\right) p^k q^k A_{2k}\left(\frac{1}{p}, \frac{1}{q}\right) = (pq)^k \widetilde{A}_{2k+1}\left(\frac{1}{p}, \frac{1}{q}\right).$$

The last part is to prove that $\widetilde{A}_{2k}(p,q) = \widetilde{A}_{2k}(q,p)$, that is,

$$\sum_{\pi\in\mathfrak{S}_{2k}} p^{\mathrm{odes}\,(\pi)}[q^{\mathrm{edes}\,(\pi)} + q^{\mathrm{edes}\,(\pi)+1}] = \sum_{\pi\in\mathfrak{S}_{2k}} q^{\mathrm{odes}\,(\pi)}[p^{\mathrm{edes}\,(\pi)} + p^{\mathrm{edes}\,(\pi)+1}].$$

Let $\mathfrak{S}'_{2k} = \{\pi(2k+1), \pi 0 \mid \pi \in \mathfrak{S}_{2k}\}$, $\mathfrak{S}''_{2k} = \{(2k+1)\pi, 0\pi \mid \pi \in \mathfrak{S}_{2k}\}$, and let $\pi = \pi_1\pi_2\cdots\pi_{2k} \in \mathfrak{S}_{2k}$. Define a map $\psi : \mathfrak{S}'_{2k} \to \mathfrak{S}''_{2k}$ by

$$\psi(\pi x) = \begin{cases} (2k+1)(2k+1-\pi_{2k})(2k+1-\pi_{2k-1})\cdots(2k+1-\pi_1), & \text{if } x = 0, \\ 0(2k+1-\pi_{2k})(2k+1-\pi_{2k-1})\cdots(2k+1-\pi_1), & \text{if } x = 2k+1. \end{cases}$$

Given a permutation $\pi \in \mathfrak{S}_{2k}$, it is no hard to see that

$$\begin{aligned} &\mathrm{odes}\,(\pi(2k+1)) = \mathrm{odes}\,(\pi), & \mathrm{edes}\,(\pi(2k+1)) &= \mathrm{edes}\,(\pi), \\ &\mathrm{odes}\,(\pi 0) = \mathrm{odes}\,(\pi), & \mathrm{edes}\,(\pi 0) &= \mathrm{edes}\,(\pi)+1, \\ &\mathrm{odes}\,((2k+1)\pi) = \mathrm{edes}\,(\pi)+1, & \mathrm{edes}\,((2k+1)\pi) &= \mathrm{odes}\,(\pi), \\ &\mathrm{odes}\,(0\pi) = \mathrm{edes}\,(\pi), & \mathrm{edes}\,(0\pi) &= \mathrm{odes}\,(\pi). \end{aligned}$$

Thus

$$\begin{aligned} &\mathrm{odes}\,(\psi(\pi(2k+1))) = \mathrm{odes}\,(0\pi^{rc}) = \mathrm{edes}\,(\pi^{rc}), \\ &\mathrm{edes}\,(\psi(\pi(2k+1))) = \mathrm{edes}\,(0\pi^{rc}) = \mathrm{odes}\,(\pi^{rc}), \\ &\mathrm{odes}\,(\psi(\pi 0)) = \mathrm{odes}\,((2k+1)\pi^{rc}) = \mathrm{edes}\,(\pi^{rc})+1, \\ &\mathrm{edes}\,(\psi(\pi 0)) = \mathrm{edes}\,((2k+1)\pi^{rc}) = \mathrm{odes}\,(\pi^{rc}). \end{aligned}$$

