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Jacobi-Type Continued Fractions for the Ordinary Generating Functions of Generalized Factorial Functions

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Abstract

The article studies a class of generalized factorial functions and symbolic product sequences through Jacobi-type continued fractions (J-fractions) that formally enumerate the typically divergent ordinary generating functions of these sequences. The rational convergents of these generalized J-fractions provide formal power series approximations to the ordinary generating functions that enumerate many specific classes of factorial-related integer product sequences. The article also provides applications to a number of specific factorial sum and product identities, new integer congruence relations satisfied by generalized factorial-related product sequences, the Stirling numbers of the first kind, and the r-order harmonic numbers, as well as new generating functions for the sequences of binomials, $m^p - 1$, among several other notable motivating examples given as applications of the new results proved in the article.

1 Notation and other conventions in the article

1.1 Notation and special sequences

Most of the conventions in the article are consistent with the notation employed within the *Concrete Mathematics* reference, and the conventions defined in the introduction to the first article [20]. These conventions include the following particular notational variants:

- **Extraction of formal power series coefficients**. The special notation for formal power series coefficient extraction, $[z^n]\left(\sum_k f_k z^k\right) :\mapsto f_n$;

- **Iverson's convention**. The more compact usage of Iverson's convention, $[i=j]_\delta \equiv \delta_{i,j}$, in place of Kronecker's delta function where $[n=k=0]_\delta \equiv \delta_{n,0}\delta_{k,0}$;

- **Bracket notation for the Stirling and Eulerian number triangles**. We use the alternate bracket notation for the Stirling number triangles, $\left[{n \atop k}\right] = (-1)^{n-k} s(n,k)$ and $\left\{{n \atop k}\right\} = S(n,k)$, as well as $\left\langle{n \atop m}\right\rangle$ to denote the first-order Eulerian number triangle, and $\left\langle\!\left\langle{n \atop m}\right\rangle\!\right\rangle$ to denote the second-order Eulerian numbers;

- **Harmonic number sequences**. Use of the notation for the first-order harmonic numbers, H_n or $H_n^{(1)}$, which defines the sequence

  $$H_n := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n},$$

  and the notation for the partial sums for the more general cases of the r-order harmonic numbers, $H_n^{(r)}$, defined as

  $$H_n^{(r)} := 1 + 2^{-r} + 3^{-r} + \cdots + n^{-r},$$

  when $r, n \geq 1$ are integer-valued and where $H_n^{(r)} \equiv 0$ for all $n \leq 0$;

- **Rising and falling factorial functions**. We use the convention of denoting the falling factorial function by $x^{\underline{n}} = x!/(x-n)!$, the rising factorial function as $x^{\overline{n}} = \Gamma(x+n)/\Gamma(x)$, or equivalently by the Pochhammer symbol, $(x)_n = x(x+1)(x+2)\cdots(x+n-1)$;

- **Shorthand notation in integer congruences and modular arithmetic**. Within the article the notation $g_1(n) \equiv g_2(n) \pmod{N_1, N_2, \ldots, N_k}$ is understood to mean that the congruence, $g_1(n) \equiv g_2(n) \pmod{N_j}$, holds modulo any of the bases, N_j, for $1 \leq j \leq k$.

Within the article, the standard set notation for $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ denote the sets of integers, rational numbers, and real numbers, respectively, where the set of natural numbers, $\mathbb{N}$, is defined by $\mathbb{N} := \{0, 1, 2, \ldots\} = \mathbb{Z}^+ \bigcup \{0\}$. Other more standard notation for the special functions cited within the article is consistent with the definitions employed within the *NIST Handbook* reference.

1.2 Mathematica summary notebook document and computational reference information

The article is prepared with a more extensive set of computational data and software routines released as open source software to accompany the examples and numerous other applications suggested as topics for future research and investigation within the article. It is highly encouraged, and expected, that the interested reader obtain a copy of the summary notebook reference and computational documentation prepared in this format to assist with computations in a multitude of special case examples cited as particular applications of the new results.

The prepared summary notebook file, `multifact-cfracs-summary.nb` , attached to the submission of this manuscript[1] contains the working *Mathematica* code to verify the formulas, propositions, and other identities cited within the article [21]. Given the length of the article, the *Mathematica* summary notebook included with this submission is intended to help the reader with verifying and modifying the examples presented as applications of the new results cited below. The summary notebook also contains numerical data corresponding to computations of multiple examples and congruences specifically referenced in several places by the applications and tables given in the next sections of the article.

2 Introduction

The primary focus of the new results established by this article is to enumerate new properties of the generalized symbolic product sequences, $p_n(\alpha, R)$ defined by (1), which are generated by the convergents to *Jacobi-type continued fractions* (*J-fractions*) that represent formal power series expansions of the otherwise divergent ordinary generating functions (OGFs) for these sequences.

$$\begin{aligned} p_n(\alpha, R) &:= \prod_{0 \leq j < n} (R + \alpha j) \, [n \geq 1]_\delta + [n = 0]_\delta \\ &= R(R+\alpha)(R+2\alpha) \times \cdots \times (R + (n-1)\alpha) \, [n \geq 1]_\delta + [n = 0]_\delta \, . \end{aligned} \tag{1}$$

The related integer-valued cases of the multiple factorial sequences, or α-*factorial* functions, $n!_{(\alpha)}$, of interest in the applications of this article are defined recursively for any fixed $\alpha \in \mathbb{Z}^+$ and $n \in \mathbb{N}$ by the following equation [20, §2]:

$$n!_{(\alpha)} = \begin{cases} n \cdot (n-\alpha)!_{(\alpha)}, & \text{if } n > 0; \\ 1, & \text{if } -\alpha < n \leq 0; \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

[1] An official copy of the original summary notebook is also linked here on the *Journal of Integer Sequences* website.

The particular new results studied within the article generalize the known series proved in the references [8, 9], including expansions of the series for generating functions enumerating the *rising* and *falling factorial* functions, $x^{\overline{n}} = (-1)^n (-x)^{\underline{n}}$ and $x^{\underline{n}} = x!/(x-n)! = p_n(-1, x)$, and the *Pochhammer symbol*, $(x)_n = p_n(1, x)$, expanded by the *Stirling numbers of the first kind*, $\left[{n \atop k}\right]$, as (A130534):

$$\begin{aligned}(x)_n &= x(x+1)(x+2)\cdots(x+n-1)\,[n \geq 1]_\delta + [n=0]_\delta \\ &= \left(\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k\right) \times [n \geq 1]_\delta + [n=0]_\delta .\end{aligned}$$

The generalized rising and falling factorial functions denote the products, $(x|\alpha)^{\overline{n}} = (x)_{n,\alpha}$ and $(x|\alpha)^{\underline{n}} = (x)_{n,-\alpha}$, defined in the reference, where the products, $(x)_n = (x|1)^{\overline{n}}$ and $x^{\underline{n}} = (x|1)^{\underline{n}}$, correspond to the particular special cases of these functions cited above [20, §2]. Roman's *Umbral Calculus* book employs the alternate, and less standard, notation of $\left(\frac{x}{a}\right)_n := \frac{x}{a}\left(\frac{x}{a}-1\right) \times \cdots \times \left(\frac{x}{a}-n+1\right)$ to denote the sequences of *lower factorial polynomials*, and $x^{(n)}$ in place of the Pochhammer symbol to denote the *rising factorial polynomials* [19, §1.2]. We do not use this convention within the article.

The generalized product sequences in (1) also correspond to the definition of the *Pochhammer α-symbol*, $(x)_{n,\alpha} = p_n(\alpha, x)$, defined as in ([6], [14, Examples] and [12, cf. §5.4]) for any fixed $\alpha \neq 0$ and non-zero indeterminate, $x \in \mathbb{C}$, by the following analogous expansions involving the generalized α-factorial coefficient triangles defined in (4) of the next section of this article [20, §3]:

$$\begin{aligned}(x)_{n,\alpha} &= x(x+\alpha)(x+2\alpha)\cdots(x+(n-1)\alpha)\,[n \geq 1]_\delta + [n=0]_\delta \\ &= \left(\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \alpha^{n-k} x^k\right) \times [n \geq 1]_\delta + [n=0]_\delta \\ &= \left(\sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_\alpha (x-1)^k\right) \times [n \geq 1]_\delta + [n=0]_\delta .\end{aligned}$$

We are especially interested in using the new results established in this article to formally generate the factorial-function-like product sequences, $p_n(\alpha, R)$ and $p_n(\alpha, \beta n + \gamma)$, for some fixed parameters $\alpha, \beta, \gamma \in \mathbb{Q}$, when the initially fixed symbolic indeterminate, R, depends linearly on n. The particular forms of the generalized product sequences of interest in the applications of this article are related to the *Gould polynomials*, $G_n(x; a, b) = \frac{x}{x-an} \cdot \left(\frac{x-an}{b}\right)^{\underline{n}}$, in the form of the following equation ([20, §3.4.2],[19, §4.1.4]):

$$p_n(\alpha, \beta n + \gamma) = \frac{(-\alpha)^{n+1}}{\gamma - \alpha - \beta} \times G_{n+1}(\gamma - \alpha - \beta; -\beta, -\alpha) . \tag{3}$$

Whereas the first results proved in the articles [8, 9] are focused on establishing properties of divergent forms of the ordinary generating functions for a number of special sequence cases through more combinatorial interpretations of these continued fraction series, the emphasis in this article is more enumerative in flavor. The new identities involving the integer-valued cases of the multiple, α-factorial functions, $n!_{(\alpha)}$, defined in (2) obtained by this article extend the study of these sequences motivated by the distinct symbolic polynomial expansions of these functions originally considered in the reference [20]. This article extends a number of the examples considered as applications of the results from the 2010 article [20] briefly summarized in the next subsection.

2.1 Polynomial expansions of generalized α-factorial functions

For any fixed integer $\alpha \geq 1$ and $n, k \in \mathbb{N}$, the coefficients defined by the triangular recurrence relation in (4) provide one approach to enumerating the symbolic polynomial expansions of the generalized factorial function product sequences defined as special cases of (1) and (2).

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\alpha} = (\alpha n + 1 - 2\alpha) \begin{bmatrix} n-1 \\ k \end{bmatrix}_{\alpha} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{\alpha} + [n = k = 0]_{\delta} \tag{4}$$

The combinatorial interpretations of these coefficients as generalized Stirling numbers of the first kind motivated in the reference [20] leads to polynomial expansions in n of the multiple factorial function sequence variants in (2) that generalize the known formulas for the single and double factorial functions, $n!$ and $n!!$, involving the unsigned Stirling numbers of the first kind, $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] = \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_1 = (-1)^{n-k} s(n,k)$, which are expanded in the forms of the following equations ([12, §6], [17, §26.8], A000142, A006882):

$$\begin{aligned} n! &= \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} (-1)^{n-m} n^m, \ \forall n \geq 1 \\ n!! &= \sum_{m=0}^{n} \begin{bmatrix} \lfloor \frac{n+1}{2} \rfloor \\ m \end{bmatrix} (-2)^{\lfloor \frac{n+1}{2} \rfloor - m} n^m, \ \forall n \geq 1 \\ n!_{(\alpha)} &= \sum_{m=0}^{n} \begin{bmatrix} \lceil n/\alpha \rceil \\ m \end{bmatrix} (-\alpha)^{\lceil \frac{n}{\alpha} \rceil - m} n^m, \ \forall n \geq 1, \alpha \in \mathbb{Z}^+. \end{aligned} \tag{5}$$

The Stirling numbers of the first kind similarly provide non-polynomial exact finite sum formulas for the single and double factorial functions in the following forms for $n \geq 1$ where $(2n)!! = 2^n \times n!$ ([12, §6.1],[4, §5.3]):

$$n! = \sum_{k=0}^{n} \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] \quad \text{and} \quad (2n-1)!! = \sum_{k=0}^{n} \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] 2^{n-k}.$$

Related finite sums generating the single factorial, double factorial, and α-factorial functions are expanded respectively through the *first-order Euler numbers*, $\left\langle\begin{smallmatrix} n \\ m \end{smallmatrix}\right\rangle$, as $n! = \sum_{k=0}^{n-1} \left\langle\begin{smallmatrix} n \\ k \end{smallmatrix}\right\rangle$ [17, §26.14(iii)], through the *second-order Euler numbers*, $\left\langle\left\langle\begin{smallmatrix} n \\ m \end{smallmatrix}\right\rangle\right\rangle$, as $(2n-1)!! = \sum_{k=0}^{n-1} \left\langle\left\langle\begin{smallmatrix} n \\ k \end{smallmatrix}\right\rangle\right\rangle$ [12, §6.2], and by the generalized cases of these triangles [20, §6.2.4].

