On The Pfaffians and Determinants of Some Skew-Centrosymmetric Matrices

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Abstract

This paper shows that the Pfaffians and determinants of some skew centrosymmetric matrices can be computed by a paired two-term recurrence relation, or a general number sequence of second order. As a result, the complexities of the formulas are of order $n$. Furthermore, the formulas have no divisions at all, i.e., they fall into the class of breakdown-free algorithms.
1 Introduction

The **determinant** is one of the basic parameters in matrix theory. The **determinant** of a square matrix $A = (a_{i,j}) \in \mathbb{C}^{n \times n}$ is defined as

$$
\text{det}(A) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)},
$$

where the symbol $S_n$ denotes the group of permutations of sets with $n$ elements and the symbol $\text{sgn}(\sigma)$ denotes the signature of $\sigma \in S_n$.

The **Pfaffian** of a skew-symmetric matrix $A = (a_{i,j}) \in \mathbb{C}^{2k \times 2k}$ is defined by

$$
Pf(A) = \frac{1}{2^k k!} \sum_{\sigma \in S_{2k}} \text{sgn}(\sigma) \prod_{i=1}^{k} a_{\sigma(2i-1),\sigma(2i)},
$$

and is closely related to the determinant. In fact, Cayley’s theorem states that the square of the Pfaffian of a matrix is equal to the determinant of the matrix, i.e.,

$$
\text{det}(A) = (\text{Pf}(A))^2.
$$

Matrix $A$ is called a **centrosymmetric** matrix if $A = JAJ^{-1}$, where $J$ is the anti-diagonal matrix whose anti-diagonal elements are one with all others being zero. If $A = -JAJ^{-1}$, the matrix is said to be **skew-centrosymmetric**. Skew-centrosymmetric matrices arise in many fields of science including numerical solutions of certain differential equations, digital signal processing, information theory, statistics, linear systems theory, and some Markov processes [1, 2, 3, 4, 5, 6].

In general, the complexities of the Pfaffian and the determinant are of the order $O(n^3)$. This paper describes efficient computational formulas for the Pfaffians and determinants of special matrices for which the complexities of the formulas are of the order $O(n)$. The formulas have no divisions at all, i.e., the formulas fall into the class of breakdown-free algorithms.

2 Pfaffians of skew-centrosymmetric matrices

**Definition 1.** $A_n = (a_{i,j})$ and $B_n = (b_{i,j})$ denote $n$-by-$n$ matrices with the following elements:

$$
a_{i,j} = \begin{cases} 
a, & \text{if } j = i + 1; 
-a, & \text{if } i = j + 1; 
0, & \text{otherwise},
\end{cases}
$$

$$
b_{i,j} = \begin{cases} 
(-1)^{i+1}b, & \text{if } i + j = n + 1; 
0, & \text{otherwise},
\end{cases}
$$
where $1 \leq i, j \leq n$.

**Definition 2.** $F_n$ and $G_n$ denote 2-by-2 block matrices of the following form:

$$F_n = \begin{pmatrix} A_k & B_k \\ (-1)^k B_k & A_k \end{pmatrix}, \quad G_n = \begin{pmatrix} A_k & -B_k \\ (-1)^{k+1} B_k & A_k \end{pmatrix},$$

where $n = 2^k$.

For example, if $n = 10$, it follows from Definition 2 that

$$F_{10} = \begin{pmatrix} A_5 & B_5 \\ (-1)^5 B_5 & A_5 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b \\ -a & 0 & a & 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & -a & 0 & a & 0 & 0 & b & 0 & 0 & 0 \\ 0 & 0 & -a & 0 & a & 0 & -b & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -b & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & -a & 0 & a & 0 & 0 \\ 0 & 0 & -b & 0 & 0 & 0 & -a & 0 & a & 0 \\ 0 & b & 0 & 0 & 0 & 0 & 0 & -a & 0 & a \\ -b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a & 0 \end{pmatrix}.$$ 

We now describe algorithms for computing the Pfaffians of $F_n$ and $G_n$.

