Journal of Integer Sequences, Vol. 21 (2018),
Article 18.3.2

On the Reciprocal Sums of Products of Fibonacci Numbers

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Abstract

In this paper we study the reciprocal sums of products of two different Fibonacci numbers. We obtain some identities related to the numbers $\lfloor (\sum_{k=n}^{\infty} 1/F_k F_{k+m})^{-1} \rfloor$, $m \geq 1$, where $\lfloor \cdot \rfloor$ indicates the floor function.

1 Introduction

As is well known, the Fibonacci numbers F_n are generated from the recurrence relation

$$F_n = F_{n-1} + F_{n-2} \ \ (n \geq 2),$$

with initial condition $F_0 = 0$ and $F_1 = 1$.

Recently Ohtsuka and Nakamura [7] found interesting properties of the Fibonacci numbers and proved Theorem 1 below.

Theorem 1. *For the Fibonacci numbers, the following identities hold:*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_n - F_{n-1}, & \text{if } n \geq 2 \text{ and } n \text{ is even;} \\ F_n - F_{n-1} - 1, & \text{if } n \geq 3 \text{ and } n \text{ is odd,} \end{cases} \tag{1}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_{n-1}F_n - 1, & \text{if } n \geq 2 \text{ and } n \text{ is even;} \\ F_{n-1}F_n, & \text{if } n \geq 3 \text{ and } n \text{ is odd.} \end{cases} \tag{2}$$

Following the paper of Ohtsuka and Nakamura [7], diverse results in the same direction have been reported in the literature [1, 2, 3], [5], [8, 9, 10, 11, 12, 13]. Among them, Liu and Wang [5] considered the product of two reciprocal Fibonacci numbers, and obtained several interesting results. For example, they proved Theorem 2 below for the products of two consecutive Fibonacci numbers.

Theorem 2. *Let $m \geq 2$. Then*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_k F_{k+1}} \right)^{-1} \right\rfloor = \begin{cases} F_n^2, & \text{if } n \geq 2 \text{ and } n \text{ is even;} \\ F_n^2 - 1, & \text{if } n \geq 3 \text{ and } n \text{ is odd.} \end{cases} \tag{3}$$

Motivated by Theorem 2, we study the reciprocal sums of products of two different Fibonacci numbers in this paper. We obtain some identities related to the numbers

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k F_{k+m}} \right)^{-1} \right\rfloor, \quad m \geq 1.$$

Remark 3. The following identity was conjectured by Ohtsuka and proved by Bruckman [6]:

$$\left(\sum_{k=n}^{\infty} \frac{1}{F_k F_{k+m}} \right)^{-1} = \sum_{k=1}^{n-1} F_k F_{k+m} - \frac{1}{3} F_{m-2(-1)^n} + O\left(\frac{1}{F_n^2} \right), \quad m \geq 0.$$

For the case where $m = 0$ and n is large, (2) also can be derived from the above result.

2 Main results

We will use Lemma 4 below to prove our main results.

Lemma 4 (Koshy [4]). *For the Fibonacci numbers, we have*

$$F_m F_n - F_{m+k} F_{n-k} = (-1)^{n-k} F_{m-n+k} F_k.$$

Our main results are stated in the following theorem.

Theorem 5. *For the Fibonacci numbers, (a), (b) and (c) below hold:*

(a) Let $m \geq 1$. If

$$\frac{2F_m - F_{m+1}}{3} \notin \mathbb{Z},$$

then there exist positive integers n_0 and n_1 such that

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k F_{k+m}} \right)^{-1} \right\rfloor = \begin{cases} F_{n+m-1} F_n + g_m - 1, & \text{if } n \geq n_0 \text{ and } n \text{ is even;} \\ F_{n+m-1} F_n - g_m, & \text{if } n \geq n_1 \text{ and } n \text{ is odd,} \end{cases} \tag{4}$$

where

$$g_m = \left\lfloor \frac{2F_m - F_{m+1}}{3} \right\rfloor + 1.$$

(b) For $m = 2$*,*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k F_{k+m}} \right)^{-1} \right\rfloor = F_{n+m-1}F_n, \ \textit{for } n \geq 1. \tag{5}$$