The polynomial expansions of the first two classical sequences of the previous equations in (5) are generalized to the more general α-factorial function cases through the triangles defined as in (4) from the reference [20] through the next explicit finite sum formulas when $n \geq 1$.

$$n!_{(\alpha)} = \sum_{m=0}^{n} \left[\begin{matrix} \lfloor \frac{n-1+\alpha}{\alpha} \rfloor + 1 \\ m+1 \end{matrix}\right]_{\alpha} (-1)^{\lfloor \frac{n-1+\alpha}{\alpha} \rfloor - m} (n+1)^m, \ \forall n \geq 1, \alpha \in \mathbb{Z}^+ \tag{6}$$

The polynomial expansions in n of the generalized α-factorial functions, $(\alpha n - d)!_{(\alpha)}$, for fixed $\alpha \in \mathbb{Z}^+$ and integers $0 \leq d < \alpha$, are obtained similarly from (6) through the generalized coefficients in (4) as follows [20, cf. §2]:

$$\begin{aligned}
(\alpha n - d)!_{(\alpha)} &= (\alpha - d) \times \sum_{m=1}^{n} \left[\begin{matrix} n \\ m \end{matrix}\right]_{\alpha} (-1)^{n-m} (\alpha n + 1 - d)^{m-1} \\
&= \sum_{m=0}^{n} \left[\begin{matrix} n+1 \\ m+1 \end{matrix}\right]_{\alpha} (-1)^{n-m} (\alpha n + 1 - d)^{m}, \ \forall n \geq 1, \alpha \in \mathbb{Z}^+, 0 \leq d < \alpha.
\end{aligned} \tag{7}$$

A binomial-coefficient-themed phrasing of the products underlying the expansions of the more general factorial function sequences of this type (each formed by dividing through by a normalizing factor of $n!$) is suggested in the following expansions of these coefficients by the Pochhammer symbol [12, §5]:

$$\binom{\frac{s-1}{\alpha}}{n} = \frac{(-1)^n}{n!} \cdot \left(\frac{s-1}{\alpha}\right)_n = \frac{1}{\alpha^n \cdot n!} \prod_{j=0}^{n-1} (s - 1 - \alpha j)\,. \tag{8}$$

When the initially fixed indeterminate $s := s_n$ is considered modulo α in the form of $s_n := \alpha n + d$ for some fixed least integer residue, $0 \leq d < \alpha$, the prescribed setting of this offset d completely determines the numerical α-factorial function sequences of the forms in (7) generated by these products (see, for example, the examples cited below in Section 3.2 and the tables given in the reference [20, cf. §6.1.2, Table 6.1]).

For any lower index $n \geq 1$, the binomial coefficient formulation of the multiple factorial function products in (8) provides the next several expansions by the exponential generating functions for the generalized coefficient triangles in (4), and their corresponding generalized Stirling polynomial analogs, $\sigma_k^{(\alpha)}(x)$, defined in the references ([20, §5],[12, cf. §6, §7.4]):

$$\begin{aligned}
\binom{\frac{s-1}{\alpha}}{n} &= \sum_{m=0}^{n} \left[\begin{matrix} n+1 \\ n+1-m \end{matrix}\right]_{\alpha} \frac{(-1)^m s^{n-m}}{\alpha^n n!} \\
&= \sum_{m=0}^{n} \frac{(-1)^m \cdot (n+1) \sigma_m^{(\alpha)}(n+1)}{\alpha^m} \times \frac{(s/\alpha)^{n-m}}{(n-m)!}
\end{aligned} \tag{9}$$

$$\binom{\frac{s-1}{\alpha}}{n} = [z^n] \left(e^{(s-1+\alpha)z/\alpha} \left(\frac{-ze^{-z}}{e^{-z}-1} \right)^{n+1} \right) \tag{10}$$

$$= [z^n w^n] \left(-\frac{z \cdot e^{(s-1+\alpha)z/\alpha}}{1+wz-e^z}\right).$$

The generalized forms of the *Stirling convolution polynomials*, $\sigma_n(x) \equiv \sigma_n^{(1)}(x)$, and the α-*factorial polynomials*, $\sigma_n^{(\alpha)}(x)$, studied in the reference are defined for each $n \geq 0$ through the triangle in (4) as follows [20, §5.2]:

$$\begin{aligned} x \cdot \sigma_n^{(\alpha)}(x) &:= \begin{bmatrix} x \\ x-n \end{bmatrix}_\alpha \frac{(x-n-1)!}{(x-1)!} \qquad \text{(\underline{Generalized Stirling Polynomials})} \\ &= [z^n] \left(e^{(1-\alpha)z} \left(\frac{\alpha z e^{\alpha z}}{e^{\alpha z}-1} \right)^x \right). \end{aligned}$$

A more extensive treatment of the properties and generating function relations satisfied by the triangular coefficients defined by (4), including their similarities to the Stirling number triangles, Stirling convolution polynomial sequences, and the generalized Bernoulli polynomials, among relations to several other notable special sequences, is provided in the references.

2.2 Divergent ordinary generating functions approximated by the convergents to infinite Jacobi-type and Stieltjes-type continued fraction expansions

2.2.1 Infinite J-fraction expansions generating the rising factorial function

Another approach to enumerating the symbolic expansions of the generalized α-factorial function sequences outlined above is constructed as a new generalization of the continued fraction series representations of the ordinary generating function for the rising factorial function, or Pochhammer symbol, $(x)_n = \Gamma(x+n)/\Gamma(x)$, first proved by Flajolet [8, 9]. For any fixed non-zero indeterminate, $x \in \mathbb{C}$, the ordinary power series enumerating the rising factorial sequence is defined through the next infinite Jacobi-type J-fraction expansion [8, §2, p. 148]:

$$R_0(x,z) := \sum_{n \geq 0} (x)_n z^n = \cfrac{1}{1 - xz - \cfrac{1 \cdot xz^2}{1-(x+2)z - \cfrac{2(x+1)z^2}{1-(x+4)z - \cfrac{3(x+2)z^2}{\cdots}}}}. \qquad (11)$$

Since we know symbolic polynomial expansions of the functions, $(x)_n$, through the Stirling numbers of the first kind, we notice that the terms in a convergent power series defined by (11) correspond to the normalized coefficients of the following well-known two-variable "*double*", or "*super*", exponential generating functions (EGFs) for the Stirling number triangle when x

is taken to be a fixed, formal parameter with respect to these series ([12, §7.4],[17, §26.8(ii)], [8, cf. Prop. 9]):

$$\sum_{n\geq 0} (x)_n \frac{z^n}{n!} = \frac{1}{(1-z)^x} \qquad \text{and} \qquad \sum_{n\geq 0} x^{\underline{n}} \cdot \frac{z^n}{n!} = (1+z)^x.$$

For natural numbers $m \geq 1$ and fixed $\alpha \in \mathbb{Z}^+$, the coefficients defined by the generalized triangles in (4) are enumerated similarly by the generating functions [20, cf. §3.3]

$$\sum_{m,n\geq 0} \begin{bmatrix} n+1 \\ m+1 \end{bmatrix}_{\alpha} \frac{x^m z^n}{n!} = (1-\alpha z)^{-(x+1)/\alpha}. \tag{12}$$

When x depends linearly on n, the ordinary generating functions for the numerical factorial functions formed by $(x)_n$ do not converge for $z \neq 0$. However, the convergents of the continued fraction representations of these series still lead to partial, truncated series approximations generating these generalized product sequences, which in turn immediately satisfy a number of combinatorial properties, recurrence relations, and other established integer congruence properties implied by the rational convergents to the first continued fraction expansion given in (11).

2.2.2 Examples

Two particular divergent ordinary generating functions for the single factorial function sequences, $f_1(n) := n!$ and $f_2(n) := (n+1)!$, are cited in the references as examples of the Jacobi-type J-fraction results proved in Flajolet's articles ([8, 9],[16, cf. §5.5]). The next pair of series expansions serve to illustrate the utility to enumerating each sequence formally with respect to z required by the results in this article [8, Thm. 3A; Thm. 3B].

$$F_{1,\infty}(z) := \sum_{n\geq 0} n! \cdot z^n = \cfrac{1}{1 - z - \cfrac{1^2 \cdot z^2}{1 - 3z - \cfrac{2^2 z^2}{\cdots}}} \qquad \text{(\underline{\textit{Single Factorial J-Fractions}})}$$

$$F_{2,\infty}(z) := \sum_{n\geq 0} (n+1)! \cdot z^n = \cfrac{1}{1 - 2z - \cfrac{1 \cdot 2z^2}{1 - 4z - \cfrac{2 \cdot 3z^2}{\cdots}}}$$

In each of these respective formal power series expansions, we immediately see that for each finite $h \geq 1$, the h^{th} convergent functions, denoted $F_{i,h}(z)$ for $i = 1, 2$, satisfy $f_i(n) = [z^n]F_{i,h}(z)$ whenever $1 \leq n < 2h$. We also have that $f_i(n) \equiv [z^n]F_{i,h}(z) \pmod{p}$ for any $n \geq 0$ whenever p is a divisor of h ([9],[16, cf. §5]).

Similar expansions of other factorial-related continued fraction series are given in the references ([8],[16, cf. §5.9]). For example, the next known *Stieltjes-type* continued fractions

(*S-fractions*), formally generating the double factorial function, $(2n-1)!!$, and the *Catalan numbers*, C_n, respectively, are expanded through the convergents of the following infinite continued fractions ([8, Prop. 5; Thm. 2], [16, §5.5], A001147, A000108):

$$F_{3,\infty}(z) := \sum_{n\geq 0} \underbrace{1\cdot 3\cdots(2n-1)}_{(2n-1)!!} \times z^{2n} \quad = \cfrac{1}{1-\cfrac{1\cdot z^2}{1-\cfrac{2\cdot z^2}{1-\cfrac{3\cdot z^2}{\cdots}}}} \qquad (\underline{\textit{Double Factorial S-Fractions}})$$

$$F_{4,\infty}(z) := \sum_{n\geq 0} \underbrace{\frac{2^n(2n-1)!!}{(n+1)!}}_{C_n = \binom{2n}{n}\frac{1}{(n+1)}} \times z^{2n} \quad = \cfrac{1}{1-\cfrac{z^2}{1-\cfrac{z^2}{1-\cfrac{z^2}{\cdots}}}}.$$

For comparison, some related forms of regularized ordinary power series in z generating the single and double factorial function sequences from the previous examples are stated in terms of the *incomplete gamma function*, $\Gamma(a,z) = \int_z^\infty t^{a-1}e^{-t}dt$, as follows [17, §8]:

$$\begin{aligned}\sum_{n\geq 0} n!\cdot z^n &= -\frac{e^{-1/z}}{z} \times \Gamma\left(0, -\frac{1}{z}\right) \\ \sum_{n\geq 0}(n+1)!\cdot z^n &= -\frac{e^{-1/z}}{z^2} \times \Gamma\left(-1, -\frac{1}{z}\right) \\ \sum_{n\geq 1}(2n-1)!!\cdot z^n &= -\frac{e^{-1/2z}}{(-2z)^{3/2}} \times \Gamma\left(-\frac{1}{2}, -\frac{1}{2z}\right). \end{aligned} \tag{13}$$

Since $p_n(\alpha, R) = \alpha^n (R/\alpha)_n$, the exponential generating function for the generalized product sequences corresponds to the series ([12, cf. §7.4, eq. (7.55)],[14]):

$$\widehat{P}(\alpha, R; z) := \sum_{n=0}^{\infty} p_n(\alpha, R)\frac{z^n}{n!} = (1-\alpha z)^{-R/\alpha}, \qquad (\underline{\textit{Generalized Product Sequence EGFs}})$$

where for each fixed $\alpha \in \mathbb{Z}^+$ and $0 \leq r < \alpha$, we have the identities, $(\alpha n - r)!_{(\alpha)} = p_n(\alpha, \alpha - r) = \alpha^n \left(1 - \frac{r}{\alpha}\right)_n$. The form of this exponential generating function then leads to the next forms of the regularized sums by applying a *Laplace transform* to the generating functions in the previous equation (see Remark 19) ([7, §B.14],[17, cf. §8.6(i)]).

$$\begin{aligned}\widetilde{B}_{\alpha,-r}(z) &:= \sum_{n\geq 0} (\alpha n - r)!_{(\alpha)} z^n \\ &= \int_0^\infty \frac{e^{-t}}{(1-\alpha t z)^{1-r/\alpha}} dt\end{aligned}$$

$$= \frac{e^{-\frac{1}{\alpha z}}}{(-\alpha z)^{1-r/\alpha}} \times \Gamma\left(\frac{r}{\alpha}, -\frac{1}{\alpha z}\right)$$

The remarks given in Section 4.3 suggest similar approximations to the α-factorial functions generated by the generalized convergent functions defined in the next section, and their relations to the confluent hypergeometric functions and the associated Laguerre polynomial sequences ([17, cf. §18.5(ii)], [19, §4.3.1]).