**Theorem 3.** Let $\{f_n\}$ and $\{g_n\}$ be the recursively defined sequences below:

$$f_n = bg_{n-1} + a^2 f_{n-2} \quad \text{for } f_1 = b,$$

$$g_n = -bf_{n-1} + a^2 g_{n-2} \quad \text{for } g_1 = -b.$$

Then, for $n = 2^k$, we obtain

$$f_k = \text{Pf}(F_n) \quad \text{and} \quad g_k = \text{Pf}(G_n),$$

where $f_{-1} = 0, f_0 = 1$ and $g_{-1} = 0, g_0 = 1$.

**Proof.** The proof is done by induction on $k$. For $k = 1$,

$$F_2 = \begin{pmatrix} A_1 & B_1 \\ -B_1 & A_1 \end{pmatrix} = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \quad \text{and} \quad G_2 = \begin{pmatrix} A_1 & -B_1 \\ B_1 & A_1 \end{pmatrix} = \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}.$$ 

The definition of the Pfaffian in (1) clearly indicates that $\text{Pf}(F_2) = b$ and $\text{Pf}(G_2) = -b$. Thus, $f_1 = b = \text{Pf}(F_2), g_1 = -b = \text{Pf}(G_2).$
Let us assume that the recurrence relations hold for all \( t \leq k \). Then we show that they hold for \( k = t + 1 \).

\[
F_{2t+2} = \begin{pmatrix}
A_{t+1} & B_{t+1} \\
(-1)^{t+1}B_{t+1} & A_{t+1}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
0 & a & 0 & \cdots & 0 & b \\
a & 0 & A_t & -B_t & \vdots \\
\vdots & (-1)^t B_t & A_t & 0 & a \\
0 & -b & 0 & \cdots & 0 & -a
\end{pmatrix}.
\tag{2}
\]

From the expansion formula along with 2t + 2 column of (2), it follows that

\[
Pf(F_{2t+2}) = bPf(G_{2t}) + aPf(M_{2t}) = bg_t + aPf(M_{2t}),
\tag{3}
\]

where

\[
M_{2t} = \begin{pmatrix}
0 & a & 0 & \cdots & 0 \\
-a & 0 & A_{t-1} & B_{t-1} & \vdots \\
\vdots & 0 & (-1)^{t-1}B_{t-1} & A_{t-1} & 0 \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}.
\tag{4}
\]

From the expansion formula along with the first row of (4), it follows that

\[
Pf(M_{2t}) = aPf(F_{2t-2}) = af_{t-1}.
\tag{5}
\]

From (3) and (5), we have

\[
f_{t+1} = bg_t + a^2 f_{t-1}.
\]

The recurrence relation for \( g_{t+1} \) can be obtained similarly. \( \Box \)

**Corollary 4.** \( f_n = (-1)^{n-1}bf_{n-1} + a^2 f_{n-2} \) with \( f_{-1} = 0 \) and \( f_1 = 1 \).

Corollary 4 shows that the computational costs of \( Pf(F_n) \) and \( \det(F_n)(= Pf(F_n)^2) \) are of the order \( O(n) \). Furthermore, the recurrences in Corollary 4 have no divisions. Thus, no breakdown occurs during the computation.

### 3 Determinant of the skew-centrosymmetric matrix

In this section, we consider the determinant of the matrix \( F_n \) with \( n = 2k \). It is well known from [3] that the determinant of the 2-by-2 block matrix holds

\[
\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \det(AD - CB)
\]
if \( AC = CA \). Applying the above formula to \( F_n \) in Definition 2, the determinant of matrix \( F_n \) equals that of \( T_k := A_k^2 - (-1)^k B_k^2 \). Thus, we have

\[
|F_n| = |T_k| = \det \begin{pmatrix}
-a^2 + b^2 & 0 & a^2 & & \\
0 & -2a^2 + b^2 & 0 & & \\
a^2 & 0 & \ddots & \ddots & a^2 \\
& \ddots & \ddots & -2a^2 + b^2 & 0 \\
a^2 & 0 & \cdots & -a^2 + b^2 & \\
\end{pmatrix}_{k \times k}.
\]

The matrix \( T_k \) belongs to the set of \( k \)-tridiagonal matrices. Sogabe and El-Mikkawy [8] considered a fast block diagonalization of \( k \)-tridiagonal matrices using permutation matrices. Exploiting the block diagonalization method, we can rearrange the matrix \( T_k \) as below.

