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On The Pfaffians and Determinants of Some Skew-Centrosymmetric Matrices

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Abstract

This paper shows that the Pfaffians and determinants of some skew centrosymmetric matrices can be computed by a paired two-term recurrence relation, or a general number sequence of second order. As a result, the complexities of the formulas are of order n. Furthermore, the formulas have no divisions at all, i.e., they fall into the class of breakdown-free algorithms.

1 Introduction

The *determinant* is one of the basic parameters in matrix theory. The *determinant* of a square matrix $A = (a_{i,j}) \in \mathbb{C}^{n\times n}$ is defined as

$$\det(A) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)},$$

where the symbol S_n denotes the group of permutations of sets with n elements and the symbol $\operatorname{sgn}(\sigma)$ denotes the signature of $\sigma \in S_n$.

The *Pfaffian* of a skew-symmetric matrix $A = (a_{i,j}) \in \mathbb{C}^{2k\times 2k}$ is defined by

$$\operatorname{Pf}(A) = \frac{1}{2^k k!} \sum_{\sigma\in S_{2k}} \operatorname{sgn}(\sigma) \prod_{i=1}^{k} a_{\sigma(2i-1),\sigma(2i)}, \tag{1}$$

and is closely related to the determinant. In fact, Cayley's theorem states that the square of the Pfaffian of a matrix is equal to the determinant of the matrix, i.e.,

$$\det(A) = \operatorname{Pf}(A)^2.$$

Matrix A is called a *centrosymmetric* matrix if $A = JAJ^{-1}$, where J is the anti-diagonal matrix whose anti-diagonal elements are one with all others being zero. If $A = -JAJ^{-1}$, the matrix is said to be *skew-centrosymmetric*. Skew-centrosymmetric matrices arise in many fields of science including numerical solutions of certain differential equations, digital signal processing, information theory, statistics, linear systems theory, and some Markov processes [1, 2, 3, 4, 5, 6].

In general, the complexities of the Pfaffian and the determinant are of the order $\mathcal{O}(n^3)$. This paper describes efficient computational formulas for the Pfaffians and determinants of special matrices for which the complexities of the formulas are of the order $\mathcal{O}(n)$. The formulas have no divisions at all, i.e., the formulas fall into the class of breakdown-free algorithms.

2 Pfaffians of skew-centrosymmetric matrices

Definition 1. $A_n = (a_{i,j})$ and $B_n = (b_{i,j})$ denote n-by-n matrices with the following elements:

$$a_{i,j} = \begin{cases} a, & \text{if } j = i+1; \\ -a, & \text{if } i = j+1; \\ 0, & \text{otherwise}, \end{cases}$$

$$b_{i,j} = \begin{cases} (-1)^{i+1} b, & \text{if } i+j = n+1; \\ 0, & \text{otherwise}, \end{cases}$$

where $1 \le i, j \le n$.

Definition 2. $\mathcal{F}_n$ and $\mathcal{G}_n$ denote 2-by-2 block matrices of the following form:

$$\mathcal{F}_n = \begin{pmatrix} A_k & B_k \\ (-1)^k B_k & A_k \end{pmatrix}, \quad \mathcal{G}_n = \begin{pmatrix} A_k & -B_k \\ (-1)^{k+1} B_k & A_k \end{pmatrix},$$

where $n = 2k$.

For example, if $n = 10$, it follows from Definition 2 that

$$\mathcal{F}_{10} = \begin{pmatrix} A_5 & B_5 \\ (-1)^5 B_5 & A_5 \end{pmatrix}$$

$$= \left(\begin{array}{ccccc|ccccc} 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b \\ -a & 0 & a & 0 & 0 & 0 & 0 & 0 & -b & 0 \\ 0 & -a & 0 & a & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & -a & 0 & a & 0 & -b & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & b & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -b & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & -a & 0 & a & 0 & 0 \\ 0 & 0 & -b & 0 & 0 & 0 & -a & 0 & a & 0 \\ 0 & b & 0 & 0 & 0 & 0 & 0 & -a & 0 & a \\ -b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a & 0 \end{array}\right).$$

We now describe algorithms for computing the Pfaffians of $\mathcal{F}_n$ and $\mathcal{G}_n$.

