Journal of Integer Sequences, Vol. 21 (2018),
Article 18.7.4

Combinatorial Identities Associated with a Multidimensional Polynomial Sequence

Isabel Cação and Helmuth R. Malonek
CIDMA and Department of Mathematics
University of Aveiro
3810-193 Aveiro
Portugal
isabel.cacao@ua.pt
hrmalon@ua.pt

M. Irene Falcão
CMAT and Department of Mathematics and Applications
University of Minho
4710-057 Braga
Portugal
mif@math.uminho.pt

Graça Tomaz
CIDMA and Research Unit for Inland Development
Department of Mathematics
Polytechnic of Guarda
6300-659 Guarda
Portugal
gtomaz@ipg.pt

Abstract

In this paper we combine the knowledge of different structures of a special Appell multidimensional polynomial sequence with the problem of establishing combinatorial identities. The elements of this special polynomial sequence have values in a Clifford algebra, are homogeneous hypercomplex differentiable functions of different degrees and their coefficients properties can be used to stress interesting matrix and combinatorial relations.

1 Motivation

The construction of polynomials with values in a Clifford algebra, based on Appell's concept of power-like polynomials, has been considered in a series of articles by authors of this paper [1, 6, 7, 12, 13, 21]. The consideration of these polynomials was motivated by the fact that positive integer powers of the usually considered hypercomplex variable are not holomorphic, except for the complex case, while hypercomplex Appell polynomials are hyperholomorphic and have the behavior of power-like functions under hypercomplex differentiation. The study of sets of Appell polynomials became a focus of many authors attention, due to their theoretical importance and interesting applications [2, 3, 4, 5, 9, 11, 17, 18, 19, 25].

The knowledge of different hypercomplex polynomial bases systems can be combined with the problem of establishing and proving new combinatorial identities by bijective methods, linking in this way Clifford analysis and combinatorics. This idea was considered in the works [8, 10, 14, 23], illustrating that the use of the underlying noncommutative algebra can lead to generalizations of known results but, at the same time, opens the way for deriving new useful formulas and combinatorial identities.

In this work we pursued the idea of considering (non-standard) applications of Clifford algebras in the solution of problems of combinatorial nature, pointing out new combinatorial aspects of generalized Appell polynomials associated with two different representations. The paper is organized as follows: in Section 2 we introduce the necessary notation from Clifford analysis and describe briefly a set of homogeneous polynomials $(\mathcal{P}_k^n(x))$ generalizing the complex power function z^k. Two different representations of the polynomials $\mathcal{P}_k^n(x)$ allow to derive, in Section 3, interesting matrix and combinatorial relations between their coefficients. In Section 4, a family of Pascal trapezoids closely related to the problem considered in the previous section is introduced and several patterns in such structure are studied.

2 A multidimensional polynomial sequence

Let $\{e_1, e_2, \ldots, e_n\}, n \geq 2$, be an orthonormal basis of the Euclidean vector space $\mathbb{R}^n$ with a non-commutative product given according to the multiplication rules

$$e_i e_j + e_j e_i = -2\delta_{ij}, \quad i, j = 1, 2, \ldots, n,$$

where δ_{ij} is the Kronecker symbol. A basis for the associative 2^n-dimensional real Clifford algebra $\mathcal{C}\ell_{0,n}$ is the set $\{e_A : A \subseteq \{1, \ldots, n\}\}$ with

$$e_A = e_{h_1} e_{h_2} \cdots e_{h_r}, \quad 1 \leq h_1 < \cdots < h_r \leq n, \quad e_\emptyset = e_0 = 1.$$

In general, the vector space $\mathbb{R}^{n+1}$ is embedded in $\mathcal{C}\ell_{0,n}$ by identifying the element $(x_0, \ldots, x_n)$ in $\mathbb{R}^{n+1}$ with the so-called paravector

$$x = x_0 + \sum_{k=1}^{n} e_k x_k = x_0 + \underline{x} \in \mathcal{A}_n := \operatorname{span}_{\mathbb{R}}\{1, e_1, \ldots, e_n\} \subset \mathcal{C}\ell_{0,n}.$$

The conjugate $\bar{x}$ and the norm $|x|$ of x are given by $\bar{x} = x_0 - \underline{x}$ and $|x| = (x\bar{x})^{1/2} = (\bar{x}x)^{1/2} = \left(\sum_{k=0}^{n} x_k^2\right)^{1/2}$.

The generalized Cauchy-Riemann operator in $\mathbb{R}^{n+1}$ is defined by

$$\overline{\partial} := \frac{1}{2}(\partial_0 + \partial_{\underline{x}}),$$

with $\partial_0 := \frac{\partial}{\partial x_0}$ and $\partial_{\underline{x}} := \sum_{k=1}^{n} e_k \frac{\partial}{\partial x_k}$. The conjugate generalized Cauchy-Riemann operator, also called the hypercomplex differential operator, is denoted by

$$\partial := \frac{1}{2}(\partial_0 - \partial_{\underline{x}}).$$

Functions defined in an open subset $\Omega \subseteq \mathbb{R}^{n+1} \cong \mathcal{A}_n$ with values in the Clifford algebra $\mathcal{C}\ell_{0,n}$ are of the form $f(z) = \sum_A f_A(z) e_A$, with $f_A(z)$ real valued.

A function f is called *left* (*right*) *monogenic* in Ω if it is a solution of the differential equation $\overline{\partial} f = 0$ ($f\overline{\partial} = 0$).

The hypercomplex differentiability as generalization of complex differentiability has to be understood in the following way: a function f defined in an open domain $\Omega \subseteq \mathbb{R}^{n+1}$ is hypercomplex differentiable if it has a uniquely defined areolar derivative f' in each point of Ω. Then, f is real differentiable and $f' = \partial f$. Furthermore, f is hypercomplex differentiable in Ω if and only if f is monogenic. In addition, the monogenicity of f implies that $f' = \partial_0 f = -\partial_{\underline{x}} f$ (see [16, 20]).

Throughout this paper, a sequence of real numbers is denoted by $(a_k)_k$ or simply (a_k). A finite k-tuple of numbers can be seen as an infinite sequence by adding zeros to the end.

We consider particular polynomial sequences $\left(\mathcal{P}_k^n(x)\right)_k$ whose terms are homogeneous monogenic polynomials of degree k, taking their values in $\mathcal{A}_n$, defined by

$$\mathcal{P}_k^n(x) = \sum_{s=0}^{k} T_s^k(n) x^{k-s} \bar{x}^s, \tag{1}$$

where

$$T_s^k(n) = \binom{k}{s} \frac{\left(\frac{n+1}{2}\right)_{k-s} \left(\frac{n-1}{2}\right)_s}{(n)_k}, \tag{2}$$

and $(a)_r$ denotes the Pochhammer symbol, i.e., $(a)_r := \frac{\Gamma(a+r)}{\Gamma(a)}$, $r \geq 1$, and $(a)_0 := 1$. Such polynomials were introduced initially [13, 21] as functions of a paravector variable x and its conjugate $\bar{x}$ and the coefficients $T_s^k(n)$ were calculated in such a way that $(\mathcal{P}_k^n(x))$ is a generalized Appell sequence of monogenic homogeneous polynomials with respect to ∂, i.e.,

1. $\mathcal{P}_0^n(x) = 1$,

2. $\partial \mathcal{P}_k^n(x) = k \mathcal{P}_{k-1}^n(x)$, $k = 1, 2, \ldots$.