Obviously, the map ψ is an involution. Then

$$\begin{aligned}
&\sum_{\pi\in\mathfrak{S}_{2k}} p^{\operatorname{odes}(\pi)}\left[q^{\operatorname{edes}(\pi)}+q^{\operatorname{edes}(\pi)+1}\right]\\
&=\sum_{\pi\in\mathfrak{S}_{2k}} p^{\operatorname{odes}(\pi(2k+1))}q^{\operatorname{edes}(\pi(2k+1))}+\sum_{\pi\in\mathfrak{S}_{2k}} p^{\operatorname{odes}(\pi 0)}q^{\operatorname{edes}(\pi 0)}\\
&=\sum_{\pi\in\mathfrak{S}_{2k}} p^{\operatorname{odes}(\psi(\pi(2k+1)))}q^{\operatorname{edes}(\psi(\pi(2k+1)))}+\sum_{\pi\in\mathfrak{S}_{2k}} p^{\operatorname{odes}(\psi(\pi 0))}q^{\operatorname{edes}(\psi(\pi 0))}\\
&=\sum_{\pi\in\mathfrak{S}_{2k}} p^{\operatorname{edes}(\pi^{rc})}q^{\operatorname{odes}(\pi^{rc})}+\sum_{\pi\in\mathfrak{S}_{2k}} p^{\operatorname{edes}(\pi^{rc})+1}q^{\operatorname{odes}(\pi^{rc})}\\
&=\sum_{\pi\in\mathfrak{S}_{2k}} q^{\operatorname{odes}(\pi)}\left[p^{\operatorname{edes}(\pi)}+p^{\operatorname{edes}(\pi)+1}\right].
\end{aligned}$$

Thus for any $k=1,2,\ldots,$ the polynomial $\widetilde{A}_{2k}(p,q)$ is palindromic of darga k. This completes the proof.

3 The case $p=1$ and the case $q=1$

If $q=1$, the polynomial $A_n(p,1)$ is the generating function for the statistic odes over the set $\mathfrak{S}_n$, and if $p=1$, the polynomial $A_n(1,q)$ is the generating function for the statistic edes over the set $\mathfrak{S}_n$. More precisely, we have

Proposition 2. *Let n be a positive integer. Then*

$$\sum_{\pi\in\mathfrak{S}_n} p^{\operatorname{odes}(\pi)} = A_n(p,1) = \frac{n!}{2^{\lfloor\frac{n}{2}\rfloor}}(1+p)^{\lfloor\frac{n}{2}\rfloor}, \tag{1}$$

and

$$\sum_{\pi\in\mathfrak{S}_n} q^{\operatorname{edes}(\pi)} = A_n(1,q) = \frac{n!}{2^{\lfloor\frac{n-1}{2}\rfloor}}(1+q)^{\lfloor\frac{n-1}{2}\rfloor}. \tag{2}$$

Proof. It is easy to verify that the equalities 1 and 1 are true for $n=1$ and $n=2$. Let $n\geq 3$ and let $\pi=\pi_1\pi_2\cdots\pi_n\in\mathfrak{S}_n$. For any $i=1,2,\ldots,\lfloor n/2\rfloor$, define a map $\varphi_i:\mathfrak{S}_n\to\mathfrak{S}_n$ by

$$\varphi_i(\pi)=\pi_1\pi_2\cdots\pi_{2i}\pi_{2i-1}\cdots\pi_n,$$

i.e., $\varphi_i(\pi)$ is obtained by swapping π_{2i} with π_{2i-1} in π. Obviously, the map φ_i is an involution, $i=1,2,\ldots,\lfloor n/2\rfloor$, and φ_i and φ_j commute for all $i,j\in\{1,2,\ldots,\lfloor n/2\rfloor\}$. For any subset $S\subseteq\{1,2,\ldots,\lfloor n/2\rfloor\}$, we define a map $\varphi_S:\mathfrak{S}_n\to\mathfrak{S}_n$ by

$$\varphi_S(\pi)=\prod_{i\in S}\varphi_i(\pi).$$

The group $\mathbb{Z}_2^{\lfloor n/2 \rfloor}$ acts on $\mathfrak{S}_n$ via the maps $\varphi_S, S \subseteq \{1, 2, \ldots, \lfloor n/2 \rfloor\}$. For any $\pi \in \mathfrak{S}_n$, let $\mathrm{Orb}^*(\pi)$ denote the orbit including π under the group action. There is a unique permutation in $\mathrm{Orb}^*(\pi)$, denoted by $\hat{\pi}$, such that

$$\hat{\pi}_1 < \hat{\pi}_2,\ \hat{\pi}_3 < \hat{\pi}_4,\ \ \ldots,\ \hat{\pi}_{2\lfloor n/2 \rfloor - 1} < \hat{\pi}_{2\lfloor n/2 \rfloor}.$$