(i) We consider the case where \( k \) is odd. Let us define the following matrices:

\[
H_{\frac{k-1}{2}} = (h_{i,j}) = \begin{cases}
-a^2 + b^2, & \text{if } i = j; \\
a^2, & \text{if } i = j + 1 \text{ or } j = i + 1; \\
0, & \text{otherwise}
\end{cases}
\]

and

\[
K_{\frac{k+1}{2}} = (k_{i,j}) = \begin{cases}
-a^2 + b^2, & \text{if } i = j = 1 \text{ or } i = j = \frac{k+1}{2}; \\
-2a^2 + b^2, & \text{if } i = j = 2 \ldots \frac{k-1}{2}; \\
a^2, & \text{if } i = j + 1 \text{ or } j = i + 1; \\
0, & \text{otherwise}.
\end{cases}
\]

Then,

\[
P^T T_k P = \begin{pmatrix}
H_{\frac{k-1}{2}} & 0 \\
0 & K_{\frac{k+1}{2}}
\end{pmatrix},
\]

where the permutation matrix \( P \) is determined by using the method in [8]. Obviously,

\[
\det(P^T T_k P) = \det T_k = \det F_n = \det(H_{\frac{k-1}{2}}) \det(K_{\frac{k+1}{2}}).
\]

(ii) We consider the case where \( k \) is even. Let us define

\[
N_{\frac{k}{2}} = (n_{i,j}) = \begin{cases}
-a^2 + b^2, & \text{if } i = j = \frac{k}{2}; \\
-2a^2 + b^2, & \text{if } i = j = 1 \ldots \frac{k}{2} - 1; \\
a^2, & \text{if } i = j + 1 \text{ or } j = i + 1; \\
0, & \text{otherwise}
\end{cases}
\]
and

\[ Q_k = (q_{i,j}) = \begin{cases} 
-a^2 + b^2, & \text{if } i = j = 1; \\
-2a^2 + b^2, & \text{if } i = j = 2, \ldots, k; \\
a^2, & \text{if } i = j + 1 \text{ or } j = i + 1; \\
0, & \text{otherwise.}
\end{cases} \]

Then,

\[ P^T T_k P = \begin{pmatrix} N_{k/2} & 0 \\ 0 & Q_{k/2} \end{pmatrix}. \]

Obviously,

\[ \det(P^T T_k P) = \det T_k = \det F_n = \det(N_{k/2}) \det(Q_{k/2}). \]

It can be seen that \( \det(N_{k/2}) = \det(Q_{k/2}). \)

El-Mikkawy [9] obtained two-term recurrence relation for the determinants of tridiagonal matrices, i.e.,

\[ v_i = \begin{vmatrix} d_1 & a_1 & 0 & \ldots & 0 \\ b_2 & d_2 & a_2 & \ddots & \vdots \\ 0 & b_3 & d_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_{i-1} \\ 0 & \ldots & 0 & b_i & d_i \end{vmatrix}, \]

where \( v_i = d_i v_{i-1} - b_i a_{i-1} v_{i-2} \) for \( v_0 = 1 \) and \( v_{-1} = 0 \). Using this relation and the Laplace expansion, we obtain the result. If \( k \) is even, then

\[ \det(N_{k/2}) = \det(Q_{k/2}) = (-a^2 + b^2) w_{k/2-1} - a^4 w_{k/2-2}. \]

If \( k \) is odd, then

\[ \det(K_{k+1}) = (-a^2 + b^2)^2 w_{k+3} - 2a^4(-a^2 + b^2) w_{k+5} + a^8 w_{k+7}, \]

\[ \det(H_{k+1}) = w_{k+1}, \]

where

\[ w_i = \begin{vmatrix} -2a^2 + b^2 & a^2 & \ldots & 0 \\ a^2 & -2a^2 + b^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a^2 \\ 0 & \ldots & a^2 & -2a^2 + b^2 \end{vmatrix}. \]

Here \( w_i = (-2a^2 + b^2) w_{i-1} - a^4 w_{i-2} \) for \( w_0 = 1 \) and \( w_{-1} = 0 \).