2.3 Generalized convergent functions generating factorial-related integer product sequences

2.3.1 Definitions of the generalized J-fraction expansions and the generalized convergent function series

We state the next definition as a generalization of the result for the rising factorial function due to Flajolet cited in (11) to form the analogous series enumerating the multiple, or α-factorial, product sequence cases defined by (1) and (2).

Definition 1 (Generalized J-Fraction Convergent Functions)**.** Suppose that the parameters $h \in \mathbb{N}$, $\alpha \in \mathbb{Z}^+$ and $R := R(n)$ are defined in the notation of the product-wise sequences from (1). For $h \geq 0$ and $z \in \mathbb{C}$, we let the component numerator and denominator convergent functions, denoted $\mathrm{FP}_h(\alpha, R; z)$ and $\mathrm{FQ}_h(\alpha, R; z)$, respectively, be defined by the recurrence relations in the next equations.

$$\mathrm{FP}_h(\alpha, R; z) := \begin{cases} (1-(R+2\alpha(h-1))z)\,\mathrm{FP}_{h-1}(\alpha,R;z)-\alpha(R+\alpha(h-2))(h-1)z^2\,\mathrm{FP}_{h-2}(\alpha,R;z) & \text{if } h \geq 2; \\ 1, & \text{if } h = 1; \\ 0, & \text{otherwise.} \end{cases} \tag{14}$$

$$\mathrm{FQ}_h(\alpha, R; z) := \begin{cases} (1-(R+2\alpha(h-1))z)\,\mathrm{FQ}_{h-1}(\alpha,R;z)-\alpha(R+\alpha(h-2))(h-1)z^2\,\mathrm{FQ}_{h-2}(\alpha,R;z) & \text{if } h \geq 2; \\ 1 - Rz, & \text{if } h = 1; \\ 1, & \text{if } h = 0; \\ 0, & \text{otherwise.} \end{cases} \tag{15}$$

The corresponding convergent functions, $\mathrm{Conv}_h(\alpha, R; z)$, defined in the next equation then provide the rational, formal power series approximations in z to the divergent ordinary generating functions of many factorial-related sequences formed as special cases of the symbolic products in (1).

$$\mathrm{Conv}_h(\alpha, R; z) := \cfrac{1}{1 - R\cdot z - \cfrac{\alpha R\cdot z^2}{1-(R+2\alpha)\cdot z - \cfrac{2\alpha(R+\alpha)\cdot z^2}{1-(R+4\alpha)\cdot z - \cfrac{3\alpha(R+2\alpha)\cdot z^2}{\cfrac{\cdots}{1-(R+2(h-1)\alpha)\cdot z}}}}}$$

$$= \frac{\mathrm{FP}_h(\alpha, R; z)}{\mathrm{FQ}_h(\alpha, R; z)} = \sum_{n=0}^{2h-1} p_n(\alpha, R) z^n + \sum_{n=2h}^{\infty} \widetilde{e}_{h,n}(\alpha, R) z^n \tag{16}$$

The first series coefficients on the right-hand-side of (16) generate the products, $p_n(\alpha, R)$, from (1), where the remaining forms of the power series coefficients, $\widetilde{e}_{h,n}(\alpha, R)$, correspond to "error terms" in the truncated formal series approximations to the exact sequence generating functions obtained from these convergent functions, which are defined such that $p_n(\alpha, R) \equiv \widetilde{e}_{h,n}(\alpha, R) \pmod{h}$ for all $h \geq 2$ and $n \geq 2h$.

2.3.2 Properties of the generalized J-fraction convergent functions

A number of the immediate, noteworthy properties satisfied by these generalized convergent functions are apparent from inspection of the first few special cases provided in Table 9.1 (page 67) and in Table 9.2 (page 68). The most important of these properties relevant to the new interpretations of the α-factorial function sequences proved in the next sections of the article are briefly summarized in the points stated below.

- **Rationality of the convergent functions in α, R, and z:**
  For any fixed $h \geq 1$, it is easy to show that the component convergent functions, $\mathrm{FP}_h(z)$ and $\mathrm{FQ}_h(z)$, defined by (14) and (15), respectively, are polynomials of finite degree in each of z, R, and α satisfying

  $$\deg_{z,R,\alpha}\{\mathrm{FP}_h(\alpha, R; z)\} = h - 1 \quad \text{and} \quad \deg_{z,R,\alpha}\{\mathrm{FQ}_h(\alpha, R; z)\} = h.$$

  For any $h, n \in \mathbb{Z}^+$, if $R := R(n)$ denotes some linear function of n, the product sequences, $p_n(\alpha, R)$, generated by the generalized convergent functions always correspond to polynomials in n (in R) of predictably finite degree with integer coefficients determined by the choice of $n \geq 1$.

- **Expansions of the denominator convergent functions by special functions**:
  For all $h \geq 0$, and fixed non-zero parameters α and R, the power series in z generated by the generalized h^{th} convergents, $\mathrm{Conv}_h(\alpha, R; z)$, are characterized by the representations of the convergent denominator functions, $\mathrm{FQ}_h(\alpha, R; z)$, through the

confluent hypergeometric functions, $U(a,b,w)$ and $M(a,b,w)$, and the associated Laguerre polynomial sequences, $L_n^{(\beta)}(x)$, as follows (see Section 4.3) ([17, §13; §18], [19, §4.3.1]):

$$\underbrace{z^h \cdot \mathrm{FQ}_h\left(\alpha, R; z^{-1}\right)}_{\widetilde{\mathrm{FQ}}_h(\alpha,R;z)} = \alpha^h \times U\left(-h, \frac{R}{\alpha}, \frac{z}{\alpha}\right) \tag{17}$$

$$= (-\alpha)^h \left(R/\alpha\right)_h \times M\left(-h, \frac{R}{\alpha}, \frac{z}{\alpha}\right)$$

$$= (-\alpha)^h \cdot h! \times L_h^{(R/\alpha-1)}\left(\frac{z}{\alpha}\right).$$

The special function expansions of the reflected convergent denominator function sequences above lead to the statements of addition theorems, multiplication theorems, and several additional auxiliary recurrence relations for these functions proved in Section 5.1.

- **Corollaries: New exact formulas and congruence properties for the α-factorial functions and the generalized product sequences**:
  If some ordering of the h zeros of (17) is fixed at each $h \geq 1$, we can define the next sequences which form special cases of the zeros studied in the references [10, 3]. In particular, each of the following special zero sequence definitions given as ordered sets, or ordered lists of zeros, provide factorizations over z of the denominator sequences, $\mathrm{FQ}_h(\alpha, R; z)$, parameterized by α and R [17, cf. §13.9, §18.16]:

$$\left(\ell_{h,j}(\alpha,R)\right)_{j=1}^{h} := \left\{z_j : \alpha^h \times U\left(-h, R/\alpha, \frac{z}{\alpha}\right) = 0,\ 1 \leq j \leq h\right\} \quad (\underline{\textit{Special Function Zeros}})$$

$$= \left\{z_j : \alpha^h \times L_h^{(R/\alpha-1)}\left(\frac{z}{\alpha}\right) = 0,\ 1 \leq j \leq h\right\}.$$

Let the sequences, $c_{h,j}(\alpha, R)$, denote a shorthand for the coefficients corresponding to an expansion of the generalized convergent functions, $\mathrm{Conv}_h(\alpha, R; z)$, by partial fractions in z, i.e., the coefficients defined so that [17, §1.2(iii)]

$$\mathrm{Conv}_h(\alpha, R; z) := \sum_{j=1}^{h} \frac{c_{h,j}(\alpha,R)}{(1 - \ell_{h,j}(\alpha,R) \cdot z)}.$$

For $n \geq 1$ and any fixed integer $\alpha \neq 0$, these rational convergent functions provide the following formulas exactly generating the respective sequence cases in (1) and (2):

$$p_n(\alpha, R) = \sum_{j=1}^{n} c_{n,j}(\alpha,R) \times \ell_{n,j}(\alpha,R)^n \tag{18}$$

$$n!_{(\alpha)} = \sum_{j=1}^{n} c_{n,j}(-\alpha,n) \times \ell_{n,j}(-\alpha,n)^{\lfloor \frac{n-1}{\alpha} \rfloor}.$$

The corresponding congruences satisfied by each of these generalized sequence cases obtained from the h^{th} convergent function expansions in z are stated similarly modulo any prescribed integers $h \geq 2$ and fixed $\alpha \geq 1$ in the next forms.

$$\begin{aligned} p_n(\alpha, R) &\equiv \sum_{j=1}^{h} c_{h,j}(\alpha, R) \times \ell_{h,j}(\alpha, R)^n && \pmod{h} \\ n!_{(\alpha)} &\equiv \sum_{j=1}^{h} c_{h,j}(-\alpha, n) \times \ell_{h,j}(-\alpha, n)^{\lfloor \frac{n-1}{\alpha} \rfloor} && \pmod{h, h\alpha, \cdots, h\alpha^h} \end{aligned} \tag{19}$$

Section 3.3 and Section 6.2 provide several particular special case examples of the new congruence properties expanded by (19).

3 New results proved within the article

3.1 A summary of the new results and outline of the article topics

J-Fractions for generalized factorial function sequences (Section 4 on page 20)

The article contains a number of new results and new examples of applications of the results from Section 4 in the next subsections. The Jacobi-type continued fraction expansions formally enumerating the generalized factorial functions, $p_n(\alpha, R)$, proved in Section 4 are new, and moreover, follow easily from the known continued fractions for the series generating the *rising factorial function*, $(x)_n = p_n(1, x)$, established by Flajolet [8].

Properties of the generalized convergent functions (Section 5 on page 24)

In Section 5 we give proofs of new properties, expansions, recurrence relations, and exact closed-form representations by special functions satisfied by the convergent numerator and denominator subsequences, $\mathrm{FP}_h(\alpha, R; z)$ and $\mathrm{FQ}_h(\alpha, R; z)$. The consideration of the convergent function approximations to these infinite continued fraction expansions is a new topic not previously explored in the references which leads to new integer congruence results for factorial functions as well as new applications to generating function identities enumerating factorial-related integer sequences.

Applications and motivating examples (Section 6 on page 32)

Specific consequences of the convergent function properties we prove in Section 5 of the article include new congruences for and new rational generating function representations enumerating the Stirling numbers of the first kind, $\left[{n \atop k}\right]$, modulo fixed integers $m \geq 2$ and the scaled r-order harmonic numbers, $n!^r \cdot H_n^{(r)}$, modulo m, which are easily extended to formulate analogous congruence results for the α-factorial functions, $n!_{(\alpha)}$, and the generalized factorial product functions, $p_n(\alpha, R)$. These particular special cases of the new congruence results are proved in Section 6.2. Section 6.2 also contains proofs of new representations of exact formulas for factorial functions expanded by the special zeros of the *Laguerre polynomials* and *confluent hypergeometric* functions.