In the context of this paper, the representation of the Appell polynomials (1) in the form

$$\mathcal{P}_k^n(x) = \sum_{s=0}^{k} \binom{k}{s} c_s(n)\, x_0^{k-s}\, \underline{x}^s, \tag{3}$$

will play an important role. Such representation has been studied in a series of papers (see, e.g., [13, 14, 22]), where, in particular, it is proved that the coefficients are given by

$$c_s(n) = \begin{cases} \frac{s!!(n-2)!!}{(n+s-1)!!}, & \text{if } s \text{ is odd;} \\ c_{s-1}(n), & \text{if } s \text{ is even,} \end{cases} \tag{4}$$

and satisfy

$$c_k(n) = \sum_{s=0}^{k} (-1)^s T_s^k(n), \ \ k = 0, 1, \ldots. \tag{5}$$

In the relation (4), $n!!$ denotes the usual double factorial of n, which (we recall) is defined as $n!! = n(n-2)!!$ for $n > 1$, and $0!! = 1!! = 1$.

It is also worth mentioning that the numbers $T_s^k(n)$, for each degree k, constitute a partition of unity, i.e.,

$$1 = \sum_{s=0}^{k} T_s^k(n), \ \ k = 0, 1, \ldots, \tag{6}$$

(cf. [14, Thm. 3.7]).

The family of numbers composed by $T_s^k(n)$, can be represented as a triangular table with lines of height $k = 0, 1, \ldots$ and ordered from $s = 0$ up to $s = k$. Table 1 shows its first lines followed by the first terms of the sequences $(c_k(n))$. Table 2 contains the first numbers associated with $(T_s^k(2))$ and $(T_s^k(3))$, while in Table 3 we collect the first elements of the sequences $c_{2k}(n)$, (for $n = 2, \ldots, 7$). In the latter two tables one can see connections between $(c_{2k}(n))$ and $(T_s^k(n))$ with sequences from The On-Line Encyclopedia of Integer Sequences (OEIS) [26].

In the recent work [8], authors of this paper constructed a sequence (m_k) closely related to $(c_k(2))$. Analyzing the sequence $(T_0^k(2))$ (cf. Table 2)

$$1, \tfrac{3}{4}, \tfrac{5}{8}, \tfrac{35}{64}, \tfrac{63}{128}, \tfrac{231}{512}, \tfrac{429}{1024}, \tfrac{6435}{16384}, \tfrac{12155}{32768}, \tfrac{46189}{131072}, \tfrac{88179}{262144}, \ldots$$

one can see that it consists of rational numbers whose denominators are strictly increasing powers of 2. The sequence (m_k) of the corresponding exponents,

$$0, 2, 3, 6, 7, 9, 10, 14, 15, 17, 18, \ldots$$

is called, in [8], the *minimal exponent integer sequence* with respect to 2, since it represents the *least non-negative integer* m_k such that $2^{m_k} T_0^k \in \mathbb{N}$. The general expression of m_k reads as follows:

$$m_k = k + \left\lfloor \tfrac{k+1}{2} \right\rfloor + \left\lfloor \tfrac{k+1}{2^2} \right\rfloor + \cdots + \left\lfloor \tfrac{k+1}{2^m} \right\rfloor,$$

Table 1: The triangles $(T_s^k(n))$ and the sequences $(c_k(n))$

$T_s^k(n)$					
1					
$\frac{n+1}{2n}$	$\frac{n-1}{2n}$				
$\frac{n+3}{4n}$	$\frac{n-1}{2n}$	$\frac{n-1}{4n}$			
$\frac{(n+5)(n+3)}{8n(n+2)}$	$\frac{3(n+3)(n-1)}{8n(n+2)}$	$\frac{3(n^2-1)}{8n(n+2)}$	$\frac{(n+3)(n-1)}{8n(n+2)}$		
$\frac{(n+5)(n+7)}{16n(n+2)}$	$\frac{(n+5)(n-1)}{4n(n+2)}$	$\frac{3(n^2-1)}{8n(n+2)}$	$\frac{(n^2-1)}{4n(n+2)}$	$\frac{(n+5)(n-1)}{16n(n+2)}$	
$\frac{(n+5)(n+7)(n+9)}{32n(n+2)(n+4)}$	$\frac{5(n+7)(n+5)(n-1)}{32n(n+2)(n+4)}$	$\frac{5(n^2-1)(n+5)}{16n(n+2)(n+4)}$	$\frac{5(n^2-1)(n+3)}{16n(n+2)(n+4)}$	$\frac{5(n^2-1)(n+5)}{32n(n+2)(n+4)}$	$\frac{(n+5)(n-1)(n+7)}{32n(n+2)}$
$\vdots$					

$c_k(n) = 1, \frac{1}{n}, \frac{1}{n}, \frac{3}{n^2+2n}, \frac{3}{n^2+2n}, \frac{15}{n(n^2+6n+8)}, \frac{15}{n(n^2+6n+8)}, \frac{105}{n(n+2)(n+4)(n+6)}, \frac{105}{n(n+2)(n+4)(n+6)}, \ldots$

Table 2: First rows of $(T_s^k(2))$ and $(T_s^k(3))$ and the links to OEIS

$n=2$					$n=3$				
1					1				
$\frac{3}{4}$	$\frac{1}{4}$				$\frac{2}{3}$	$\frac{1}{3}$			
$\frac{10}{16}$	$\frac{4}{16}$	$\frac{2}{16}$			$\frac{3}{6}$	$\frac{2}{6}$	$\frac{1}{6}$		
$\frac{35}{64}$	$\frac{15}{64}$	$\frac{9}{64}$	$\frac{5}{64}$		$\frac{4}{10}$	$\frac{3}{10}$	$\frac{2}{10}$	$\frac{1}{10}$	
$\frac{126}{256}$	$\frac{56}{256}$	$\frac{36}{256}$	$\frac{24}{256}$	$\frac{14}{256}$	$\frac{5}{15}$	$\frac{4}{15}$	$\frac{3}{15}$	$\frac{2}{15}$	$\frac{1}{15}$
$\vdots$					$\vdots$				

$T_s^k(2) = \frac{\text{A078817}(k(k+1)/2+s)}{\text{A000302}(k)}$ $T_s^k(3) = \frac{\text{A004736}(k(k+1)/2+s)}{\text{A026741}(k+2)}$

Table 3: First elements of $(c_{2k}(n))$, $(n = 2, \ldots, 7)$ and the links to OEIS