(c) Let $m \geq 3$*. If*

$$\frac{2F_m - F_{m+1}}{3} \in \mathbb{Z},$$

then there exist positive integers n_2 *and* n_3 *such that*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k F_{k+m}} \right)^{-1} \right\rfloor = \begin{cases} F_{n+m-1}F_n + \hat{g}_m - 1, & \textit{if } n \geq n_2 \textit{ and } n \textit{ is even;} \\ F_{n+m-1}F_n - \hat{g}_m - 1, & \textit{if } n \geq n_3 \textit{ and } n \textit{ is odd,} \end{cases} \tag{6}$$

where

$$\hat{g}_m = \frac{2F_m - F_{m+1}}{3}.$$

Proof. (a) To prove (4), consider

$$\begin{aligned} X_1 &= \frac{1}{F_{n+m-1}F_n + (-1)^n g_m} - \frac{1}{F_{n+m+1}F_{n+2} + (-1)^n g_m} - \frac{1}{F_n F_{n+m}} - \frac{1}{F_{n+1}F_{n+m+1}} \\ &= \frac{\hat{X}_1}{\{F_{n+m-1}F_n + (-1)^n g_m\}\{F_{n+m+1}F_{n+2} + (-1)^n g_m\} F_n F_{n+m} F_{n+1} F_{n+m+1}}, \end{aligned}$$

where, by the identity $F_{n+m+1}F_{n+2} - F_{n+m-1}F_n = F_n F_{n+m} + F_{n+1}F_{n+m+1}$

$$\hat{X}_1 = (F_n F_{n+m} + F_{n+1}F_{n+m+1})\tilde{X}_1,$$

with

$$\begin{aligned} \tilde{X}_1 &= F_n F_{n+1} F_{n+m} F_{n+m+1} - F_{n+m-1}F_{n+m+1}F_n F_{n+2} \\ &\quad -(-1)^n g_m (F_{n+m-1}F_n + F_{n+m+1}F_{n+2}) - g_m^2. \end{aligned}$$

From Lemma 4, we have

$$\begin{aligned} F_{n+1}F_{n+m} - F_{n+m+1}F_n &= (-1)^n F_m, \\ F_{n+m+1}F_n - F_{n+m-1}F_{n+2} &= (-1)^n (F_m - F_{m+1}), \\ F_{n+m+1}F_{n-1} - F_{n+m}F_n &= (-1)^n F_{m+1}. \end{aligned}$$

Then

$$\begin{aligned} & F_nF_{n+1}F_{n+m}F_{n+m+1} - F_{n+m-1}F_{n+m+1}F_nF_{n+2} \\ = \; & F_{n+m+1}F_n\Big\{F_{n+m+1}F_n + (-1)^nF_m\Big\} \\ & -F_{n+m+1}F_n\Big\{F_{n+m+1}F_n + (-1)^n(F_{m+1} - F_m)\Big\} \\ = \; & (-1)^nF_{n+m+1}F_n(2F_m - F_{m+1}), \end{aligned}$$

and

$$\begin{aligned} F_{n+m-1}F_n + F_{n+m+1}F_{n+2} &= 3F_{n+m+1}F_n + F_{n+m+1}F_{n-1} - F_{n+m}F_n \\ &= 3F_{n+m+1}F_n + (-1)^nF_{m+1}. \end{aligned}$$

Hence

$$\tilde{X}_1 = (-1)^nF_{n+m+1}F_n(2F_m - F_{m+1} - 3g_m) - g_mF_{m+1} - g_m^2.$$

Assume that n is even. Since $g_m > 0$ and $2F_m - F_{m+1} - 3g_m < 0$, then $X_1 < 0$ and

$$\frac{1}{F_{n+m-1}F_n + g_m} - \frac{1}{F_{n+m+1}F_{n+2} + g_m} < \frac{1}{F_nF_{n+m}} + \frac{1}{F_{n+1}F_{n+m+1}}.$$

Repeatedly applying the above inequality, we have

$$\frac{1}{F_{n+m-1}F_n + g_m} < \sum_{k=n}^{\infty} \frac{1}{F_kF_{k+m}}, \text{ if } n \geq 2 \text{ and } n \text{ is even.} \tag{7}$$

Similarly, if n is odd, then there exists a positive integer m_1 such that, for $n \geq m_1$, $X_1 > 0$ and

$$\frac{1}{F_nF_{n+m}} + \frac{1}{F_{n+1}F_{n+m+1}} < \frac{1}{F_{n+m-1}F_n - g_m} - \frac{1}{F_{n+m+1}F_{n+2} - g_m},$$

from which we obtain

$$\sum_{k=n}^{\infty} \frac{1}{F_kF_{k+m}} < \frac{1}{F_{n+m-1}F_n - g_m}, \text{ if } n \geq m_1 \text{ and } n \text{ is odd.} \tag{8}$$