The subsections of Section 6 also provide specific sequence examples and new sequence generating function identities that demonstrate the utility and breadth of new applications implied by the convergent-based rational and hybrid-rational generating functions we rigorously treat first in Section 4 and Section 5. For example, in Section 6.3, we are the first to notice several specific integer sequence applications of new convergent-function-based *Hadamard product* identities that effectively provide truncated series approximations to the formal *Laplace-Borel transformation* where multiples of the *rational* convergents, $\mathrm{Conv}_n(\alpha, R; z)$, generate the sequence multiplier, $n!$, in place of more standard integral representations of the transformation. In Section 6.4 through Section 6.7, we focus on expanding particular examples of convergent-function-based generating functions enumerating special factorial-related integer sequences and combinatorial identities. The next few subsections of the article provide several special case examples of the new applications, new congruences, and other examples of the new results established in Section 6.

3.2 Examples of factorial-related finite product sequences enumerated by the generalized convergent functions

3.2.1 Generating functions for arithmetic progressions of the α-factorial functions

There are a couple of noteworthy subtleties that arise in defining the specific numerical forms of the α-factorial function sequences defined by (2) and (6). First, since the generalized convergent functions generate the distinct symbolic products that characterize the forms of these expansions, we see that the following convergent-based enumerations of the multiple factorial sequence variants hold at each $\alpha, n \in \mathbb{Z}^+$, and some fixed choice of the prescribed offset, $0 \leq d < \alpha$:

$$\begin{aligned}(\alpha n - d)!_{(\alpha)} &= \underbrace{(-\alpha)^n \cdot \left(\frac{d}{\alpha} - n\right)_n}_{p_n(-\alpha, \alpha n - d)} = [z^n]\,\mathrm{Conv}_n\left(-\alpha, \alpha n - d; z\right) \\ &= \underbrace{\alpha^n \cdot \left(1 - \frac{d}{\alpha}\right)_n}_{p_n(\alpha, \alpha - d)} \qquad = [z^n]\,\mathrm{Conv}_n\left(\alpha, \alpha - d; z\right).\end{aligned} \tag{20}$$

For example, some variants of the arithmetic progression sequences formed by the single factorial and double factorial functions, $n!$ and $n!!$, in Section 6.4 are generated by the particular shifted inputs to these functions highlighted by the special cases in the next equations (A000142, A000165, A001147):

$$\begin{aligned}
(n!)_{n=1}^{\infty} &= ((1)_n)_{n=1}^{\infty} & \xrightarrow{\texttt{A000142}} & \quad (1, 2, 6, 24, 120, 720, 5040, \ldots) \\
((2n)!!)_{n=1}^{\infty} &= (2^n \cdot (1)_n)_{n=1}^{\infty} & \xrightarrow{\texttt{A001147}} & \quad (2, 8, 48, 384, 3840, 46080, \ldots) \\
((2n-1)!!)_{n=1}^{\infty} &= (2^n \cdot (1/2)_n)_{n=1}^{\infty} & \xrightarrow{\texttt{A000165}} & \quad (1, 3, 15, 105, 945, 10395, \ldots).
\end{aligned}$$

The next few special case variants of the α-factorial function sequences corresponding to $\alpha := 3, 4$, also expanded in Section 6.4, are given in the following sequence forms (A032031, A008544, A007559, A047053, A007696):

$$\begin{aligned}
((3n)!!!)_{n=1}^{\infty} &= (3^n \cdot (1)_n)_{n=1}^{\infty} & \xrightarrow{\texttt{A032031}} & \quad (3, 18, 162, 1944, 29160, \ldots) \\
((3n-1)!!!)_{n=1}^{\infty} &= (3^n \cdot (2/3)_n)_{n=1}^{\infty} & \xrightarrow{\texttt{A008544}} & \quad (2, 10, 80, 880, 12320, 209440, \ldots) \\
((3n-2)!!!)_{n=1}^{\infty} &= (3^n \cdot (1/3)_n)_{n=1}^{\infty} & \xrightarrow{\texttt{A007559}} & \quad (1, 4, 28, 280, 3640, 58240, \ldots) \\
\left((4n)!_{(4)}\right)_{n=0}^{\infty} &= (4^n \cdot (1)_n)_{n=0}^{\infty} & \xrightarrow{\texttt{A047053}} & \quad (1, 4, 32, 384, 6144, 122880, \ldots) \\
\left((4n+1)!_{(4)}\right)_{n=0}^{\infty} &= (4^n \cdot (5/4)_n)_{n=0}^{\infty} & \xrightarrow{\texttt{A007696}} & \quad (1, 5, 45, 585, 9945, 208845, \ldots).
\end{aligned}$$

For each $n \in \mathbb{N}$ and prescribed constants $r, c \in \mathbb{Z}$ defined such that $c \mid n + r$, we also obtain rational convergent-based generating functions enumerating the modified single factorial function sequences given by

$$\left(\tfrac{n+r}{c}\right)! = [z^n] \operatorname{Conv}_h\left(-c, n+r; \frac{z}{c}\right) + \left[\frac{r}{c} = 0\right]_{\delta} [n = 0]_{\delta}, \ \forall\, h \geq \lfloor (n+r)/c \rfloor. \tag{21}$$

3.2.2 Generating functions for multi-valued integer product sequences

Likewise, given any $n \geq 1$ and fixed $\alpha \in \mathbb{Z}^+$, we can enumerate the somewhat less obvious full forms of the generalized α-factorial function sequences defined piecewise for the distinct residues, $n \in \{0, 1, \ldots, \alpha - 1\}$, modulo α by (2) and in (20). The multi-valued products defined by (1) for these functions are generated as follows:

$$\begin{aligned}
n!_{(\alpha)} &= \left[z^{\lfloor (n+\alpha-1)/\alpha \rfloor}\right] \operatorname{Conv}_n\left(-\alpha, n; z\right) && (22) \\
&= [z^n] \left(\sum_{0 \leq d < \alpha} z^{-d} \cdot \operatorname{Conv}_n\left(\alpha, \alpha - d; z^{\alpha}\right)\right) \\
&= [z^{n+\alpha-1}] \left(\frac{1 - z^{\alpha}}{1 - z} \times \operatorname{Conv}_n\left(-\alpha, n; z^{\alpha}\right)\right). && (23)
\end{aligned}$$

The complete sequences over the multi-valued symbolic products formed by the special cases of the double factorial function, the *triple factorial* function, $n!!!$, the *quadruple factorial* function, $n!!!! = n!_{(4)}$, the *quintuple factorial* (5-*factorial*) function, $n!_{(5)}$, and the 6-*factorial* function, $n!_{(6)}$, respectively, are generated by the convergent generating function approximations expanded in the next equations (A006882, A007661, A007662, A085157, A085158).

$$\begin{aligned}
(n!!)_{n=1}^{\infty} &= \left(\left[z^{\lfloor (n+1)/2 \rfloor}\right] \operatorname{Conv}_n(-2, n; z)\right)_{n=1}^{\infty} && \xrightarrow{\text{A006882}} && (1, 2, 3, 8, 15, 48, 105, 384, \ldots) \\
(n!!!)_{n=1}^{\infty} &= \left(\left[z^{\lfloor (n+2)/3 \rfloor}\right] \operatorname{Conv}_n(-3, n; z)\right)_{n=1}^{\infty} && \xrightarrow{\text{A007661}} && (1, 2, 3, 4, 10, 18, 28, 80, 162, \ldots) \\
\left(n!_{(4)}\right)_{n=1}^{\infty} &= \left(\left[z^{\lfloor (n+3)/4 \rfloor}\right] \operatorname{Conv}_n(-4, n; z)\right)_{n=1}^{\infty} && \xrightarrow{\text{A007662}} && (1, 2, 3, 4, 5, 12, 21, 32, 45, \ldots) \\
\left(n!_{(5)}\right)_{n=1}^{\infty} &= \left(\left[z^{\lfloor (n+4)/5 \rfloor}\right] \operatorname{Conv}_n(-5, n; z)\right)_{n=1}^{\infty} && \xrightarrow{\text{A085157}} && (1, 2, 3, 4, 5, 6, 14, 24, 36, 50, \ldots) \\
\left(n!_{(6)}\right)_{n=1}^{\infty} &= \left(\left[z^{\lfloor (n+5)/6 \rfloor}\right] \operatorname{Conv}_n(-6, n; z)\right)_{n=1}^{\infty} && \xrightarrow{\text{A085158}} && (1, 2, 3, 4, 5, 6, 7, 16, 27, 40, \ldots)
\end{aligned}$$

3.2.3 Examples of new convergent-based generating function identities for binomial coefficients

The rationality in z of the generalized convergent functions, $\operatorname{Conv}_h(\alpha, R; z)$, for all $h \geq 1$ also provides several of the new forms of generating function identities for many factorial-related product sequences and related expansions of the binomial coefficients that are easily proved from the diagonal-coefficient, or Hadamard product, generating function results established in Section 6.3. For example, for natural numbers $n \geq 1$, the next variants of the binomial-coefficient-related product sequences are enumerated by the following coefficient identities (A009120, A001448):

$$\begin{aligned}
\frac{(4n)!}{(2n)!} &= \frac{4^{4n} \cancel{(1)_n} \cancel{\left(\frac{2}{4}\right)_n} \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{2^{2n} \cancel{(1)_n} \cancel{\left(\frac{1}{2}\right)_n}} \qquad (24) \\
&= 4^n \times 4^n (1/4)_n \times 4^n (3/4)_n \\
&= [x_1^0 z^n] \left(\operatorname{Conv}_n\left(4, 3; \frac{4z}{x_1}\right) \operatorname{Conv}_n(4, 1; x_1)\right) \\
&= 4^n \times (4n-3)!_{(4)} (4n-1)!_{(4)} \\
&= [x_1^0 z^n] \left(\operatorname{Conv}_n\left(-4, 4n-3; \frac{4z}{x_1}\right) \operatorname{Conv}_n(-4, 4n-1; x_1)\right) \\
\binom{4n}{2n} &= [x_1^0 x_2^0 z^n] \left(\operatorname{Conv}_n\left(4, 3; \frac{4z}{x_2}\right) \operatorname{Conv}_n\left(4, 1; \frac{x_2}{x_1}\right) \times \underbrace{\cosh\left(\sqrt{x_1}\right)}_{\widehat{E}_2(x_1) = E_{2,1}(x_1)}\right) \\
&= [x_1^0 x_2^0 z^n] \left(\operatorname{Conv}_n\left(-4, 4n-3; \frac{4z}{x_2}\right) \operatorname{Conv}_n\left(-4, 4n-1; \frac{x_2}{x_1}\right) \times \underbrace{\cosh\left(\sqrt{x_1}\right)}_{E_{2,1}(x_1)}\right).
\end{aligned}$$

The examples given in Section 6.3.2 provide examples of related constructions of the hybrid rational convergent-based generating function products that generate the central binomial coefficients and several other notable cases of related sequence expansions. We can similarly generate the sequences of binomials, $(a+b)^n$ and $c^n - 1$ for fixed non-zero $a, b, c \in \mathbb{R}$, which we consider in Section 6.7, using the binomial theorem and a rational convergent-based approximation to the formal Laplace-Borel transform as follows:

$$\begin{aligned}
(a+b)^n &= n! \times \sum_{k=0}^{n} \frac{a^k}{k!} \cdot \frac{b^{n-k}}{(n-k)!} \\
&= [x^0][z^n] \operatorname{Conv}_n\left(1, 1; \frac{z}{x}\right) e^{(a+b)x} \\
&= [x^0][z^n] \operatorname{Conv}_n\left(-1, n; \frac{z}{x}\right) e^{(a+b)x} \\
c^n - 1 &= n! \times \sum_{k=0}^{n-1} \frac{(c-1)^{k+1}}{(k+1)!} \cdot \frac{1}{(n-1-k)!} \\
&= [x^0][z^n] \operatorname{Conv}_n\left(1, 1; \frac{z}{x}\right) \left(e^{(c-1)x} - 1\right) e^x \\
&= [x^0][z^n] \operatorname{Conv}_n\left(-1, n; \frac{z}{x}\right) \left(e^{(c-1)x} - 1\right) e^x.
\end{aligned}$$

3.3 Examples of new congruences for the α-factorial functions, the Stirling numbers of the first kind, and the r-order harmonic number sequences

3.3.1 Congruences for the α-factorial functions modulo 2

For any fixed $\alpha \in \mathbb{Z}^+$ and natural numbers $n \geq 1$, the generalized multiple, α-factorial functions, $n!_{(\alpha)}$, defined by (2) satisfy the following congruences modulo 2 (and 2α):

$$\begin{aligned}
n!_{(\alpha)} &\equiv \frac{n}{2}\left(\left(n-\alpha+\sqrt{\alpha(\alpha-n)}\right)^{\left\lfloor \frac{n-1}{\alpha} \right\rfloor} + \left(n-\alpha-\sqrt{\alpha(\alpha-n)}\right)^{\left\lfloor \frac{n-1}{\alpha} \right\rfloor}\right) \pmod{2, 2\alpha} \\
&= [z^n]\left(\frac{(z^\alpha-1)(1+(2\alpha-n)z^\alpha)}{z(1-z)\left((\alpha-n)(n \cdot z^\alpha-2)z^\alpha-1\right)}\right). \qquad (25)
\end{aligned}$$

Given that the definition of the single factorial function implies that $n! \equiv 0 \pmod{2}$ whenever $n \geq 2$, the statement of (25) provides somewhat less obvious results for the generalized α-factorial function sequence cases when $\alpha \geq 2$. Table 9.3 (page 71) provides specific listings of the result in (25) satisfied by the α-factorial functions, $n!_{(\alpha)}$, for $\alpha := 1, 2, 3, 4$. The corresponding, closely-related new forms of congruence properties satisfied by these functions expanded by exact algebraic formulas modulo 3 (3α) and modulo 4 (4α) are also cited as special cases in the examples given in the next subsection (see Section 6.2.2).