$c_{2k}(2) = 1, \frac{1}{2}, \frac{3}{8}, \frac{5}{16}, \frac{35}{128}, \frac{63}{256}, \frac{231}{1024}, \ldots$	$= \frac{\text{A001790}(k)}{\text{A046161}(k)}$
$c_{2k}(3) = 1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}, \frac{1}{13}, \ldots$	$= \frac{1}{\text{A005408}(k)}$
$c_{2k}(4) = 1, \frac{1}{4}, \frac{1}{8}, \frac{5}{64}, \frac{7}{128}, \frac{21}{512}, \frac{33}{1024}, \ldots$	$= \frac{\text{A098597}(k)}{\text{A120777}(k)}$
$c_{2k}(5) = \frac{3}{3}, \frac{3}{15}, \frac{3}{35}, \frac{3}{63}, \frac{3}{99}, \frac{3}{143}, \frac{3}{195}, \ldots$	$= \frac{3}{\text{A000466}(k+1)}$
$c_{2k}(6) = 1, \frac{1}{6}, \frac{1}{16}, \frac{1}{32}, \frac{7}{384}, \frac{3}{256}, \frac{33}{4096}, \ldots$	$= \frac{\text{A099398}(k)}{\text{A099399}(k)}$
$c_{2k}(7) = \frac{5}{5}, \frac{5}{35}, \frac{5}{105}, \frac{5}{231}, \frac{5}{429}, \frac{5}{715}, \frac{5}{1105}, \ldots$	$= \frac{5}{\text{A162540}(k)}$

with $m \le \log_2(k+1)$, (see [8, Thm. 6] for details). The sequence, now listed in OEIS as A283208, allows to write

$$T_0^k(2) = 2c_{2k+1}(2) = \frac{\text{A001790}(k+1)}{2^{\text{A283208}(k)}}.$$

3 Connection-like identities

Connection-like is here understood as the problem of linking the coefficients $T_s^k(n)$ and $c_s(n)$ associated with the two representations (1) and (3) of the same polynomial sequence $(\mathcal{P}_k^n(x))$. In this section we describe bijections between those coefficients and present corresponding matrix relations.

We let $\mathbf{T}_k(n)$ and $\mathbf{C}_k(n)$ denote the $(k+1)$-dimensional vectors

$$\mathbf{T}_k(n) = \begin{bmatrix} T_0^k(n) & T_1^k(n) & \cdots & T_{k-1}^k(n) & T_k^k(n) \end{bmatrix}^T$$

and

$$\mathbf{C}_k(n) = \begin{bmatrix} c_0(n) & c_1(n) & \cdots & c_{k-1}(n) & c_k(n) \end{bmatrix}^T.$$

We start with a technical result relating the numbers (2) in the row $k-i$ with their distant neighbors in the row k.

Lemma 1. *For any fixed value of $k \ge 1$, the following relation holds:*

$$T_s^{k-i}(n) = \sum_{j=0}^{i} A(k,i,s,j) T_{s+j}^k(n); \; i = 1, \ldots, k, \; s = 0, \ldots, k-i, \tag{7}$$

where

$$A(k,i,s,j) = \binom{i}{j} \frac{(k-i-s+1)_{i-j}(s+1)_j}{(k-i+1)_i}. \tag{8}$$

Proof. The result is true for $i = 1$, since (7) reads as

$$T_s^{k-1}(n) = \tfrac{k-s}{k} T_s^k(n) + \tfrac{s+1}{k} T_{s+1}^k(n), \quad s = 0, \ldots, k-1. \tag{9}$$

This relation between an element in row $k-1$ and two adjacent elements in the row k was already proved in [14, Thm. 3.5].

The repeated use of (9) allows to express an element in row $k-i$ in terms of $i+1$ adjacent elements in the row k. In fact, for $s = 0, \ldots, k-2$ one can write

$$\begin{aligned} T_s^{k-2}(n) &= \tfrac{k-s-1}{k-1} T_s^{k-1}(n) + \tfrac{s+1}{k-1} T_{s+1}^{k-1}(n) \\ &= \tfrac{k-s-1}{k-1}\Big(\tfrac{k-s}{k} T_s^k(n) + \tfrac{s+1}{k} T_{s+1}^k(n)\Big) + \tfrac{s+1}{k-1}\Big(\tfrac{k-s-1}{k} T_{s+1}^k(n) + \tfrac{s+2}{k} T_{s+2}^k(n)\Big) \\ &= \tfrac{(k-s-1)_2}{(k-1)_2} T_s^k(n) + \tfrac{2(k-s-1)(s+1)}{(k-1)_2} T_{s+1}^k(n) + \tfrac{(s+1)_2}{(k-1)_2} T_{s+2}^k(n). \end{aligned}$$

Applying the same reasoning we have

$$\begin{aligned} T_s^{k-3}(n) &= \tfrac{(k-s-2)_3}{(k-2)_3} T_s^k(n) + \tfrac{3(k-s-2)_2(s+1)}{(k-2)_3} T_{s+1}^k(n) + \tfrac{3(k-s-2)(s+1)_2}{(k-2)_3} T_{s+2}^k(n) + \tfrac{(s+1)_3}{(k-2)_3} T_{s+3}^k(n) \\ &\;\vdots \\ T_s^{k-i}(n) &= \sum_{j=0}^{i} \tbinom{i}{j} \tfrac{(k-s-i+1)_{i-j}(1+s)_j}{(k-i+1)_i} T_{s+j}^k(n) \end{aligned}$$

and the result is proved. $\square$

The next result is a generalization to arbitrary dimensions of Theorem 2 in [21] (see also [12]).

Theorem 2. *The components $T_s^k(n)$ and $T_s^{k-1}(n)$ of the vectors $\mathbf{T}_k(n)$ and $\mathbf{T}_{k-1}(n)$, respectively, satisfy the $(k+1)\times(k+1)$ system of algebraic equations represented in matrix form as*

$$M_k \mathbf{T}_k(n) = \left[\begin{array}{c} k\mathbf{T}_{k-1}(n) \\ \hdashline c_k(n) \end{array}\right],$$

where M_k is given by

$$M_k := \begin{bmatrix} k & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & k-1 & 2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & k-2 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & k-1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & k \\ 1 & -1 & 1 & -1 & \cdots & (-1)^{k-1} & (-1)^k \end{bmatrix}, \quad k \geq 1. \tag{10}$$

Proof. It is a simple consequence of relation (9) together with the alternating sum formula (5) for $c_k(n)$. □

Lemma 3. *The matrix* M_k $(k = 1, 2, \dots)$ *in* (10) *is nonsingular and*

$$\det M_k = (-2)^k k!.$$

Proof. Developing the determinant along the last row, we obtain

$$\begin{aligned}\det M_k =&(-1)^{k+2}k! - (-1)^{k+3}k\tfrac{k!}{1!} + (-1)^{k+4}k(k-1)\tfrac{k!}{2!} - (-1)^{k+5}k(k-1)(k-2)\tfrac{k!}{3!} \\ &+ \cdots + (-1)^k k(k-1)(k-2)\cdots 1\tfrac{k!}{k!} \\ =&(-1)^k k!\left(1 + \tfrac{k}{1!} + \tfrac{k(k-1)}{2!} + \tfrac{k(k-1)(k-2)}{3!} + \cdots + \tfrac{k!}{k!}\right) \\ =&(-1)^k k! \sum_{j=0}^{k} \binom{k}{j} = (-1)^k k! 2^k.\end{aligned}$$

□

Remark 4. We point out that $\det M_k = (-1)^k$ A000165(k).