Next, consider

$$\begin{aligned} X_2 &= \frac{1}{F_{n+m-1}F_n + (-1)^ng_m - 1} - \frac{1}{F_{n+m}F_{n+1} + (-1)^{n+1}g_m - 1} - \frac{1}{F_nF_{n+m}} \\ &= \frac{\hat{X}_2}{\{F_{n+m-1}F_n + (-1)^ng_m - 1\}\{F_{n+m}F_{n+1} + (-1)^{n+1}g_m - 1\}F_nF_{n+m}}, \end{aligned}$$

where

$$\begin{aligned}\hat{X}_2 &= F_nF_{n+m}^2F_{n+1} - F_{n+m}F_{n+m-1}F_nF_{n+1} - F_n^2F_{n+m-1}F_{n+m}\\ &\quad -(-1)^n g_m(2F_nF_{n+m} - F_{n+m-1}F_n + F_{n+m}F_{n+1})\\ &\quad +F_{n+m-1}F_n + F_{n+m}F_{n+1} + g_m^2 - 1.\end{aligned}$$

From Lemma 4, we have

$$F_{n+m-1}F_n - F_{n+m-2}F_{n+1} = (-1)^{n+1}F_{m-2} = (-1)^n(F_{m+1} - 2F_m).$$

Then

$$\begin{aligned}&F_nF_{n+m}F_{n+1}F_{n+m} - F_{n+m}F_{n+m-1}F_nF_{n+1} - F_n^2F_{n+m-1}F_{n+m}\\ =\ &F_nF_{n+m}(F_{n+1}F_{n+m-2} - F_nF_{n+m-1})\\ =\ &(-1)^nF_nF_{n+m}(2F_m - F_{m+1}),\end{aligned}$$

and

$$\begin{aligned}&2F_nF_{n+m} + F_{n+m}F_{n+1} - F_{n+m-1}F_n\\ =\ &3F_nF_{n+m} + F_{n+m}F_{n-1} - F_{n+m-1}F_n\\ =\ &3F_nF_{n+m} + (-1)^n(2F_{m+2} - F_{m+3}).\end{aligned}$$

Hence

$$\begin{aligned}\hat{X}_2 &= (-1)^nF_nF_{n+m}(2F_m - F_{m+1} - 3g_m) + F_{n+m-1}F_n + F_{n+m}F_{n+1}\\ &\quad -g_m(2F_{m+2} - F_{m+3}) + g_m^2 - 1.\end{aligned}$$

Suppose that n is even. Since

$$-2 \le 2F_m - F_{m+1} - 3g_m \le -1,$$

then

$$\begin{aligned}&F_nF_{n+m}(2F_m - F_{m+1} - 3g_m) + (F_{n+m-1}F_n + F_{n+m}F_{n+1})\\ \ge\ &-2F_nF_{n+m} + F_nF_{n+m-1} + F_{n+1}F_{n+m}\\ =\ &(F_{n-1} - F_n)(F_{n+m-1} + F_{n-m-2}) + F_nF_{n+m-1}\\ =\ &F_{n-1}F_{n+m-1} - F_{n-2}F_{n+m-2}\\ >\ &0,\end{aligned}$$

and there exists a positive integer m_2 such that, for $n \ge m_2$, $X_2 > 0$ and

$$\frac{1}{F_nF_{n+m}} < \frac{1}{F_{n+m-1}F_n + (-1)^ng_m - 1} - \frac{1}{F_{n+m}F_{n+1} + (-1)^{n+1}g_m - 1}.$$

Repeatedly applying the above inequality, we have

$$\sum_{k=n}^{\infty} \frac{1}{F_k F_{k+m}} < \frac{1}{F_{n+m-1}F_n + g_m - 1}, \text{ if } n \geq m_2 \text{ and } n \text{ is even.} \tag{9}$$

On the other hand,

$$\begin{aligned} X_3 &= \frac{1}{F_{n+m-1}F_n + (-1)^n g_m + 1} - \frac{1}{F_{n+m}F_{n+1} + (-1)^{n+1} g_m + 1} - \frac{1}{F_n F_{n+m}} \\ &= \frac{\hat{X}_3}{\{F_{n+m-1}F_n + (-1)^n g_m + 1\}\{F_{n+m-1}F_{n+1} + (-1)^{n+1} g_m + 1\}F_n F_{n+m}}, \end{aligned}$$

where

$$\begin{aligned} \hat{X}_3 &= \hat{X}_2 - 2(F_{n+m-1}F_n + F_{n+m}F_{n+1}) \\ &= (-1)^n F_n F_{n+m}(2F_m - F_{m+1} - 3g_m) - F_{n+m-1}F_n - F_{n+1}F_{n+1} \\ &\quad -g_m(2F_{m+2} - F_{m+3}) + g_m^2 - 1. \end{aligned}$$