3.3.2 New forms of congruences for the α-factorial functions modulo 3, modulo 4, and modulo 5

To simplify notation, we first define the next shorthand for the respective (distinct) roots, $r_{p,i}^{(\alpha)}(n)$ for $1 \leq i \leq p$, corresponding to the special cases of the convergent denominator functions, $\mathrm{FQ}_p(\alpha, R; z)$, factorized over z for any fixed integers $n, \alpha \geq 1$ when $p := 3, 4, 5$ [17, §1.11(iii); cf. §4.43]:

$$\begin{aligned}
\left(r_{3,i}^{(\alpha)}(n)\right)_{i=1}^{3} &:= \big\{z_i : z_i^3 - 3z_i^2(2\alpha+n) + 3z_i(\alpha+n)(2\alpha+n) \\
&\qquad - n(\alpha+n)(2\alpha+n) = 0,\ 1 \leq i \leq 3\big\} \\
\left(r_{4,j}^{(\alpha)}(n)\right)_{j=1}^{4} &:= \big\{z_j : z_j^4 - 4z_j^3(3\alpha+n) + 6z_j^2(2\alpha+n)(3\alpha+n) - 4z_j(\alpha+n)(2\alpha+n)(3\alpha+n) \\
&\qquad + n(\alpha+n)(2\alpha+n)(3\alpha+n) = 0,\ 1 \leq j \leq 4\big\} \\
\left(r_{5,k}^{(\alpha)}(n)\right)_{k=1}^{5} &:= \big\{z_k : z_k^5 - 5(4\alpha+n)z_k^4 + 10(3\alpha+n)(4\alpha+n)z_k^3 \\
&\qquad - 10(2\alpha+n)(3\alpha+n)(4\alpha+n)z_k^2 \\
&\qquad + 5(\alpha+n)(2\alpha+n)(3\alpha+n)(4\alpha+n)z_k \\
&\qquad - n(\alpha+n)(2\alpha+n)(3\alpha+n)(4\alpha+n) = 0,\ 1 \leq k \leq 5\big\}.
\end{aligned} \tag{26}$$

Similarly, we define the following functions for any fixed $\alpha \in \mathbb{Z}^+$ and $n \geq 1$ to simplify the notation in stating next the congruences in (27) below:

$$\begin{aligned}
\widetilde{R}_3^{(\alpha)}(n) &:= \frac{\left(6\alpha^2 + \alpha\left(6r_{3,1}^{(-\alpha)}(n) - 4n\right) + \left(n - r_{3,1}^{(-\alpha)}(n)\right)^2\right) r_{3,1}^{(-\alpha)}(n)^{\lfloor \frac{n-1}{\alpha} \rfloor + 1}}{\left(r_{3,1}^{(-\alpha)}(n) - r_{3,2}^{(-\alpha)}(n)\right)\left(r_{3,1}^{(-\alpha)}(n) - r_{3,3}^{(-\alpha)}(n)\right)} \\
&\quad + \frac{\left(6\alpha^2 + \alpha\left(6r_{3,3}^{(-\alpha)}(n) - 4n\right) + \left(n - r_{3,3}^{(-\alpha)}(n)\right)^2\right) r_{3,3}^{(-\alpha)}(n)^{\lfloor \frac{n-1}{\alpha} \rfloor + 1}}{\left(r_{3,3}^{(-\alpha)}(n) - r_{3,1}^{(-\alpha)}(n)\right)\left(r_{3,3}^{(-\alpha)}(n) - r_{3,2}^{(-\alpha)}(n)\right)} \\
&\quad + \frac{\left(6\alpha^2 + \alpha\left(6r_{3,2}^{(-\alpha)}(n) - 4n\right) + \left(n - r_{3,2}^{(-\alpha)}(n)\right)^2\right) r_{3,2}^{(-\alpha)}(n)^{\lfloor \frac{n-1}{\alpha} \rfloor + 1}}{\left(r_{3,2}^{(-\alpha)}(n) - r_{3,1}^{(-\alpha)}(n)\right)\left(r_{3,2}^{(-\alpha)}(n) - r_{3,3}^{(-\alpha)}(n)\right)} \\
C_{4,i}^{(\alpha)}(n) &:= 24\alpha^3 - 18\alpha^2\left(n - 2 \cdot r_{4,i}^{(-\alpha)}(n)\right) + \alpha\left(7n - 12 \cdot r_{4,i}^{(-\alpha)}(n)\right)\left(n - r_{4,i}^{(-\alpha)}(n)\right) \\
&\qquad - \left(n - r_{4,i}^{(-\alpha)}(n)\right)^3, \quad \text{for } 1 \leq i \leq 4 \\
C_{5,k}^{(\alpha)}(n) &:= 120\alpha^4 + 2\alpha^2\left(23n^2 - 79n \cdot r_{5,k}^{(-\alpha)}(n) + 60 \cdot r_{5,k}^{(-\alpha)}(n)^2\right) + 48\alpha^3(2n - 5 \cdot r_{5,k}^{(-\alpha)}(n)) \\
&\qquad + \alpha(11n - 20r_{5,k}^{(-\alpha)}(n))(n - r_{5,k}^{(-\alpha)}(n))^2 + (n - r_{5,k}^{(-\alpha)}(n))^4, \quad \text{for } 1 \leq k \leq 5.
\end{aligned}$$

For fixed $\alpha \in \mathbb{Z}^+$ and $n \geq 0$, we obtain the following analogs to the first congruence result modulo 2 expanded by (25) for the α-factorial functions, $n_{(\alpha)}$, when $n \geq 1$ (see Section 6.2.2):

$$\begin{aligned} n!_{(\alpha)} &\equiv \widetilde{R}_3^{(\alpha)}(n) && \pmod{3, 3\alpha} \quad (27) \\ n!_{(\alpha)} &\equiv \underbrace{\sum_{1 \leq i \leq 4} \frac{C_{4,i}^{(\alpha)}(n)}{\prod_{j \neq i} \left(r_{4,i}^{(-\alpha)}(n) - r_{4,j}^{(-\alpha)}(n)\right)} r_{4,i}^{(-\alpha)}(n)^{\left\lfloor \frac{n+\alpha-1}{\alpha} \right\rfloor}}_{:= R_4^{(\alpha)}(n)} && \pmod{4, 4\alpha} \\ n!_{(\alpha)} &\equiv \underbrace{\sum_{1 \leq k \leq 5} \frac{C_{5,k}^{(\alpha)}(n)}{\prod_{j \neq k} \left(r_{5,k}^{(-\alpha)}(n) - r_{5,j}^{(-\alpha)}(n)\right)} r_{5,k}^{(-\alpha)}(n)^{\left\lfloor \frac{n+\alpha-1}{\alpha} \right\rfloor}}_{:= R_5^{(\alpha)}(n)} && \pmod{5, 5\alpha}. \end{aligned}$$

Several particular concrete examples illustrating the results cited in (25) modulo 2 (and 2α), and in (27) modulo p (and $p\alpha$) for $p := 3, 4, 5$, corresponding to the first few cases of $\alpha \geq 1$ and $n \geq 1$ appear in Table 9.3 (page 71) Further computations of the congruences given in (27) modulo $p\alpha^i$ (for some $0 \leq i \leq p$) are contained in the *Mathematica* summary notebook included as a supplementary file with the submission of this article (see Section 1.2 and the reference document [21]). The results in Section 6.2 provide statements of these new integer congruences for fixed $\alpha \neq 0$ modulo any integers $p \geq 2$. The analogous formulations of the new relations for the factorial-related product sequences modulo any p and $p\alpha$ are easily established for the subsequent cases of integers $p \geq 6$ from the partial fraction expansions of the convergent functions, $\mathrm{Conv}_h(\alpha, R; z)$, cited in the particular listings in Table 9.1 (page 67) and in Table 9.2 (page 68), and through the generalized rational convergent function properties proved in Section 5.

3.3.3 New congruence properties for the Stirling numbers of the first kind

The results given in Section 6.2 also provide new congruences for the generalized Stirling number triangles in (4), as well as several new forms of rational generating functions that enumerate the scaled factorial-power variants of the *r-order harmonic numbers*, $(n!)^r \times H_n^{(r)}$, modulo integers $p \geq 2$ [12, §6.3] (see Section 6.2.1). For example, the known congruences for the Stirling numbers of the first kind proved by the generating function techniques enumerated in the reference [25, §4.6] imply the next new congruence results satisfied by the binomial coefficients modulo 2 (A087755).

$$\binom{\lfloor \frac{n}{2} \rfloor}{m - \lceil \frac{n}{2} \rceil} [1 \leq m \leq 6]_\delta$$

$$\equiv [n > m]_\delta \times \begin{cases} \frac{2^n}{4}, & \text{if } m = 1; \\ \frac{3\cdot 2^n}{16}(n-1), & \text{if } m = 2; \\ \frac{2^n}{128}(9n-20)(n-1), & \text{if } m = 3; \\ \frac{2^n}{512}(3n-10)(3n-7)(n-1), & \text{if } m = 4; \\ \frac{2^n}{8192}(27n^3-279n^2+934n-1008)(n-1), & \text{if } m = 5; \\ \frac{2^n}{163840}(9n^2-71n+120)(3n-14)(3n-11)(n-1), & \text{if } m = 6; \\ 0, & \text{otherwise.} \end{cases}$$

$$+ [1 \leq m \leq 6]_\delta [n = m]_\delta \pmod{2}$$

3.3.4 New congruences and rational generating functions for the r-order harmonic numbers

The next results state several additional new congruence properties satisfied by the first-order, second-order, and third-order harmonic number sequences, each expanded by the rational generating functions enumerating these sequences modulo the first few small cases of integer-valued p constructed from the generalized convergent functions in Section 6.2.1 (A001008, A002805, A007406, A007407, A007408, A007409).

$$(n!)^3 \times H_n^{(3)} \equiv [z^n] \left(\frac{z\left(1-7z+49z^2-144z^3+192z^4\right)}{(1-8z)^2} \right) \pmod{2}$$

$$(n!)^2 \times H_n^{(2)} \equiv [z^n] \left(\frac{z\left(1-61z+1339z^2-13106z^3+62284z^4-144264z^5+151776z^6-124416z^7+41472z^8\right)}{(1-6z)^3(1-24z+36z^2)^2} \right) \pmod{3}$$

$$\begin{aligned} (n!) \times H_n^{(1)} &\equiv [z^{n+1}] \left(\frac{36z^2-48z+325}{576} + \frac{17040z^2+1782z+6467}{576(24z^3-36z^2+12z-1)} + \frac{78828z^2-33987z+3071}{288(24z^3-36z^2+12z-1)^2} \right) && \pmod{4} \\ &\equiv [z^n] \left(\frac{3z-4}{48} + \frac{1300z^2+890z+947}{96(24z^3-36z^2+12z-1)} + \frac{24568z^2-10576z+955}{96(24z^3-36z^2+12z-1)^2} \right) && \pmod{4} \\ &\equiv [z^{n-1}] \left(\frac{1}{16} + \frac{-96z^2+794z+397}{48(24z^3-36z^2+12z-1)} + \frac{5730z^2-2453z+221}{24(24z^3-36z^2+12z-1)^2} \right) && \pmod{4}. \end{aligned}$$

4 The Jacobi-type J-fractions for generalized factorial function sequences

4.1 Enumerative properties of Jacobi-type J-fractions

To simplify the exposition in this article, we adopt the notation for the Jacobi-type continued fractions, or J-fractions, in ([8, 9], [17, cf. §3.10], [16, cf. §5.5]). Given some application-specific choices of the prescribed sequences, $\{a_k, b_k, c_k\}$, we consider the formal power series whose coefficients are generated by the rational convergents, $J^{[h]}(z) := J^{[h]}(\{a_k, b_k, c_k\}; z)$, to the infinite continued fractions, $J(z) := J^{[\infty]}(\{a_k, b_k, c_k\}; z)$, defined as follows:

$$J(z) = \cfrac{1}{1 - c_0 z - \cfrac{a_0 b_1 z^2}{1 - c_1 z - \cfrac{a_1 b_2 z^2}{\cdots}}}. \tag{28}$$

We briefly summarize the other enumerative properties from the references that are relevant in constructing the new factorial-function-related results given in the following subsections ([9, 8], [16, §5.5], [12, cf. §6.7]).