As a consequence of the previous results one can derive recursive matrix relations between the vectors $\mathbf{T}_k(n)$ and $\mathbf{C}_k(n)$.

Theorem 5. *The values of* $T_s^k(n)$ *and* $c_s(n)$, $k = 0, 1, \dots,$ $s = 0, 1, \dots, k$, *are related by the following matrix relation*

$$\mathbf{T}_k(n) = N_k \mathbf{C}_k(n), \tag{11}$$

where N_k *is recursively defined by*

$$\begin{aligned} N_0 =&1 \\ N_k =&M_k^{-1} \left[\begin{array}{c:c} kN_{k-1} & \mathbf{0} \\ \hdashline \mathbf{0} & 1 \end{array}\right], \quad k = 1, 2, \dots. \end{aligned} \tag{12}$$

Moreover, if we denote by $\widetilde{N}_k$ *the matrix* N_k^{-1} *then we have*

$$\mathbf{C}_k(n) = \widetilde{N}_k \mathbf{T}_k(n), \tag{13}$$

where

$$\begin{aligned} \widetilde{N}_0 =&1 \\ \widetilde{N}_k =& \left[\begin{array}{c:c} \frac{1}{k}\widetilde{N}_{k-1} & \mathbf{0} \\ \hdashline \mathbf{0} & 1 \end{array}\right] M_k, \quad k = 1, 2, \dots. \end{aligned} \tag{14}$$

Proof. To prove the relations (11)-(12) we use an inductive process on k. Taking into account Theorem 2 and the fact that $T_0^0(n) = 1 = c_0(n)$, one can write, for $k = 1$,

$$\mathbf{T}_1(n) = M_1^{-1}\left[\begin{array}{c}\mathbf{T}_0(n)\\ \hdashline c_1(n)\end{array}\right] = M_1^{-1}\left[\begin{array}{c:c}1 & 0\\ \hdashline 0 & 1\end{array}\right]\left[\begin{array}{c}c_0(n)\\ \hdashline c_1(n)\end{array}\right] = N_1\mathbf{C}_1(n),$$

with $N_1 = M_1^{-1}\left[\begin{array}{c:c}1N_0 & 0\\ \hdashline 0 & 1\end{array}\right]$.

Now assume that $\mathbf{T}_k(n) = N_k\mathbf{C}_k(n)$ holds for k. For $k+1$ one has

$$\begin{aligned}\mathbf{T}_{k+1}(n) &= M_{k+1}^{-1}\left[\begin{array}{c}(k+1)\mathbf{T}_k(n)\\ \hdashline c_{k+1}(n)\end{array}\right] = M_{k+1}^{-1}\left[\begin{array}{c}(k+1)N_k\mathbf{C}_k(n)\\ \hdashline c_{k+1}(n)\end{array}\right]\\ &= M_{k+1}^{-1}\left[\begin{array}{c:c}(k+1)N_k & \mathbf{0}\\ \hdashline \mathbf{0} & 1\end{array}\right]\left[\begin{array}{c}\mathbf{C}_k(n)\\ \hdashline c_{k+1}(n)\end{array}\right] = N_{k+1}\mathbf{C}_{k+1}(n),\end{aligned}$$

with $N_{k+1} = M_{k+1}^{-1}\left[\begin{array}{c:c}(k+1)N_k & \mathbf{0}\\ \hdashline \mathbf{0} & 1\end{array}\right]$.

To prove the relations (13)-(14), observe that N_k is invertible, since

$$\begin{aligned}\det N_k =& k^k \det M_k^{-1} \det N_{k-1} = \tfrac{k^k}{k!(-2)^k} \det N_{k-1}\\ =&(-\tfrac{k}{2})^k \tfrac{1}{k!}(-\tfrac{k-1}{2})^{k-1}\tfrac{1}{(k-1)!}\det N_{k-2} = \cdots = \prod_{m=1}^{k}(-\tfrac{m}{2})^m \tfrac{1}{m!}.\end{aligned}$$

The inversion of N_k leads to the desired result. □

The next result allows to express each component of the vector $\mathbf{T}_k(n)$ as a linear combination of $c_0(n), \ldots, c_k(n)$.

Theorem 6. *The numbers $T_s^k(n)$, $k = 0, 1, \ldots,$ $s = 0, 1, \ldots, k,$ can be written as*

$$T_s^k(n) = \tfrac{1}{2^k}\binom{k}{s}\sum_{j=0}^{k}\sigma_{s,j}^k c_j(n), \tag{15}$$

where

$$\sigma_{s,j}^k = \sum_{m=0}^{s}(-1)^m\binom{s}{m}\binom{k-s}{j-m}. \tag{16}$$

Proof. Replacing $x_0 = \frac{x+\bar{x}}{2}$ and $\underline{x} = \frac{x-\bar{x}}{2}$ in (3) we obtain successively[1]

$$\begin{aligned}
\mathcal{P}_k^n(x) =& \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j} c_j(n)\,(x+\bar{x})^{k-j}\,(x-\bar{x})^j\\
=& \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j} c_j(n)\left(\sum_{t=0}^{k-j}\binom{k-j}{t}x^{k-j-t}\bar{x}^t\right)\left(\sum_{l=0}^{j}(-1)^l\binom{j}{l}x^{j-l}\bar{x}^l\right)\\
=& \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j} c_j(n)\,\sum_{s=0}^{k}\sum_{m=0}^{s}(-1)^{s-m}\binom{k-j}{m}\binom{j}{s-m}x^{k-s}\bar{x}^s\\
=& \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j} c_j(n)\,\sum_{s=0}^{k}\sum_{m=0}^{s}(-1)^{m}\binom{k-j}{s-m}\binom{j}{m}x^{k-s}\bar{x}^s.
\end{aligned}$$

By comparing last expression with (1) we obtain

$$T_s^k(n) = \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}c_j(n)\sum_{m=0}^{s}(-1)^m\binom{k-j}{s-m}\binom{j}{m}.$$

The result follows, since $\binom{k}{j}\binom{k-j}{s-m}\binom{j}{m} = \binom{k}{s}\binom{s}{m}\binom{k-s}{j-m}$. □

The explicit expression of each $c_j(n)$, $j = 0,\ldots,k$, as a linear combination of $T_s^k(n)$, $s = 0,\ldots,k$, reads as follows.