It is not hard to prove that $\mathrm{odes}\,(\hat{\pi}) = 0$ and $\mathrm{odes}\,(\varphi_S(\hat{\pi})) = |S|$ for any $S \subseteq \{1, 2, \ldots, \lfloor n/2 \rfloor\}$. Then

$$\sum_{\sigma \in \mathrm{Orb}^*(\pi)} p^{\mathrm{odes}\,(\sigma)} = (1+p)^{\left\lfloor \frac{n}{2} \right\rfloor}.$$

Let $\mathfrak{S}_n^*$ consist of all the permutations in $\mathfrak{S}_n$ such that

$$\pi_1 < \pi_2,\ \pi_3 < \pi_4,\ \ \ldots,\ \pi_{2\lfloor n/2 \rfloor - 1} < \pi_{2\lfloor n/2 \rfloor}.$$

The cardinality of the set $\mathfrak{S}_n^*$ is

$$\binom{n}{2}\binom{n-2}{2}\cdots\binom{n+2-2\left\lfloor \frac{n}{2} \right\rfloor}{2} = \frac{n!}{2^{\left\lfloor \frac{n}{2} \right\rfloor}}.$$

Then

$$\sum_{\pi \in \mathfrak{S}_n} p^{\mathrm{odes}\,(\pi)} = A_n(p,1) = \frac{n!}{2^{\left\lfloor \frac{n}{2} \right\rfloor}}(1+p)^{\left\lfloor \frac{n}{2} \right\rfloor}.$$

Similarly, for any $i = 1, 2, \ldots, \lfloor (n-1)/2 \rfloor$, we define a map $\phi_i : \mathfrak{S}_n \to \mathfrak{S}_n$ by

$$\phi_i(\pi) = \pi_1 \cdots \pi_{2i+1}\pi_{2i} \cdots \pi_n,$$

i.e., $\phi_i(\pi)$ is obtained by swapping π_{2i} with π_{2i+1} in π. Obviously, the map ϕ_i is an involution, $i = 1, 2, \ldots, \lfloor (n-1)/2 \rfloor$, and ϕ_i and ϕ_j commute for all $i, j \in \{1, 2, \ldots, \lfloor (n-1)/2 \rfloor\}$. For any subset $S \subseteq \{1, 2, \ldots, \lfloor (n-1)/2 \rfloor\}$, we define a map $\phi_S : \mathfrak{S}_n \to \mathfrak{S}_n$ by

$$\phi_S(\pi) = \prod_{i \in S} \phi_i(\pi).$$

The group $\mathbb{Z}_2^{\lfloor (n-1)/2 \rfloor}$ acts on $\mathfrak{S}_n$ via the maps $\phi_S, S \in [\lfloor (n-1)/2 \rfloor]$. For any $\pi \in \mathfrak{S}_n$, let $\mathrm{Orb}^{**}(\pi)$ denote the orbit including π under the group action. There is a unique permutation in $\mathrm{Orb}^{**}(\pi)$, denoted by $\bar{\pi}$, such that

$$\bar{\pi}_2 < \bar{\pi}_3,\ \bar{\pi}_4 < \bar{\pi}_5,\ \ \ldots,\ \bar{\pi}_{2\lfloor (n-1)/2 \rfloor} < \bar{\pi}_{2\lfloor (n-1)/2 \rfloor + 1}.$$

It is easily obtained that $\mathrm{edes}\,(\bar{\pi}) = 0$ and $\mathrm{edes}\,(\phi_S(\bar{\pi})) = |S|$ for any $S \subseteq \{1, 2, \ldots, \lfloor (n-1)/2 \rfloor\}$. Then

$$\sum_{\sigma \in \mathrm{Orb}^{**}(\pi)} q^{\mathrm{edes}\,(\sigma)} = (1+q)^{\left\lfloor \frac{n-1}{2} \right\rfloor}.$$