- **Definitions of the h-order convergent series**:
  When $h \geq 1$, the h^{th} convergent functions, given by the equivalent notation of $J^{[h]}(z)$ and $J^{[h]}(\{a_k, b_k, c_k\}; z)$ within this section, of the infinite continued fraction expansions in (28) are defined as the ratios, $J^{[h]}(z) := P_h(z)/Q_h(z)$.

  The component functions corresponding to the convergent numerator and denominator sequences, $P_h(z)$ and $Q_h(z)$, each satisfy second-order finite difference equations (in h) of the respective forms defined by the next two equations.

$$P_h(z) = (1 - c_{h-1} z) P_{h-1}(z) - a_{h-2} b_{h-1} z^2 P_{h-2}(z) + [h = 1]_\delta$$
$$Q_h(z) = (1 - c_{h-1} z) Q_{h-1}(z) - a_{h-2} b_{h-1} z^2 Q_{h-2}(z) + (1 - c_0 z) [h = 1]_\delta + [h = 0]_\delta$$

- **Rationality of truncated convergent function approximations**:
  Let $p_n = p_n(\{a_k, b_k, c_k\}) := [z^n] J(z)$ denote the expected term corresponding to the coefficient of z^n in the formal power series expansion defined by the infinite J-fraction from (28). For all $n \geq 0$, we know that the h^{th} convergent functions have truncated power series expansions that satisfy

$$p_n(\{a_k, b_k, c_k\}) = [z^n] J^{[h]}(\{a_k, b_k, c_k\}; z), \ \forall n \leq h.$$

  In particular, the series coefficients of the h^{th} convergents are always at least h-order accurate as formal power series expansions in z that exactly enumerate the expected sequence terms, $(p_n)_{n \geq 0}$.

  The resulting "*eventually periodic*" nature suggested by the approximate sequences enumerated by the rational convergent functions in z is formalized in the congruence properties given below in (29) ([9], [16, See §2, §5.7]).

- **Congruence properties modulo integer bases**:
  Let $\lambda_k := a_{k-1} b_k$ and suppose that the corresponding bases, M_h, are formed by the products $M_h := \lambda_1 \lambda_2 \cdots \lambda_h$ for $h \geq 1$. Whenever $M_h \in \mathbb{Z}$, $N_h \mid M_h$, and $n \geq 0$, we have that

$$p_n(\{a_k, b_k, c_k\}) \equiv [z^n] J^{[h]}(\{a_k, b_k, c_k\}; z) \pmod{N_h}, \tag{29}$$

  which is also true of all partial sequence terms enumerated by the h^{th} convergent functions modulo any integer divisors of the M_h [16, cf. §5.7].

4.2 A short direct proof of the J-fraction representations for the generalized product sequence generating functions

We omit the details to a more combinatorially flavored proof that the J-fraction series defined by the convergent functions in (16) do, in fact, correctly enumerate the expected symbolic product sequences in (1). Instead, a short direct proof following from the J-fraction results given in Flajolet's first article is sketched below. Even further combinatorial interpretations of the sequences generated by these continued fraction series, their relations to the Stirling number triangles, and other properties tied to the coefficient triangles studied in depth by the article [20] based on the properties of these new J-fractions is suggested as a topic for later investigation.

Definition 2. The prescribed sequences in the J-fraction expansions defined by (28) in the previous section, corresponding to (i) the Pochhammer symbol, $(x)_n$, and (ii) the convergent functions enumerating the generalized products, $p_n(\alpha, R)$, or equivalently, the Pochhammer k-symbols, $(R)_{n,\alpha}$, over any fixed $\alpha \in \mathbb{Z}^+$ and indeterminate, R, are defined as follows:

$$\{\mathrm{as}_k(x), \mathrm{bs}_k(x), \mathrm{cs}_k(x)\} \stackrel{\text{(i)}}{:=} \{x+k, k, x+2k\} \qquad (\underline{\textit{Pochhammer Symbol}})$$

$$\{\mathrm{af}_k(\alpha,R), \mathrm{bf}_k(\alpha,R), \mathrm{cf}_k(\alpha,R)\} \stackrel{\text{(ii)}}{:=} \{R+k\alpha, k, \alpha\cdot(R+2k\alpha)\}\,. \qquad (\underline{\textit{Generalized Products}})$$

***Claim*:** We claim that the modified J-fractions, $R_0(R/\alpha, \alpha z)$, that generate the corresponding series expansions in (11) enumerate the analogous terms of the generalized symbolic product sequences, $p_n(\alpha, R)$, defined by (1).

Proof of the Claim. First, an appeal to the polynomial expansions of both the Pochhammer symbol, $(x)_n$, and then of the products, $p_n(\alpha, R)$, defined by (1) by the Stirling numbers of the first kind, yields the following sums:

$$\begin{aligned} p_n(\alpha,R)\times z^n &= \left(\prod_{j=0}^{n-1}(R+j\alpha)\,[n\geq 1]_\delta + [n=0]_\delta\right)\times z^n \\ &= \left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\alpha^{n-k}R^k\right)\times z^n \\ &= \underbrace{\alpha^n\cdot\left(\frac{R}{\alpha}\right)_n\times z^n}_{(R)_{n,\alpha}\times z^n}. \end{aligned} \tag{30}$$

Finally, after a parameter substitution of $x :\mapsto R/\alpha$ together with the change of variable $z :\mapsto \alpha z$ in the first results from Flajolet's article [8], we obtain identical forms of the convergent-based definitions for the generalized J-fraction definitions given in Definition 1. □

4.3 Alternate exact expansions of the generalized convergent functions

The notational inconvenience introduced in the inner sums of (18) and (19), implicitly determined by the shorthand for the coefficients, $c_{h,j}(\alpha, R)$, each of which depend on the more difficult terms of the convergent numerator function sequences, $\mathrm{FP}_h(\alpha, R; z)$, is avoided in place of alternate recurrence relations for each finite h^{th} convergent function, $\mathrm{Conv}_h(\alpha, R; z)$, involving paired products of the denominator polynomials, $\mathrm{FQ}_h(\alpha, R; z)$. These denominator function sequences are related to generalized forms of the Laguerre polynomial sequences as follows (see Section 5.1):

$$\mathrm{FQ}_h(\alpha, R; z) = (-\alpha z)^h \cdot h! \cdot L_h^{(R/\alpha-1)}\left((\alpha z)^{-1}\right). \tag{31}$$

The expansions provided by the formula in (31) suggest useful alternate formulations of the congruence results given below in Section 6.2 when the Laguerre polynomial, or corresponding confluent hypergeometric function, zeros are considered to be less complicated in form than the more involved sums expanded through the numerator functions, $\mathrm{FP}_h(\alpha, R; z)$.

Example 3 (Recurrence Relations for the Convergent Functions and Laguerre Polynomials)**.** The well-known cases of the enumerative properties satisfied by the expansions of the convergent function sequences given in the references immediately yield the following relations ([8, §3], [17, §1.12(ii)]):

$$\begin{aligned}
\mathrm{Conv}_h(\alpha, R; z) &= \mathrm{Conv}_{h-k}(\alpha, R; z) \\
&\quad + \sum_{i=0}^{k-1} \frac{\alpha^{h-i-1}(h-i-1)! \cdot p_{h-i-1}(\alpha, R) \cdot z^{2(h-i-1)}}{\mathrm{FQ}_{h-i}(\alpha, R; z)\, \mathrm{FQ}_{h-i-1}(\alpha, R; z)}, \ h > k \geq 1 \\
\mathrm{Conv}_h(\alpha, R; z) &= \sum_{i=0}^{h-1} \frac{\alpha^{h-i-1}(h-i-1)! \cdot p_{h-i-1}(\alpha, R) \cdot z^{2(h-i-1)}}{\mathrm{FQ}_{h-i}(\alpha, R; z)\, \mathrm{FQ}_{h-i-1}(\alpha, R; z)} \\
&= \sum_{i=0}^{h-1} \binom{\frac{R}{\alpha}+i-1}{i} \times \frac{(-\alpha z)^{-1}}{(i+1) \cdot L_i^{(R/\alpha-1)}\left((\alpha z)^{-1}\right) L_{i+1}^{(R/\alpha-1)}\left((\alpha z)^{-1}\right)}, \ h \geq 2.
\end{aligned} \tag{32}$$

The convergent function recurrences expanded by (32) provide identities for particular cases of the generalized Laguerre polynomial sequences, $L_n^{(\beta)}(x)$, expressed as finite sums over paired reciprocals of the sequence at the same choices of the x and β.

Topics aimed at finding new results obtained from other known, strictly continued-fraction-related properties of the convergent function sequences beyond the proofs given in Section 5 are suggested as a further avenue to approach the otherwise divergent ordinary generating functions for these more general cases of the integer-valued factorial-related product sequences [13, cf. §10].

Remark 4 (Related Convergent Function Expansions). For $h-i \geq 0$, we have a noteworthy Rodrigues-type formula satisfied by the Laguerre polynomial sequences in (31) stated by the reference as follows [17, §18.5(ii)]:

$$(-\alpha z)^{h-i} \cdot (h-i)! \times L_{h-i}^{(\beta)}\left(\frac{1}{\alpha z}\right) = \alpha^{h-i} \cdot z^{2h-2i+\beta+1} e^{1/\alpha z} \times \left\{\frac{e^{-1/\alpha z}}{z^{\beta+1}}\right\}^{(h-i)}.$$

The multiple derivatives implicit to the statement in the previous equation then have the additional expansions through the product rule analog provided by the formula of Halphen from the reference given in the form of the following equation for natural numbers $n \geq 0$, where the notation for the functions, $F(z)$ and $G(z)$, employed in the previous equation corresponds to any prescribed choice of these functions that are each n times continuously differentiable at the point $z \neq 0$ [5, §3 Exercises, p. 161]:

$$\left\{F\left(\frac{1}{z}\right) G(z)\right\}^{(n)} = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{z^k} \cdot F^{(k)}\left(\frac{1}{z}\right) \left\{\frac{G(z)}{z^k}\right\}^{(n-k)}. \qquad (\underline{\textit{Halphen's Product Rule}})$$

The next particular restatements of (32) then follow easily from the last two equations as

$$\begin{aligned}
\operatorname{Conv}_h(\alpha, R; z) &= \operatorname{Conv}_{h-k}(\alpha, R; z) \\
&\quad + \sum_{i=0}^{k-1} \frac{(h-i-1)!}{(\alpha z) \cdot (R/\alpha)_{h-i}} \times \frac{1}{{}_1F_1\left(-(h-i); \frac{R}{\alpha}; \frac{1}{\alpha z}\right) {}_1F_1\left(-(h-i-1); \frac{R}{\alpha}; \frac{1}{\alpha z}\right)}, \quad 1 \leq k < h \\
\operatorname{Conv}_h(\alpha, R; z) &= \sum_{i=0}^{h-1} \frac{(h-i-1)!}{(\alpha z) \cdot (R/\alpha)_{h-i}} \times \frac{1}{{}_1F_1\left(-(h-i); \frac{R}{\alpha}; \frac{1}{\alpha z}\right) {}_1F_1\left(-(h-i-1); \frac{R}{\alpha}; \frac{1}{\alpha z}\right)} \\
&= \sum_{i=1}^{h} \binom{\frac{R}{\alpha}+i-1}{i-1}^{-1} \times \frac{(-Rz)^{-1}}{{}_1F_1\left(1-i; \frac{R}{\alpha}; \frac{1}{\alpha z}\right) {}_1F_1\left(-i; \frac{R}{\alpha}; \frac{1}{\alpha z}\right)},
\end{aligned} \qquad (33)$$

where the h^{th} convergents, $\operatorname{Conv}_h(\alpha, R; z)$, are rational in z for all $h \geq 1$. The functions ${}_1F_1(a; b; z)$, or $M(a, b, z) = \sum_{s \geq 0} \frac{(a)_s}{(b)_s s!} z^s$, in the previous equations denote *Kummer's confluent hypergeometric function* ([17, §13.2],[12, §5.5]).