Theorem 7. *For any fixed value of k, the coefficients $c_{k-i}(n)$ can be written as*

$$c_{k-i}(n) = \frac{1}{\binom{k}{i}}\sum_{s=0}^{k}(-1)^s\sigma_{s,i}^k T_s^k(n),\;\; i = 0,1,\ldots,k. \tag{17}$$

Proof. Using Lemma 1 together with (5) we obtain

$$\begin{aligned}
c_{k-i}(n) &= \sum_{s=0}^{k-i}(-1)^s T_s^{k-i}(n) = \sum_{s=0}^{k-i}(-1)^s\sum_{j=0}^{i}A(k,i,s,j)T_{s+j}^k\\
&= \sum_{j=0}^{i}A(k,i,0,j)T_j^k - \sum_{j=1}^{i}A(k,i,1,j-1)T_j^k + \cdots + (-1)^{k-i}\sum_{j=k-i}^{k}A(k,i,k-i,j-k+i)T_j^k.
\end{aligned}$$

Observe that if we allow in the expression (8) of $A(k,i,s,j)$ s and j to be arbitrary integers, then $A(k,i,s,j)$ vanishes whenever $j > i$ or $j < 0$ or $k \le s+i-1$. Therefore we can write

$$\begin{aligned}
c_{k-i}(n) &= \sum_{j=0}^{k}A(k,i,0,j)T_j^k - \sum_{j=0}^{k}A(k,i,1,j-1)T_j^k + \cdots + (-1)^{k-i}\sum_{j=0}^{k}A(k,i,k-i,j-k+i)T_j^k\\
&= \sum_{s=0}^{k}\sum_{m=0}^{k-i}(-1)^m A(k,i,m,s-m)T_s^k(n)
\end{aligned}$$

[1]For convenience we include binomial coefficients $\binom{n}{k}$ for nonnegative integer n and integer k such that $k < 0$ or $k > n$, which as usually are considered to be zero.

or, equivalently

$$c_{k-i}(n)=\begin{cases}\sum_{s=0}^{k}\Big(\sum_{m=0}^{s}(-1)^m A(k,i,m,s-m)+\sum_{m=s+1}^{k-i}(-1)^m A(k,i,m,s-m)\Big)T_s^k(n), & \text{if } k-i\geq s;\\ \sum_{s=0}^{k}\Big(\sum_{m=0}^{s}(-1)^m A(k,i,m,s-m)-\sum_{m=k-i+1}^{s}(-1)^m A(k,i,m,s-m)\Big)T_s^k(n), & \text{if } k-i<s.\end{cases}$$

Since $A(k,i,m,s-m)=0$, for $m>s$, the sum $\sum_{m=s+1}^{k-i}(-1)^m A(k,i,m,s-m)$ is zero. On the other hand, we also have $A(k,i,m,s-m)=0$ in $\sum_{m=k-i+1}^{s}(-1)^m A(k,i,m,s-m)$, since $k\leq m+i-1$. We have proved that

$$c_{k-i}(n)=\sum_{s=0}^{k}\vartheta_{i,s}^k T_s^k(n),\ \ i=0,\dots,k,$$

where

$$\vartheta_{i,s}^k=\sum_{m=0}^{s}(-1)^m A(k,i,m,s-m),\ \ i=0,\dots,k.$$

The use of (8) allows to write

$$\begin{aligned}\vartheta_{i,s}^k&=\sum_{m=0}^{s}(-1)^m\tfrac{i!}{(i-s+m)!(s-m)!}\tfrac{(k-i-m+1)\dots(k-s)(m+1)\dots s}{(k-i+1)\dots k}\\&=\sum_{m=0}^{s}(-1)^m\tfrac{i!(k-i)!}{k!}\tfrac{(k-s)!}{(i-s+m)!(k-i-m)!}\tfrac{s!}{(s-m)!m!}=\tfrac{(-1)^s}{\binom{k}{i}}\sigma_{s,i}^k\end{aligned}$$

(cf. (16)) and the result follows. □

Remark 8. Since $\sigma_{s,k}^k=(-1)^s$, Theorem 7 contains as a particular case ($i=k$) the already known property (6).

Theorem 5 can be used together with Theorems 6 and 7 to obtain the main result of this section: the explicit form of the matrix that relates vectors $\mathbf{T}_k(n)$ and $\mathbf{C}_k(n)$.

Theorem 9. *Consider the coefficients $\sigma_{s,j}^k$ introduced in (16). Then the following relations are true:*

i) $\mathbf{T}_k(n)=N_k\mathbf{C}_k(n)$, *where N_k is the $(k+1)\times(k+1)$ matrix whose elements n_{ij}^k are*

$$n_{ij}^k=\frac{1}{2^k}\binom{k}{i-1}\sigma_{i-1,j-1}^k,\quad i,j=1,\dots,k+1.$$

ii) $\mathbf{C}_k(n)=\widetilde{N}_k\mathbf{T}_k(n)$, *where $\widetilde{N}_k$ is the $(k+1)\times(k+1)$ matrix whose elements $\widetilde{n}_{ij}^k$ are*

$$\widetilde{n}_{ij}^k=\tfrac{(-1)^{j-1}}{\binom{k}{k-i+1}}\sigma_{j-1,k-i+1}^k,\quad i,j=1,\dots,k+1.$$

Proof. Observe that Theorem 6 can be expressed in matrix form as

$$\mathbf{T}_k(n) = \tfrac{1}{2^k} S_k \mathbf{C}_k(n),$$

where S_k is the $(k+1) \times (k+1)$ matrix whose elements s_{ij} are such that

$$s_{ij} = \binom{k}{i-1} \sigma^k_{i-1,j-1}, \; i,j = 1, \dots, k+1.$$

Result i) follows at once from (11), while result ii) is a simple consequence of Theorem 7. □

The last theorem points out that the connection between $\mathbf{T}_k(n)$ and $\mathbf{C}_k(n)$ is provided by the matrices N_k and $\widetilde{N}_k$. However, a deeper observation permits to recognize that such connection is actually done by the coefficients $\sigma^k_{s,j}$, since they are the common factors in the coefficients of linear combinations (15)-(17). Examples of matrices $N_k, \widetilde{N}_k$, and vectors $\mathbf{T}_k(n), \mathbf{C}_k(n),\ k = 1, \dots, 4$, are given in Table 4.

4 A family of Pascal trapezoids

In addition to providing the link between $T^k_s(n)$ and $c_s(n)$, the coefficients $\sigma^k_{i,j}$ (cf. (16)) in the identities (15) and (17) have several other interesting properties. In fact, considering i, j, k arbitrary nonnegative integers such that $j \leq k$, the numbers $\sigma^k_{i,j}$ can be represented, for each fixed i, as a triangle with rows k ($k = 0, 1, \dots$) and ordered from $j = 0$ to $j = k$, as illustrated in Table 5. This section is dedicated to the study of various patterns in such structure and to the discussion of properties and combinatorial identities.

We observe that

1. for $i = 0$, we obtain the Pascal triangle, since $\sigma^k_{0,j} = \binom{k}{j}$, (see first triangle in Table 5);
2. for $i = 1$, $\sigma^k_{1,j} = \binom{k-1}{j} - \binom{k-1}{j-1} = \frac{k-2j}{k}\binom{k}{j}$, which corresponds to the Catalan triangle numbers $C_{k,j} = \frac{k-2j}{k}\binom{k}{j}$ mentioned by Miana et al. [24].

In general, for each fixed value of i, $\sigma^k_{i,j}$ can be obtained recursively, as stated in the following result.