Let $\mathfrak{S}_n^{**}$ consist of all the permutations in $\mathfrak{S}_n$ such that

$$\pi_2 < \pi_3,\ \pi_4 < \pi_5,\ \ \ldots,\ \pi_{2\lfloor (n-1)/2 \rfloor} < \pi_{2\lfloor (n-1)/2\rfloor + 1}.$$

The cardinality of the set $\mathfrak{S}_n^{**}$ is

$$\begin{cases} \binom{n}{2}\binom{n-2}{2}\cdots\binom{n+2-2\lfloor\frac{n-1}{2}\rfloor}{2} = \frac{n!}{2^{\lfloor\frac{n-1}{2}\rfloor}}, & \text{if } n \text{ is odd,} \\ 2\binom{n}{2}\binom{n-2}{2}\cdots\binom{n+2-2\lfloor\frac{n-1}{2}\rfloor}{2} = \frac{n!}{2^{\lfloor\frac{n-1}{2}\rfloor}}, & \text{if } n \text{ is even.} \end{cases}$$

Then

$$\sum_{\pi\in\mathfrak{S}_n} q^{\operatorname{edes}(\pi)} = A_n(1,q) = \frac{n!}{2^{\lfloor\frac{n-1}{2}\rfloor}}(1+q)^{\lfloor\frac{n-1}{2}\rfloor}.$$

□

4 Remarks

The set of palindromic bivariate polynomials of darga k is a vector space with gamma basis

$$\mathcal{B}_k = \{(pq)^i(p+q)^j(1+pq)^{k-2i-j} \mid i,j \geq 0, 2i+j \leq k\}.$$

Thus the refined Eulerian polynomials $\widetilde{A}_n(p,q)$, $n = 1,2,\ldots$, can be expanded in terms of the gamma basis $\mathcal{B}_{\lfloor\frac{n}{2}\rfloor}$. For example,

$$\begin{aligned}
\widetilde{A}_1(p,q) &= A_1(p,q) = 1,\\
\widetilde{A}_2(p,q) &= (1+q)A_2(p,q) = (1+q)(1+p) = 1+p+q+pq\\
&= (1+pq) + (p+q),\\
\widetilde{A}_3(p,q) &= A_3(p,q) = 1+2p+2q+pq = (1+pq)+2(p+q),\\
\widetilde{A}_4(p,q) &= (1+q)A_4(p,q) = (1+q)(1+6p+5q+5p^2+6pq+p^2q)\\
&= 1+6p+6q+5p^2+12pq+5q^2+6p^2q+6pq^2+p^2q^2\\
&= (1+pq)^2+6(p+q)(1+pq)+5(p+q)^2,\\
\widetilde{A}_5(p,q) &= A_5(p,q) = 1+13p+13q+16p^2+34pq+16q^2+13p^2q+13pq^2+p^2q^2\\
&= (1+pq)^2+13(p+q)(1+pq)+16(p+q)^2,\\
\widetilde{A}_6(p,q) &= (1+q)A_6(p,q)\\
&= (1+q)(1+29p+28q+89p^2+152pq+61q^2\\
&\quad +61p^3+152p^2q+89pq^2+28p^3q+29p^2q^2+p^3q^2)\\
&= 1+29p+29q+89p^2+89q^2+181pq+61p^3+241p^2q\\
&\quad +241pq^2+61q^3+181p^2q^2+89p^3q+89pq^3+29p^3q^2+29p^2q^3+p^3q^3\\
&= (1+pq)^3+29(p+q)(1+pq)^2+89(p+q)^2(1+pq)+61(p+q)^3.
\end{aligned}$$

We conjecture that for any $n \geq 1$, all c_j are positive integers in the following expansion

$$\widetilde{A}_n(p,q) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} c_j (p+q)^j (1+pq)^{\lfloor \frac{n}{2} \rfloor - j}.$$

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