Theorem 3. *Let $\{f_n\}$ and $\{g_n\}$ be the recursively defined sequences below:*

$$\begin{aligned} f_n &= b g_{n-1} + a^2 f_{n-2} \quad \textit{for} \quad f_1 = b, \\ g_n &= -b f_{n-1} + a^2 g_{n-2} \quad \textit{for} \quad g_1 = -b. \end{aligned}$$

Then, for $n = 2k$, we obtain

$$f_k = \mathrm{Pf}(\mathcal{F}_n) \quad \textit{and} \quad g_k = \mathrm{Pf}(\mathcal{G}_n),$$

where $f_{-1} = 0,\ f_0 = 1$ and $g_{-1} = 0,\ g_0 = 1$.

Proof. The proof is done by induction on k. For $k = 1$,

$$\mathcal{F}_2 = \begin{pmatrix} A_1 & B_1 \\ -B_1 & A_1 \end{pmatrix} = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \text{ and } \mathcal{G}_2 = \begin{pmatrix} A_1 & -B_1 \\ B_1 & A_1 \end{pmatrix} = \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}.$$

The definition of the Pfaffian in (1) clearly indicates that $\mathrm{Pf}(\mathcal{F}_2) = b$ and $\mathrm{Pf}(\mathcal{G}_2) = -b$. Thus, $f_1 = b = \mathrm{Pf}(\mathcal{F}_2), g_1 = -b = \mathrm{Pf}(\mathcal{G}_2)$.

Let us assume that the recurrence relations hold for all $t \le k$. Then we show that they hold for $k = t+1$.

$$\mathcal{F}_{2t+2} = \left(\begin{array}{c|c} A_{t+1} & B_{t+1} \\ \hline (-1)^{t+1} B_{t+1} & A_{t+1} \end{array}\right)$$

$$= \left(\begin{array}{c|cccc|c} 0 & a & 0 & \cdots & 0 & b \\ \hline -a & & & & & 0 \\ 0 & & A_t & -B_t & & \vdots \\ \vdots & & (-1)^{t+1}B_t & A_t & & 0 \\ 0 & & & & & a \\ \hline -b & 0 & \cdots & 0 & -a & 0 \end{array}\right). \tag{2}$$

From the expansion formula along with $2t+2$ column of (2), it follows that

$$\mathrm{Pf}(\mathcal{F}_{2t+2}) = b\mathrm{Pf}(\mathcal{G}_{2t}) + a\mathrm{Pf}(\mathcal{M}_{2t}) = bg_t + a\mathrm{Pf}(\mathcal{M}_{2t}), \tag{3}$$

where

$$\mathcal{M}_{2t} = \left(\begin{array}{cc|cccc} 0 & a & 0 & \cdots & \cdots & 0 \\ -a & 0 & a & 0 & \cdots & 0 \\ \hline 0 & a & & & & \\ \vdots & 0 & & A_{t-1} & B_{t-1} & \\ \vdots & \vdots & & (-1)^{t-1}B_{t-1} & A_{t-1} & \\ 0 & 0 & & & & \end{array}\right). \tag{4}$$

From the expansion formula along with the first row of (4), it follows that

$$\mathrm{Pf}(\mathcal{M}_{2t}) = a\mathrm{Pf}(\mathcal{F}_{2t-2}) = af_{t-1}. \tag{5}$$

From (3) and (5), we have

$$f_{t+1} = bg_t + a^2 f_{t-1}.$$

The recurrence relation for g_{t+1} can be obtained similarly. □

Corollary 4. *$f_n = (-1)^{n-1} b f_{n-1} + a^2 f_{n-2}$ with $f_{-1} = 0$ and $f_1 = 1$.*

Corollary 4 shows that the computational costs of $\mathrm{Pf}(\mathcal{F}_n)$ and $\det(\mathcal{F}_n)(= \mathrm{Pf}(\mathcal{F}_n)^2)$ are of the order $\mathcal{O}(n)$. Furthermore, the recurrences in Corollary 4 have no divisions. Thus, no breakdown occurs during the computation.