Theorem 10 (Pascal recurrence)**.** *For each fixed value of i, the numbers $\sigma^k_{i,j}$ satisfy the following linear recurrence relation*

$$\sigma^{(k+1)}_{i,j+1} = \sigma^k_{i,j} + \sigma^k_{i,j+1}, \; (0 \leq j \leq k-1, k \geq i) \tag{18}$$

Table 4: The matrices N_k, $\widetilde{N}_k = N_k^{-1}$, and the vectors $\mathbf{T}_k(n)$, $\mathbf{C}_k(n)$, $(k = 1, \ldots, 4)$

k	N_k	$\widetilde{N}_k$	$\mathbf{T}_k$	$\mathbf{C}_k$
1	$\frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$	$\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$	$\begin{bmatrix}\frac{n+1}{2n}\\ \frac{n-1}{2n}\end{bmatrix}$	$\begin{bmatrix}1\\ \frac{1}{n}\end{bmatrix}$
2	$\frac{1}{2^2}\begin{bmatrix}1 & 2 & 1\\ 2 & 0 & -2\\ 1 & -2 & 1\end{bmatrix}$	$\begin{bmatrix}1 & 1 & 1\\ 1 & 0 & -1\\ 1 & -1 & 1\end{bmatrix}$	$\begin{bmatrix}\frac{n+3}{4n}\\ \frac{n-1}{4n}\\ \frac{n-1}{2n}\end{bmatrix}$	$\begin{bmatrix}1\\ \frac{1}{n}\\ \frac{1}{n}\end{bmatrix}$
3	$\frac{1}{2^3}\begin{bmatrix}1 & 3 & 3 & 1\\ 3 & 3 & -3 & -3\\ 3 & -3 & -3 & 3\\ 1 & -3 & 3 & -1\end{bmatrix}$	$\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & \frac{1}{3} & -\frac{1}{3} & -1\\ 1 & -\frac{1}{3} & -\frac{1}{3} & 1\\ 1 & -1 & 1 & -1\end{bmatrix}$	$\begin{bmatrix}\frac{(3+n)(5+n))}{8n(2+n)}\\ \frac{3(-1+n)(3+n)}{8n(2+n)}\\ \frac{3(-1+n)(1+n)}{8n(2+n)}\\ \frac{(-1+n)(3+n)}{8n(2+n)}\end{bmatrix}$	$\begin{bmatrix}1\\ \frac{1}{n}\\ \frac{1}{n}\\ \frac{3}{n(n+2)}\end{bmatrix}$
	$\frac{1}{2^4}\begin{bmatrix}1 & 4 & 6 & 4 & 1\\ 4 & 8 & 0 & -8 & -4\\ 6 & 0 & -12 & 0 & 6\\ 4 & -8 & 0 & 8 & -4\\ 1 & -4 & 6 & -4 & 1\end{bmatrix}$	$\begin{bmatrix}1 & 1 & 1 & 1 & 1\\ 1 & \frac{1}{2} & 0 & -\frac{1}{2} & -1\\ 1 & 0 & -\frac{1}{3} & 0 & 1\\ 1 & -\frac{1}{2} & 0 & \frac{1}{2} & -1\\ 1 & -1 & 1 & -1 & 1\end{bmatrix}$	$\begin{bmatrix}\frac{(n+5)(n+7)}{16n(n+2)}\\ \frac{(n-1)(n+5)}{4n(n+2)}\\ \frac{3(n-1)(n+1)}{8n(n+2)}\\ \frac{(n-1)(n+1)}{4n(n+2)}\\ \frac{(n-1)(n+5)}{16n(n+2)}\end{bmatrix}$	$\begin{bmatrix}1\\ \frac{1}{n}\\ \frac{1}{n}\\ \frac{3}{n(n+2)}\\ \frac{3}{n(n+2)}\end{bmatrix}$

with boundary conditions

$$\sigma_{i,0}^k = 1, \qquad \sigma_{i,k}^k = (-1)^i, \ \ (k \geq i) \tag{19}$$

and initial values

$$\sigma_{i,j}^{(i)} = \binom{i}{j}(-1)^j, \ \ j = 1, \ldots, i-1. \tag{20}$$

Proof. When $k \geq i$ and $0 \leq j \leq k-1$ we immediately obtain from the definition of $\sigma_{i,j}^k$ that (cf. (16))

$$\sigma_{i,j}^k + \sigma_{i,j+1}^k = \sum_{m=0}^{i}(-1)^{i-m}\binom{i}{m}\left(\binom{k-i}{j-m} + \binom{k-i}{j+1-m}\right) = \sum_{m=0}^{i}(-1)^{i-m}\binom{i}{m}\binom{k-i+1}{j+1-m} = \sigma_{i,j+1}^{(k+1)}.$$

When $j = 0$ the sum in the right-hand side of (16) reduces to the case $m = 0$ whereas when $j = k$ the corresponding sum reduces to the case $m = i$. Therefore

$$\sigma_{i,0}^k = 1 \qquad \text{and} \qquad \sigma_{i,k}^k = (-1)^i.$$

Table 5: Triangles associated with $\sigma^{k}_{i,j}$, $(i = 0, \ldots, 3)$

$$\begin{array}{c}
\boxed{1} \\
\boxed{1} \quad \boxed{1} \\
\boxed{1} \quad 2 \quad \boxed{1} \\
\boxed{1} \quad 3 \quad 3 \quad \boxed{1} \\
\boxed{1} \quad 4 \quad 6 \quad 4 \quad \boxed{1} \\
\boxed{1} \quad 5 \quad 10 \quad 10 \quad 5 \quad \boxed{1} \\
\boxed{1} \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad \boxed{1} \\
\boxed{1} \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad \boxed{1} \\
\iddots \quad \iddots \quad \iddots \quad \iddots \quad \ddots \quad \ddots \quad \ddots \quad \ddots
\end{array}$$

i=0

$$\begin{array}{c}
0 \\
\boxed{1} \quad \boxed{-1} \\
\boxed{1} \quad 0 \quad \boxed{-1} \\
\boxed{1} \quad 1 \quad -1 \quad \boxed{-1} \\
\boxed{1} \quad 2 \quad 0 \quad -2 \quad \boxed{-1} \\
\boxed{1} \quad 3 \quad 2 \quad -2 \quad -3 \quad \boxed{-1} \\
\boxed{1} \quad 4 \quad 5 \quad 0 \quad -5 \quad -4 \quad \boxed{-1} \\
\boxed{1} \quad 5 \quad 9 \quad 5 \quad -5 \quad -9 \quad -5 \quad \boxed{-1} \\
\iddots \quad \iddots \quad \iddots \quad \iddots \quad \ddots \quad \ddots \quad \ddots \quad \ddots
\end{array}$$