5 Properties of the generalized convergent functions

We first focus on the comparatively simple factored expressions for the series coefficients of the denominator sequences in the results proved by Section 5.1. The identification of the convergent denominator functions as special cases of the confluent hypergeometric function yields additional identities providing analogous addition and multiplication theorems for these functions with respect to the parameter z, as well as a number of further, new recurrence relations derived from established relations stated in the references, such as those provided by

Kummer's transformations. These properties form a superset of extended results beyond the immediate, more combinatorial, known relations for the J-fractions summarized in Section 4.1 and in Section 4.3.

The numerator convergent sequences considered in Section 5.2 have less obvious expansions through special functions, or otherwise more well-known polynomial sequences. A point concerning the relative simplicity of the expressions of the denominator convergent polynomials compared to the numerator convergent sequences is also mentioned in Flajolet's article [8, §3.1]. The last characterization of the generalized convergent denominator functions by the Laguerre polynomials in Proposition 5 below provides the factorizations over the zeros of the classical orthogonal polynomial sequences studied in the references [10, 3] which are required to state the results provided by (18) and (19) from Section 2.3, and more generally by (6.2.2) in Section 6.2.

5.1 The convergent denominator function sequences

In contrast to the convergent numerator functions, $\mathrm{FP}_h(\alpha, R; z)$, discussed next in Section 5.2, the corresponding denominator functions, $\mathrm{FQ}_h(\alpha, R; z)$, are readily expressed through well-known special functions. The first several special cases given in Table 9.1 (page 67) suggest the next identity, which is proved following Proposition 5 below.

$$\mathrm{FQ}_h(\alpha, R; z) = \sum_{k=0}^{h} \binom{h}{k} (-1)^k \left(\prod_{j=0}^{k-1} (R + (h-1-j)\alpha) \right) z^k \tag{34}$$

The convergent denominator functions are expanded by the *confluent hypergeometric functions*, $U(-h, b, w)$ and $M(-h, b, w)$, and equivalently by the *associated Laguerre polynomials*, $L_h^{(b-1)}(w)$, when $b :\mapsto R/\alpha$ and $w :\mapsto (\alpha z)^{-1}$ through the relations proved in the next proposition [17, cf. §18.6(iv); §13.9(ii)].

Proposition 5 (Exact Representations by Special Functions)**.** *The convergent denominator functions,* $\mathrm{FQ}_h(\alpha, R; z)$*, are expanded in terms of the confluent hypergeometric function and the associated Laguerre polynomials through the following results:*

$$\begin{aligned}
\mathrm{FQ}_h(\alpha, R; z) &= (\alpha z)^h \times U\left(-h, R/\alpha, (\alpha z)^{-1}\right) && (35)\\
&= (-\alpha z)^h \left(\frac{R}{\alpha}\right)_h \times M\left(-h, R/\alpha, (\alpha z)^{-1}\right) && (36)\\
&= (-\alpha z)^h \cdot h! \times L_h^{(R/\alpha-1)}\left((\alpha z)^{-1}\right). && (37)
\end{aligned}$$

Proof. We proceed to prove the first identity in (35) by induction. It is easy to verify by computation (see Table 9.1) that the left-hand-side and right-hand-sides of (35) coincide when $h = 0$ and $h = 1$. For $h \geq 2$, we apply the recurrence relation from (15) to write the right-hand-side of (35) as

$$\mathrm{FQ}_h(\alpha, R; z) = (1 - (R + 2\alpha(h-1))z) U\left(-h+1, R/\alpha, (\alpha z)^{-1}\right) (\alpha z)^{h-1} \tag{38}$$

$$- \alpha(R+\alpha(h-2))(h-1)z^2 U\left(-h+2, R/\alpha, (\alpha z)^{-1}\right)(\alpha z)^{h-2}.$$

The proof is completed using the known recurrence relation for the confluent hypergeometric function stated in reference as [17, §13.3(i)]

$$U(-h,b,u) = (u-b-2(h-1))U(-h+1,b,u) - (h-1)(b+h-2)U(-h+2,b,u)\,. \quad (39)$$

In particular, we can rewrite (38) as

$$\begin{aligned}\mathrm{FQ}_h(\alpha,R;z) = (\alpha z)^h \Bigg[& \left((\alpha z)^{-1} - \left(\frac{R}{\alpha} + 2(h-1)\right)\right) U\left(-h+1, R/\alpha, (\alpha z)^{-1}\right) \\ & - \left(\frac{R}{\alpha} + h - 2\right)(h-1) U\left(-h+2, R/\alpha, (\alpha z)^{-1}\right)\Bigg], \end{aligned} \quad (40)$$

which implies (35) in the special case of (39) where $(b,u) := (R/\alpha, (\alpha z)^{-1})$. The second characterizations of $\mathrm{FQ}_h(\alpha,R;z)$ by Kummer's confluent hypergeometric function, $M(a,b,z)$, in (36), and by the Laguerre polynomials stated in (37), follow from the first result whenever $h \geq 0$ [17, §13.6(v), §18.11(i)]. □

Proof of Equation (34). The first identity for the denominator functions, $\mathrm{FQ}_h(\alpha,R;z)$, conjectured from the special case table listings by (34), follows from the first statement of the previous proposition. We cite the particular expansions of $U(-n,b,z)$ when $n \geq 0$ is integer-valued involving the Pochhammer symbol, $(x)_n$, stated as follows [17, §13.2(i)]:

$$\begin{aligned} U(-n,b,z) &= \sum_{k=0}^{n} \binom{n}{k} (b+k)_{n-k} (-1)^n (-z)^k \\ &= \sum_{k=0}^{n} \binom{n}{k} (b+n-k)_k (-1)^k z^{n-k}. \end{aligned} \quad (41)$$

The second sum for the confluent hypergeometric function given in (41) then implies that the right-hand-side of (35) can be expanded as follows:

$$\begin{aligned} \mathrm{FQ}_h(\alpha,R;z) &= (\alpha z)^h \, U\left(-h, R/\alpha, (\alpha z)^{-1}\right) \\ &= (\alpha z)^h \sum_{k=0}^{h} \binom{h}{k} (-1)^k \left(\frac{R}{\alpha} + h - k\right)_k (\alpha z)^{k-h} \\ &= \sum_{k=0}^{h} \binom{h}{k} \left(\frac{R}{\alpha} + h - k\right)_k (-\alpha z)^k \\ &= \sum_{k=0}^{h} \binom{h}{k} \underbrace{\left((-1)^k \times \prod_{j=0}^{k-1} (R + (h-1-j)\alpha)\right)}_{(\pm 1)^k p_k(\mp\alpha, \pm R \pm (h-1)\alpha)} z^k. \end{aligned}$$

The last line of previous equations provides the required expansion to complete a proof of the first identity cited in (34). We obtain the alternate restatements of these coefficients of z^k provided by the following equations for $p \geq 0$:

$$\begin{aligned}
[z^k]\,\mathrm{FQ}_p(\alpha, R; z) &= \binom{p}{k}(-1)^k p_k(-\alpha, R + (p-1)\alpha) \cdot [0 \leq k \leq p]_\delta \qquad (42)\\
&= \binom{p}{k}\alpha^k \, (1 - p - R/\alpha)_k \cdot [0 \leq k \leq p]_\delta \\
[z^k]\,\mathrm{FQ}_p(\alpha, R; z) &= \binom{p}{k} p_k(\alpha, -R - (p-1)\alpha) \cdot [0 \leq k \leq p]_\delta \\
&= \binom{p}{k}(-\alpha)^k \, (R/\alpha + p - 1)^{\underline{k}} \cdot [0 \leq k \leq p]_\delta \,.
\end{aligned}$$

□

Corollary 6 (Recurrence Relations)**.** *For $h \geq 0$ and any integers $s > -h$, the convergent denominator functions, $\mathrm{FQ}_h(\alpha, R; z)$, satisfy the reflection identity, or analog to Kummer's transformation for the confluent hypergeometric function, given by*

$$\mathrm{FQ}_h(\alpha, \alpha s; z) = \mathrm{FQ}_{h+s-1}(\alpha; \alpha(2-s); z). \qquad (43)$$

Additionally, for $h \geq 0$ these functions satisfy recurrence relations of the following forms:

$$(R + (h-1)\alpha)z\,\mathrm{FP}_h(\alpha, R-\alpha; z) + ((\alpha - R)z - 1)\,\mathrm{FQ}_h(\alpha, R; z) + \mathrm{FQ}_h(\alpha, R+\alpha; z) = 0 \qquad (44)$$

$$\begin{aligned}
\mathrm{FQ}_h(\alpha, R; z) + \alpha h z\,\mathrm{FQ}_{h-1}(\alpha, R; z) - \mathrm{FQ}_h(\alpha, R-\alpha; z) &= 0 \\
(R + \alpha h)z\,\mathrm{FQ}_h(\alpha, R; z) + \mathrm{FQ}_{h+1}(\alpha, R; z) - \mathrm{FQ}_h(\alpha, R+\alpha; z) &= 0 \\
(1 - \alpha h z)\,\mathrm{FQ}_h(\alpha, R; z) - \mathrm{FQ}_h(\alpha, R+\alpha; z) - \alpha h(R + (h-1)\alpha)z^2\,\mathrm{FQ}_{h-1}(\alpha, R; z) &= 0 \\
(1 - (h+1)\alpha z)\,\mathrm{FQ}_h(\alpha, R; z) - \mathrm{FQ}_{h+1}(\alpha, R; z) - (R + (h-1)\alpha)z\,\mathrm{FQ}_h(\alpha, R-\alpha; z) &= 0 \\
\alpha(h-1)z\,\mathrm{FQ}_{h-2}(\alpha, R+2\alpha; z) - (1 - Rz)\,\mathrm{FQ}_{h-1}(\alpha, R+\alpha; z) + \mathrm{FQ}_h(\alpha, R; z) &= \delta_{h,0}.
\end{aligned}$$

Proof. The first equation results from *Kummer's transformation* for the confluent hypergeometric function, $U(a, b, z)$, given by [17, §13.2(vii)]

$$U(a, b, z) = z^{1-b} U(a - b + 1, 2 - b, z)\,.$$

In particular, when $R := \alpha s$ and $h + s - 1 \geq 0$ Proposition 5 implies that

$$\begin{aligned}
\mathrm{FQ}_h(\alpha, R; z) &= (\alpha z)^{h+R/\alpha-1}\, U\left(-(h + \frac{R}{\alpha} - 1), \frac{2\alpha - R}{\alpha}, (\alpha z)^{-1}\right) \\
&= \mathrm{FQ}_{h+R/\alpha-1}(\alpha, 2\alpha - R; z).
\end{aligned}$$

The recurrence relations stated in (44) follow similarly as consequences of the first proposition by applying the known results for the confluent hypergeometric functions cited in the reference [17, §13.3(i)]. □