Suppose that n is odd. As shown above, we have

$$-F_n F_{n+m}(2F_m - F_{m+1} - 3g_m) - F_{n+m-1}F_n - F_{n+m}F_{n+1} < F_{n-2}F_{n+m-2} - F_{n-1}F_{n+m-1}.$$

Hence there exists a positive integer m_3 such that, for $n \geq m_3$, $X_3 < 0$ and

$$\frac{1}{F_{n+m-1}F_n + (-1)^n g_m + 1} - \frac{1}{F_{n+m}F_{n+1} + (-1)^{n+1} g_m + 1} < \frac{1}{F_n F_{n+m}},$$

from which we have

$$\frac{1}{F_{n+m-1}F_n - g_m + 1} < \sum_{k=n}^{\infty} \frac{1}{F_k F_{k+m}}, \text{ if } n \geq m_3 \text{ and } n \text{ is odd.} \tag{10}$$

Then, (4) follows from (7), (8), (9) and (10).

(b) Since $F_{n+2}F_{n+3} - F_n F_{n+1} = F_n F_{n+2} + F_{n+1}F_{n+3}$, we have

$$\frac{1}{F_n F_{n+1}} - \frac{1}{F_{n+2}F_{n+3}} - \frac{1}{F_n F_{n+2}} - \frac{1}{F_{n+1}F_{n+3}} = \frac{F_{n+2}F_{n+3} - F_n F_{n+1} - (F_n F_{n+2} + F_{n+1}F_{n+3})}{F_n F_{n+1} F_{n+2} F_{n+3}} = 0,$$

i.e.,

$$\frac{1}{F_n F_{n+1}} - \frac{1}{F_{n+2}F_{n+3}} = \frac{1}{F_n F_{n+2}} + \frac{1}{F_{n+1}F_{n+3}}.$$

Repeatedly applying the above equality, we obtain (5).

(c) Let $m \geq 3$ and assume that

$$\hat{g}_m = \frac{2F_m - F_{m+1}}{3} \in \mathbb{Z}.$$

We recall the proof of (a). Replacing g_m by $\hat{g}_m$ in $\tilde{X}_1$, we have

$$\tilde{X}_1 = -\hat{g}_m F_{m+1} - \hat{g}_m^2 < 0.$$

Then $X_1 < 0$ if $n \geq 2$ and n is even or if $n \geq m_4$ and n is odd for some positive integer m_4, and we have

$$\frac{1}{F_{n+m-1}F_n + (-1)^n \hat{g}_m} < \sum_{k=n}^{\infty} \frac{1}{F_k F_{k+m}}, \text{ if } n \geq 2 \text{ (} n \text{ is even) or if } n \geq m_4 \text{ (} n \text{ is odd).} \quad (11)$$

Similarly there exist positive integers m_5 and m_6 such that $X_2 > 0$ if $n \geq m_5$ and n is even, or if $n \geq m_6$ and n is odd, from which we have

$$\sum_{k=n}^{\infty} \frac{1}{F_k F_{k+m}} < \frac{1}{F_{n+m-1}F_n + (-1)^n \hat{g}_m - 1}, \text{ if } n \geq m_5 \text{ (} n \text{ is even) or if } n \geq m_6 \text{ (} n \text{ is odd).} \quad (12)$$

Then, (6) follows from (11) and (12). □

Remark 6. From Theorem 5, we have

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k F_{k+1}} \right)^{-1} \right\rfloor = \begin{cases} F_n^2, & \text{if } n \geq 2 \text{ and } n \text{ is even;} \\ F_n^2 - 1, & \text{if } n \geq 1 \text{ and } n \text{ is odd,} \end{cases}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k F_{k+6}} \right)^{-1} \right\rfloor = \begin{cases} F_{n+5}F_n, & \text{if } n \geq 2 \text{ and } n \text{ is even;} \\ F_{n+5}F_n - 2, & \text{if } n \geq 1 \text{ and } n \text{ is odd,} \end{cases}$$

etc.

3 Acknowledgments

The author is thankful to the editor-in-chief and to the anonymous referee for their helpful comments which led to the improved presentation of the paper.

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2010 *Mathematics Subject Classification*: Primary 11B39; Secondary 11B37.
Keywords: Fibonacci number, reciprocal, floor.

(Concerned with sequence A000108.)

Received July 17 2017; revised versions received December 29 2017; January 22 2018. Published in *Journal of Integer Sequences*, March 8 2018.

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