i=1

$$\begin{array}{c}
2 \\
-2 \quad -2 \\
\boxed{1} \quad \boxed{-2} \quad \boxed{1} \\
\boxed{1} \quad -1 \quad -1 \quad \boxed{1} \\
\boxed{1} \quad 0 \quad -2 \quad 0 \quad \boxed{1} \\
\boxed{1} \quad 1 \quad -2 \quad -2 \quad 1 \quad \boxed{1} \\
\boxed{1} \quad 2 \quad -1 \quad -4 \quad -1 \quad 2 \quad \boxed{1} \\
\boxed{1} \quad 3 \quad 1 \quad -5 \quad -5 \quad 1 \quad 3 \quad \boxed{1} \\
\iddots \quad \iddots \quad \iddots \quad \iddots \quad \ddots \quad \ddots \quad \ddots \quad \ddots
\end{array}$$

i=2

$$\begin{array}{c}
0 \\
6 \quad -6 \\
-6 \quad 6 \quad -6 \\
\boxed{1} \quad \boxed{-3} \quad \boxed{3} \quad \boxed{-1} \\
\boxed{1} \quad -2 \quad 0 \quad 2 \quad \boxed{-1} \\
\boxed{1} \quad -1 \quad -2 \quad 2 \quad 1 \quad \boxed{-1} \\
\boxed{1} \quad 0 \quad -3 \quad 0 \quad 3 \quad 0 \quad \boxed{-1} \\
\boxed{1} \quad 1 \quad -3 \quad -3 \quad 3 \quad 3 \quad -1 \quad \boxed{-1} \\
\iddots \quad \iddots \quad \iddots \quad \iddots \quad \ddots \quad \ddots \quad \ddots \quad \ddots
\end{array}$$

i=3

On the other hand, if $k = i$ the same sum only has the term corresponding to $m = j$. Hence

$$\sigma^{(i)}_{i,j} = \binom{i}{j}(-1)^j, \ j = 0, \ldots, i.$$

□

Theorem 10 supports the idea that the above recurrence relation together with the boundary conditions (19) and initial conditions (20) lead, for each i, to a Pascal trapezoid which can be seen as a substructure of the triangle array $\sigma^{k}_{i,j}$. In Table 5, we can observe the trapezoids corresponding to the particular cases of $i = 1$, $i = 2$, and $i = 3$ as well as the boundary and initial conditions. These trapezoids are related to the sequences A037012, A182533, and A230206.

In the context of this paper, we use the designation of *Pascal trapezoid of order* $i = 1, 2, \ldots$ to refer to the trapezoidal array whose entries are $\sigma^k_{i,j}$, $k \geq i, 0 \leq j \leq k$. The following properties can easily be derived.

Property 11.

(i) Pascal trapezoids of even order are symmetric, i.e., $\sigma^k_{2i,j} = \sigma^k_{2i,k-j}$;

(ii) Pascal trapezoids of odd order are anti-symmetric, i.e., $\sigma^k_{2i+1,j} = -\sigma^k_{2i+1,k-j}$;

(iii) The Hockey-stick identity is valid, i.e., $\sum_{r=j}^{k} \sigma^r_{i,j} = \sigma^{k+1}_{i,j+1}$.

Proof. From (16) we have

$$\sigma^k_{i,k-j} = \sum_{m=0}^{i} (-1)^m \binom{i}{m}\binom{k-i}{k-j-m} = \sum_{m=0}^{i} (-1)^m \binom{i}{m}\binom{k-i}{j+m-i} = \sum_{m=0}^{i} (-1)^{i-m} \binom{i}{m}\binom{k-i}{j-m} = (-1)^i \sigma^k_{i,j}$$

and (i) and (ii) are proved. According to (18)-(19),

$$\begin{aligned}\sum_{r=j}^{k} \sigma^r_{i,j} &= \sigma^{(j)}_{i,j} + \sum_{r=j+1}^{k} \sigma^r_{i,j} = (-1)^i + \sum_{r=j+1}^{k} (\sigma^{(r+1)}_{i,j+1} - \sigma^r_{i,j+1}) \\ &= \sum_{r=j+1}^{k} \sigma^{(r+1)}_{i,j+1} - \sum_{r=j+1}^{k-1} \sigma^{(r+1)}_{i,j+1} = \sigma^{k+1}_{i,j+1},\end{aligned}$$

and the proof of (iii) is complete. □

Property 12. The central coefficients of the Pascal trapezoid of order i are linked to the central binomial coefficients in the following way:

$$\sigma^{2k}_{i,k} = \begin{cases} 0, & \text{if } i \text{ is odd;} \\ (-1)^r \dfrac{\binom{k}{r}}{\binom{2k}{2r}} \dbinom{2k}{k}, & \text{if } i = 2r \text{ is even.} \end{cases}$$

Proof. Observe that

$$\sigma^{2k}_{i,k} = \sum_{m=0}^{i} (-1)^m \binom{i}{m}\binom{2k-i}{k-m} = \frac{\binom{2k}{k}}{\binom{2k}{i}} \sum_{m=0}^{i} (-1)^m \binom{k}{m}\binom{k}{i-m}$$

and the result follows by using the identity [15, Formula (1.19)]

$$\sum_{m=0}^{i} (-1)^m \binom{k}{m}\binom{k}{i-m} = \begin{cases} 0, & \text{if } i \text{ is odd;} \\ (-1)^{\frac{i}{2}} \dbinom{k}{\frac{i}{2}}, & \text{if } i \text{ is even.} \end{cases}$$

It is worth noting that the odd case also follows from Theorem 6. □

Observe that $\sigma^k_{i,j}$ can also be written as (cf. (16))

$$\sigma^k_{i,j} = \sum_{m=0}^{j} (-1)^m \binom{i}{m} \binom{k-i}{j-m},$$

since $\binom{k-i}{j-m} = 0$, for $m > j$, whereas $\binom{i}{m} = 0$, for $m > i$. This leads to the conclusion that the entries of the Pascal trapezoid of order i are nothing else than the coefficients in the expansion of $(1-x)^i(1+x)^{k-i}$, i.e.,

$$(1-x)^i(1+x)^{k-i} = \sum_{j=0}^{k} \sigma^k_{i,j} x^j \tag{21}$$

which means that the family of Pascal trapezoids also contains as particular cases the sequences A230207 up to A230212. Moreover, one can deduce the following properties.

Property 13.

(i) Sum of the rows: $\sum_{j=0}^{k} \sigma^k_{i,j} = 0$, for $k \geq i > 0$;

(ii) Alternating sum of the rows: $\sum_{j=0}^{k} (-1)^j \sigma^k_{i,j} = 0$, for $k > i$;

(iii) Chu-Vandermonde identity: $\sum_{m=0}^{j} \sigma^{k_1}_{i,m} \sigma^{k_2}_{i,j-m} = \sigma^{k_1+k_2}_{2i,j}$;

(iv) Sum of the squares of the rows: $\sum_{j=0}^{k} (\sigma^k_{i,j})^2 = (-1)^i \sigma^{2k}_{2i,k}$.