Consequently, for \( n = 2k \), we obtain
(i) If $k$ is odd, 
\[
\det \mathcal{F}_n = \det \mathcal{T}_k = w_{\frac{k-1}{2}} \left( (-a^2 + b^2)^2 w_{\frac{k-3}{2}} - 2a^4(-a^2 + b^2)w_{\frac{k-5}{2}} + a^8w_{\frac{k-7}{2}} \right).
\]

(ii) If $k$ is even, \[
\det \mathcal{F}_n = \det \mathcal{T}_k = \left( (-a^2 + b^2)w_{\frac{k-1}{2}} - a^4w_{\frac{k-3}{2}} \right)^2.
\]

### 4 Examples

Some examples of the Pfaffian and the determinant of the matrix $\mathcal{F}_n$ ($n = 2k$) are shown in the following tables. Here $F_n$, $P_n$, and $J_n$ are the $n$th Fibonacci, Pell, and Jacobsthal numbers, respectively.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$a = i, b = 1$</th>
<th>$a = i, b = 2$</th>
<th>$a = i\sqrt{2}, b = 1$</th>
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<tr>
<td></td>
<td>$\text{Pf}(\mathcal{F}_{2k})$</td>
<td>$\text{Pf}(\mathcal{F}_{2k})$</td>
<td>$\text{Pf}(\mathcal{F}_{2k})$</td>
</tr>
<tr>
<td>1</td>
<td>$F_2 = 1$</td>
<td>$P_2 = 2$</td>
<td>$J_2 = 1$</td>
</tr>
<tr>
<td>2</td>
<td>$-F_3 = -2$</td>
<td>$-P_3 = -5$</td>
<td>$-J_3 = -3$</td>
</tr>
<tr>
<td>3</td>
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<td>$-P_4 = -12$</td>
<td>$-J_4 = -5$</td>
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<tr>
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<td>$F_5 = 5$</td>
<td>$P_5 = 29$</td>
<td>$J_5 = 11$</td>
</tr>
<tr>
<td>5</td>
<td>$F_6 = 8$</td>
<td>$P_6 = 70$</td>
<td>$J_6 = 21$</td>
</tr>
<tr>
<td>6</td>
<td>$-F_7 = -13$</td>
<td>$-P_7 = -169$</td>
<td>$-J_7 = -43$</td>
</tr>
<tr>
<td>7</td>
<td>$-F_8 = -21$</td>
<td>$-P_8 = -408$</td>
<td>$-J_8 = -85$</td>
</tr>
<tr>
<td>8</td>
<td>$F_9 = 34$</td>
<td>$P_9 = 985$</td>
<td>$J_9 = 171$</td>
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<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$\equiv 0, 1 \pmod{4}$</td>
<td>$F_{k+1}$</td>
<td>$P_{k+1}$</td>
<td>$J_{k+1}$</td>
</tr>
<tr>
<td>$\equiv 2, 3 \pmod{4}$</td>
<td>$-F_{k+1}$</td>
<td>$-P_{k+1}$</td>
<td>$-J_{k+1}$</td>
</tr>
</tbody>
</table>

Examples of the Pfaffians
<table>
<thead>
<tr>
<th>$k$</th>
<th>$a = i, b = 1$</th>
<th>$a = i, b = 2$</th>
<th>$a = i\sqrt{2}, b = 1$</th>
</tr>
</thead>
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<tr>
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<td>$P_2^2$</td>
<td>$J_2^2$</td>
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<td>$P_3^2$</td>
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<td>$P_{t+1}^2$</td>
<td>$J_{t+1}^2$</td>
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</table>

Examples of the determinants

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