3 Determinant of the skew-centrosymmetric matrix

In this section, we consider the determinant of the matrix $\mathcal{F}_n$ with $n = 2k$. It is well known from [3] that the determinant of the 2-by-2 block matrix holds

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \det(AD - CB)$$

if $AC = CA$. Applying the above formula to $\mathcal{F}_n$ in Definition 2, the determinant of matrix $\mathcal{F}_n$ equals that of $\mathcal{T}_k := A_k^2 - (-1)^k B_k^2$. Thus, we have

$$|\mathcal{F}_n| = |\mathcal{T}_k| = \det \begin{pmatrix} -a^2+b^2 & 0 & a^2 & & \\ 0 & -2a^2+b^2 & 0 & \ddots & \\ a^2 & 0 & \ddots & \ddots & a^2 \\ & \ddots & \ddots & -2a^2+b^2 & 0 \\ & & a^2 & 0 & -a^2+b^2 \end{pmatrix}_{k\times k}.$$

The matrix $\mathcal{T}_k$ belongs to the set of k-tridiagonal matrices. Sogabe and El-Mikkawy [8] considered a fast block diagonalization of k-tridiagonal matrices using permutation matrices. Exploiting the block diagonalization method, we can rearrange the matrix $\mathcal{T}_k$ as below.

(i) We consider the case where k is odd. Let us define the following matrices:

$$H_{\frac{k-1}{2}} = (h_{i,j}) = \begin{cases} -2a^2+b^2, & \text{if } i=j; \\ a^2, & \text{if } i=j+1 \text{ or } j=i+1; \\ 0, & \text{otherwise} \end{cases}$$

and

$$K_{\frac{k+1}{2}} = (k_{i,j}) = \begin{cases} -a^2+b^2, & \text{if } i=j=1 \text{ or } i=j=\frac{k+1}{2}; \\ -2a^2+b^2, & \text{if } i=j=2\ldots\frac{k-1}{2}; \\ a^2, & \text{if } i=j+1 \text{ or } j=i+1; \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$P^T\mathcal{T}_kP = \left(\begin{array}{c|c} H_{\frac{k-1}{2}} & 0 \\ \hline 0 & K_{\frac{k+1}{2}} \end{array}\right),$$

where the permutation matrix P is determined by using the method in [8]. Obviously,

$$\det(P^T\mathcal{T}_kP) = \det\mathcal{T}_k = \det\mathcal{F}_n = \det(H_{\frac{k-1}{2}})\det(K_{\frac{k+1}{2}}).$$

(ii) We consider the case where k is even. Let us define

$$N_{\frac{k}{2}} = (n_{i,j}) = \begin{cases} -a^2+b^2, & \text{if } i=j=\frac{k}{2}; \\ -2a^2+b^2, & \text{if } i=j=1\ldots\frac{k}{2}-1; \\ a^2, & \text{if } i=j+1 \text{ or } j=i+1; \\ 0, & \text{otherwise} \end{cases}$$

and

$$Q_{\frac{k}{2}} = (q_{i,j}) = \begin{cases} -a^2 + b^2, & \text{if } i = j = 1; \\ -2a^2 + b^2, & \text{if } i = j = 2 \dots \frac{k}{2}; \\ a^2, & \text{if } i = j + 1 \text{ or } j = i + 1; \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$P^T \mathcal{T}_k P = \left(\begin{array}{c|c} N_{\frac{k}{2}} & 0 \\ \hline 0 & Q_{\frac{k}{2}} \end{array} \right).$$

Obviously,

$$\det(P^T \mathcal{T}_k P) = \det \mathcal{T}_k = \det \mathcal{F}_n = \det(N_{\frac{k}{2}}) \det(Q_{\frac{k}{2}}).$$

It can be seen that $\det(N_{\frac{k}{2}}) = \det(Q_{\frac{k}{2}})$.