Proof. The first two properties follow at once by considering $x = 1$ and $x = -1$ in (21). The Chu-Vandermonde formula follows by noting that

$$\big((1-x)^i(1+x)^{k_1-i}\big)\big((1-x)^i(1+x)^{k_2-i}\big) = (1-x)^{2i}(1+x)^{k_1+k_2-2i}$$

and equating the coefficients in

$$\Big(\sum_{j=0}^{k_1} \sigma^{k_1}_{i,j} x^j\Big)\Big(\sum_{j=0}^{k_2} \sigma^{k_2}_{i,j} x^j\Big) = \sum_{j=0}^{k_1+k_2} \sigma^{k_1+k_2}_{2i,j} x^j.$$

Property (iv) follows from (iii) and using Property 11 (i) or (ii). □

5 Acknowledgments

The work of the third author was supported by Portuguese funds through the CMAT — Centre of Mathematics and FCT within the Project UID/MAT/00013/2013. The work of the other authors was supported by Portuguese funds through the CIDMA — Center for Research and Development in Mathematics and Applications, and the Portuguese Foundation for Science and Technology ("FCT-Fundação para a Ciência e Tecnologia"), within project PEst-OE/MAT/UI4106/2013.

References

[1] L. Aceto, H. R. Malonek, and G. Tomaz, Matrix approach to hypercomplex Appell polynomials, *Appl. Numer. Math.* **116** (2017), 2–9.

[2] S. Bock and K. Gürlebeck, On a generalized Appell system and monogenic power series, *Math. Methods Appl. Sci.* **33** (2010), 394–411.

[3] I. Cação and H. R. Malonek, On complete sets of hypercomplex Appell polynomials, in *Numerical Analysis and Applied Mathematics: ICNAAM 2008*, AIP Conference Proceedings, Vol. 1048, 2008, pp. 647–650.

[4] I. Cação, M. I. Falcão, and H. R. Malonek, Laguerre derivative and monogenic Laguerre polynomials: An operational approach, *Math. Comput. Model.* **53** (2011), 1084–1094.

[5] I. Cação and H. R. Malonek, On a hypercomplex generalization of Gould-Hopper and related Chebyshev polynomials, in *Computational Science and its Applications: ICCSA 2011*, Lect. Notes in Comp. Sci., Vol. 6784, Springer, 2011, pp. 316–326.

[6] I. Cação, M. I. Falcão, and H. R. Malonek, Matrix representations of a special polynomial sequence in arbitrary dimension, *Comput. Methods Funct. Theory* **12** (2012), 371–391.

[7] I. Cação, M. I. Falcão, and H. R. Malonek, Three-term recurrence relations for systems of Clifford algebra-valued orthogonal polynomials, *Adv. Appl. Clifford Algebr.* **27** (2017), 71–85.

[8] I. Cação, M. I. Falcão, and H. R. Malonek, Hypercomplex polynomials, Vietoris' rational numbers and a related integer numbers sequence, *Complex Anal. Oper. Theory* **11** (2017), 1059–1076.

[9] C. Cruz, M. I. Falcão, and H. R. Malonek, 3D mappings by generalized Joukowski transformations, in *Computational Science and its Applications: ICCSA 2011*, Lect. Notes in Comp. Sci., Vol. 6784, Springer, 2011, pp. 358–373.

[10] C. Cruz, M. I. Falcão, and H. R. Malonek, Monogenic pseudo-complex power functions and their applications, *Math. Methods Appl. Sci.* **37** (2014), 1723–1735.

[11] D. Eelbode, Monogenic Appell sets as representations of Heisenberg algebra, *Adv. Appl. Clifford Algebr.* **22** (2012), 1009–1023.

[12] M. I. Falcão, J. Cruz, and H. R. Malonek, Remarks on the generation of monogenic functions, in *17th Inter. Conf. on the Appl. of Computer Science and Mathematics on Architecture and Civil Engineering*, Weimar, Germany, 2006, https://e-pub.uni-weimar.de/opus4/files/2939/f81_pdfa.pdf

[13] M. I. Falcão and H. R. Malonek, Generalized exponentials through Appell sets in $\mathbb{R}^{n+1}$ and Bessel functions, in *Numerical Analysis and Applied Mathematics: ICNAAM 2007*, AIP Conference Proceedings, Vol. 936, 2007, pp. 738–741.

[14] M. I. Falcão and H. R. Malonek, A note on a one-parameter family of non-symmetric number triangles, *Opuscula Math.* **32** (2012), 661–673.

[15] H. G. Gould, Combinatorial Identities: Table I: Intermediate Techniques for Summing Finite Series, http://www.math.wvu.edu/~gould/Vol.4.PDF.

[16] K. Gürlebeck and H. R. Malonek, A hypercomplex derivative of monogenic functions in $\mathbb{R}^{m+1}$ and its applications, *Complex Variables* **39** (1999), 199–228.

[17] N. Gürlebeck, On Appell sets and the Fueter-Sce mapping, *Adv. Appl. Clifford Algebr.* **19** (1999), 51–61.

[18] R. Lávička, Canonical bases for $\mathfrak{sl}(2,\mathbb{C})$-modules of spherical monogenics in dimension 3, *Arch. Math.* **46** (2010), 339–349.

[19] R. Lávička, Complete orthogonal Appell systems for spherical monogenics, *Complex Anal. Oper. Theory* **6** (2012), 477–480.

[20] H. R. Malonek, Selected topics in hypercomplex function theory, in S.-L. Eriksson, ed., *Clifford Algebras and Potential Theory*, Vol. 7, 2004, pp. 111–150.

[21] H. R. Malonek and M. I. Falcão, Special monogenic polynomials - properties and applications, in *Numerical Analysis and Applied Mathematics: ICNAAM 2007*, AIP Conference Proceedings, Vol. 936, 2007, pp. 764–767.

[22] H. R. Malonek and M. I. Falcão, On special functions in the context of Clifford analysis, in *Numerical Analysis and Applied Mathematics: ICNAAM 2010*, AIP Conference Proceedings, Vol. 1281, 2010, pp. 1492–1495.

[23] H. R. Malonek and M. I. Falcão, Linking Clifford analysis and combinatorics through bijective methods, in *9th Inter. Conf. on Clifford Algebras and their Applications in Mathematical Physics*, Weimar, Germany, 2011.

[24] P. J. Miana, H. Ohtsuka, and N. Romero, Sums of powers of Catalan triangle numbers, *Discrete Math.* **340** (2017), 2388–2397.

[25] J. Morais and H. T. Le, Orthogonal Appell systems of monogenic functions in the cylinder, *Math. Methods Appl. Sci.* **34** (2011), 1472–1486.

[26] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, http://oeis.org.

2010 *Mathematics Subject Classification*: Primary 05A10; Secondary 05A19, 11B83, 30G35.
Keywords: binomial coefficient, special sequence, hypercomplex Appell polynomial.

(Concerned with sequences A000165, A000302, A000466, A001790, A004736, A005408, A026741, A037012, A046161, A078817, A098597, A099398, A099399, A120777, A162540, A182533, A230206, A230207, A230208, A230209, A230210, A230211, A230212, and A283208.)

Received July 21 2018; revised version received August 6 2018. Published in *Journal of Integer Sequences*, August 23 2018.

Return to Journal of Integer Sequences home page.