El-Mikkawy [9] obtained two-term recurrence relation for the determinants of tridiagonal matrices, i.e.,

$$v_i = \begin{vmatrix} d_1 & a_1 & 0 & \dots & 0 \\ b_2 & d_2 & a_2 & \ddots & \vdots \\ 0 & b_3 & d_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_{i-1} \\ 0 & \dots & 0 & b_i & d_i \end{vmatrix},$$

where $v_i = d_i v_{i-1} - b_i a_{i-1} v_{i-2}$ for $v_0 = 1$ and $v_{-1} = 0$. Using this relation and the Laplace expansion, we obtain the result. If k is even, then

$$\det(N_{\frac{k}{2}}) = \det(Q_{\frac{k}{2}}) = (-a^2 + b^2) w_{\frac{k}{2}-1} - a^4 w_{\frac{k}{2}-2}.$$

If k is odd, then

$$\begin{aligned} \det(K_{\frac{k+1}{2}}) &= \left(-a^2 + b^2\right)^2 w_{\frac{k-3}{2}} - 2a^4(-a^2 + b^2) w_{\frac{k-5}{2}} + a^8 w_{\frac{k-7}{2}}, \\ \det(H_{\frac{k-1}{2}}) &= w_{\frac{k-1}{2}}, \end{aligned}$$

where

$$w_i = \begin{vmatrix} -2a^2 + b^2 & a^2 & \dots & 0 \\ a^2 & -2a^2 + b^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a^2 \\ 0 & \dots & a^2 & -2a^2 + b^2 \end{vmatrix}.$$

Here $w_i = (-2a^2 + b^2) w_{i-1} - a^4 w_{i-2}$ for $w_0 = 1$ and $w_{-1} = 0$.

Consequently, for $n = 2k$, we obtain

(i) If k is odd,

$$\begin{aligned}\det \mathcal{F}_n &= \det \mathcal{T}_k \\ &= w_{\frac{k-1}{2}} \left(\left(-a^2+b^2\right)^2 w_{\frac{k-3}{2}} - 2a^4(-a^2+b^2) w_{\frac{k-5}{2}} + a^8 w_{\frac{k-7}{2}} \right).\end{aligned}$$

(ii) If k is even, $\det \mathcal{F}_n = \det \mathcal{T}_k = \left((-a^2+b^2) w_{\frac{k}{2}-1} - a^4 w_{\frac{k}{2}-2} \right)^2$.

4 Examples

Some examples of the Pfaffian and the determinant of the matrix $\mathcal{F}_n$ ($n = 2k$) are shown in the following tables. Here F_n, P_n, and J_n are the nth Fibonacci, Pell, and Jacobsthal numbers, respectively.

	$a = i, b = 1$	$a = i, b = 2$	$a = i\sqrt{2}, b = 1$
k	$\mathrm{Pf}(\mathcal{F}_{2k})$	$\mathrm{Pf}(\mathcal{F}_{2k})$	$\mathrm{Pf}(\mathcal{F}_{2k})$
1	$F_2 = 1$	$P_2 = 2$	$J_2 = 1$
2	$-F_3 = -2$	$-P_3 = -5$	$-J_3 = -3$
3	$-F_4 = -3$	$-P_4 = -12$	$-J_4 = -5$
4	$F_5 = 5$	$P_5 = 29$	$J_5 = 11$
5	$F_6 = 8$	$P_6 = 70$	$J_6 = 21$
6	$-F_7 = -13$	$-P_7 = -169$	$-J_7 = -43$
7	$-F_8 = -21$	$-P_8 = -408$	$-J_8 = -85$
8	$F_9 = 34$	$P_9 = 985$	$J_9 = 171$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\equiv 0, 1 \pmod 4$	F_{k+1}	P_{k+1}	J_{k+1}
$\equiv 2, 3 \pmod 4$	$-F_{k+1}$	$-P_{k+1}$	$-J_{k+1}$

Examples of the Pfaffians

	$a=i, b=1$	$a=i, b=2$	$a=i\sqrt{2}, b=1$
k	$\det(\mathcal{F}_{2k})$	$\det(\mathcal{F}_{2k})$	$\det(\mathcal{F}_{2k})$
1	F_2^2	P_2^2	J_2^2
2	F_3^2	P_3^2	J_3^2
3	F_4^2	P_4^2	J_4^2
4	F_5^2	P_5^2	J_5^2
5	F_6^2	P_6^2	J_6^2
6	F_7^2	P_7^2	J_7^2
7	F_8^2	P_8^2	J_8^2
8	F_9^2	P_9^2	J_9^2
$\vdots$	$\vdots$	$\vdots$	$\vdots$
t	F_{t+1}^2	P_{t+1}^2	J_{t+1}^2

Examples of the determinants

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