Proposition 7 (Addition and Multiplication Theorems)**.** *Let $z, w \in \mathbb{C}$ with $z \neq w$ and suppose that $z \neq 0$. For a fixed $\alpha \in \mathbb{Z}^+$ and $h \geq 0$, the following finite sums provide two addition theorem analogs satisfied by the sequences of convergent denominator functions:*

$$\mathrm{FQ}_h(\alpha, R; z-w) = \sum_{n=0}^{h} \frac{(-h)_n (-w)^n (z-w)^{h-n}}{z^h \cdot n!} \mathrm{FQ}_{h-n}(\alpha, R+\alpha n; z) \tag{45}$$

$$\mathrm{FQ}_h(\alpha, R; z-w) = \sum_{n=0}^{h} \frac{(-h)_n \left(1-h-\frac{R}{\alpha}\right)_n (\alpha w)^n}{n!} \mathrm{FQ}_{h-n}(\alpha, R; z)$$

$$= \sum_{n=0}^{h} \binom{h}{n} \binom{h+\frac{R}{\alpha}-1}{n} \times (\alpha w)^n n! \times \mathrm{FQ}_{h-n}(\alpha, R; z).$$

The corresponding multiplication theorems for the denominator functions are stated similarly for $h \geq 0$ in the forms of the following equations:

$$\mathrm{FQ}_h(\alpha, R; zw) = \sum_{n=0}^{h} \frac{(-h)_n (w-1)^n w^{h-n}}{n!} \mathrm{FQ}_{h-n}(\alpha, R+\alpha n; z) \tag{46}$$

$$\mathrm{FQ}_h(\alpha, R; zw) = \sum_{n=0}^{h} \frac{(-h)_n \left(1-h-\frac{R}{\alpha}\right)_n (1-w)^n (\alpha z)^n}{n!} \mathrm{FQ}_{h-n}(\alpha, R; z)$$

$$= \sum_{n=0}^{h} \binom{h}{n} \binom{n-h-\frac{R}{\alpha}}{n} \times (\alpha z(w-1))^n \, n! \times \mathrm{FQ}_{h-n}(\alpha, R; z)$$

Proof of the Addition Theorems. The sums stated in (45) follow from special cases of established addition theorems for the confluent hypergeometric function, $U(a, b, x+y)$, cited in the reference [17, §13.13(ii)]. The particular addition theorems required in the proof are provided as follows:

$$U(a, b, x+y) = \sum_{n=0}^{\infty} \frac{(a)_n (-y)^n}{n!} U(a+n, b+n, x), \ |y| < |x| \tag{47}$$

$$U(a, b, x+y) = \left(\frac{x}{x+y}\right)^a \sum_{n=0}^{\infty} \frac{(a)_n (1+a-b)_n y^n}{n! (x+y)^n} U(a+n, b, x), \ \Re[y/x] > -\frac{1}{2}.$$

First, observe that in the special case inputs to $U(a, b, z)$ resulting from the application of Proposition 5 involving the functions $\mathrm{FQ}_h(\alpha, R; z) = U(-h, R/\alpha, (\alpha z)^{-1})$ in the infinite sums of (47) lead to to finite sum identities corresponding to the inputs, h, to $\mathrm{FQ}_h(\alpha, R; z)$ where $h \geq 0$. More precisely, the definition of the convergent denominator sequences provided by (15) requires that $\mathrm{FQ}_h(\alpha, R; z) = 0$ whenever $h < 0$.

To apply the cited results for $U(a, b, x+y)$ in these cases, let $z \neq w$, assume that both $\alpha, z \neq 0$, and suppose the parameters corresponding to x and y in (45) are defined so that

$$x := (\alpha z)^{-1}, \ y := \frac{1}{\alpha}\left((z-w)^{-1} - z^{-1}\right), \ x+y = (\alpha(z-w))^{-1}. \tag{48}$$

Since each of the sums in (45) involve only finitely-many terms, we ignore treatment of the convergence conditions given on the right-hand-sides of the equations in (47) to justify these two restatements of the addition theorem analogs provided above. □

Proof of the Multiplication Theorems. The second pair of identities stated in (46) are formed by the multiplication theorems for $U(a, b, z)$ noted as in the reference [17, §13.13(iii)]. The proof is derived similarly from the first parameter definitions of x and y given in the addition theorem proof, with an additional adjustment employed in these cases corresponding to the change of variable $\widehat{y} \mapsto (y-1)x$, which is selected so that $x + \widehat{y} :\mapsto xy$ in the above proof. The analog to (48) that results in these two cases then yields the parameters, $x := (\alpha z)^{-1}$ and $y := (w^{-1} - 1) \cdot (\alpha z)^{-1}$, in the first identities for the confluent hypergeometric function, $U(a, b, x + y)$, given by (47). □

Remark 8. The expansions of the addition and multiplication theorem analogs to the established relations for the confluent hypergeometric function, $U(a, b, w)$, are also compared to the known expansions of the *duplication formula* for the associated Laguerre polynomial sequence stated in the following form [19, §5.1] [17, cf. §18.18(iii)]:

$$h! \times L_h^{(\beta)}(wx) = \sum_{k=0}^{h} \binom{h+\beta}{h-k} \left(\frac{h!}{k!}\right)^2 \times w^k (1-w)^{h-k} \times k! L_k^{(\beta)}(x).$$

The second expansion of the convergent denominator functions, $\mathrm{FQ}_h(\alpha, R; z)$, by the confluent hypergeometric function, $M(a, b, w)$, stated in (36) of the first proposition in this section also suggests additional identities for these sequences generated by the multiplication formula analogs in (46) when the parameter $z :\mapsto \pm 1/\alpha$, for example, as in the simplified cases of the identities expanded in the references ([12, §5.5 - §5.6; Ex. 5.29], [17, cf. §15]).

5.2 The convergent numerator function sequences

The most direct expansion of the convergent numerator functions, $\mathrm{FP}_h(\alpha, R; z)$, is obtained from the *erasing operator*, defined as in Flajolet's first article, which performs the formal power series truncation operation defined by the next equation [8, §3].

$$\mathrm{E}_m \left[\!\!\left[\sum_i g_i z^i \right]\!\!\right] := \sum_{i \leq m} g_i z^i \qquad (\underline{\textit{Erasing Operator}})$$

The numerator polynomials are then given through this notation by the expansions in the following equations:

$$\begin{aligned}\mathrm{FP}_h(\alpha, R; z) &= \mathrm{E}_{h-1}\left[\!\!\left[\mathrm{FQ}_h(\alpha, R; z) \cdot \mathrm{Conv}_h(\alpha, R; z)\right]\!\!\right] \\ &= \sum_{n=0}^{h-1} \underbrace{\left(\sum_{i=0}^{n} [z^i]\, \mathrm{FQ}_h(\alpha, R; z) \times p_{n-i}(\alpha, R)\right)}_{C_{h,n}(\alpha, R) := [z^n]\, \mathrm{FP}_h(\alpha, R; z)} \times z^n.\end{aligned}$$

The coefficients of z^n expanded in the last equation are rewritten slightly in terms of (42) and the Pochhammer symbol representations of the product sequences, $p_n(\alpha, R)$, to arrive at a pair of formulas expanded as follows:

$$C_{h,n}(\alpha, R) = \sum_{i=0}^{n} \binom{h}{i} (-1)^i p_i(-\alpha, R + (h-1)\alpha)\, p_{n-i}(\alpha, R)\,, \qquad h > n \geq 0 \tag{49a}$$

$$= \sum_{i=0}^{n} \binom{h}{i} (1 - h - R/\alpha)_i\, (R/\alpha)_{n-i} \times \alpha^n, \qquad h > n \geq 0. \tag{49b}$$

These sums are remarkably similar in form to the next binomial-type convolution formula, or *Vandermonde identity*, stated as follows [5, §1.13(I)] [26] [15, §1.2.6, Ex. 34]:

$$\begin{aligned}(x+y)_n &= \sum_{i=0}^{n} \binom{n}{i} (x)_i\, (y)_{n-i} && (\underline{\textit{Vandermonde Convolution}}) \\ &= \sum_{i=0}^{n} \binom{n}{i} x \cdot (x - iz + 1)_{i-1}\, (y + iz)_{n-i}\,,\ x \neq 0.\end{aligned}$$

A separate treatment of other properties implicit to the more complicated expansions of these convergent function subsequences is briefly explored through the definitions of the three additional forms of auxiliary coefficient sequences, denoted in respective order by $C_{h,n}(\alpha, R)$, $R_{h,k}(\alpha; z)$, and $T_h^{(\alpha)}(n, k)$, considered in the next subsection below.

Remark 9 (Reflected Convergent Numerator Function Sequences). The special cases of the reflected numerator polynomials given in Table 9.2 (page 68) also suggest a consideration of the numerator convergent functions factored with respect to powers of $\pm(z-R)$ by expanding these sequences with respect to another formal auxiliary variable, w, when $R :\mapsto z \mp w$. The tables contained in the attached summary notebook [21] provide working *Mathematica* code to expand and factor these modified forms of the reflected numerator polynomial sequences employed in stating the generalized congruence results for the α-factorial functions, $n!_{(\alpha)}$, from the examples cited in Section 3.3, and more generally by the results proved in Section 6.2, satisfied by the α-factorial functions and generalized product sequence expansions modulo integers $p \geq 2$.

5.2.1 Alternate forms of the convergent numerator function subsequences

The next results summarize three semi-triangular recurrence relations satisfied by the particular variations of the numerator function subsequences considered, respectively, as polynomials with respect to z and R. For $h \geq 2$, fixed $\alpha \in \mathbb{Z}^+$, and $n, k \geq 0$, we consider the following forms of these auxiliary numerator coefficient subsequences:

$$\begin{aligned}C_{h,n}(\alpha, R) &:= [z^n]\, \mathrm{FP}_h(\alpha, R; z)\,, \quad \text{for } 0 \leq n \leq h-1 \\ &= C_{h-1,n}(\alpha, R) - (R + 2\alpha(h-1)) C_{h-1,n-1}(\alpha, R)\end{aligned} \tag{50}$$

$$
\begin{aligned}
&\quad - \alpha(R+\alpha(h-2))(h-1)C_{h-2,n-2}(\alpha,R) \\
R_{h,k}(\alpha;z) &:= [R^k]\,\mathrm{FP}_h\,(\alpha,R;z)\,, \quad \text{for} \quad 0 \le k \le h-1 \\
&= (1-2\alpha(h-1)z)R_{h-1,k}(\alpha;z) - \alpha^2(h-1)(h-2)z^2 R_{h-2,k}(\alpha;z) \\
&\quad - zR_{h-1,k-1}(\alpha;z) - \alpha(h-1)z^2 R_{h-2,k-1}(\alpha;z) \\
T_h^{(\alpha)}(n,k) &:= [z^n R^k]\,\mathrm{FP}_h\,(\alpha,R;z)\,, \quad \text{for} \quad 0 \le n,k \le h-1 \\
&= T_{h-1}^{(\alpha)}(n,k) - T_{h-1}^{(\alpha)}(n-1,k-1) - 2\alpha(h-1)T_{h-1}^{(\alpha)}(n-1,k) \\
&\quad - \alpha(h-1)T_{h-2}^{(\alpha)}(n-2,k-1) - \alpha^2(h-1)(h-2)T_{h-2}^{(\alpha)}(n-2,k) \\
&\quad + \left([z^n R^0]\,\mathrm{FP}_h(z)\right)[h\ge 1]_\delta\,[n\ge 0]_\delta\,[k=0]_\delta\,.
\end{aligned}
$$

Each of the recurrence relations for the triangles cited in the previous equations are derived from (14) by a straightforward application of the coefficient extraction method first motivated in [20]. Table 9.5 (page 74) and Table 9.6 (page 75) list the first few special cases of the first two auxiliary forms of these component polynomial subsequences.

We also state, without proof, a number of multiple, alternating sums involving the Stirling number triangles that generate these auxiliary subsequences for reference in the next several equations. In particular, for $h \ge 1$ and $0 \le n < h$, the sequences, $C_{h,n}(\alpha,R)$, are expanded by the following sums:

$$
C_{h,n}(\alpha,R) = \sum_{\substack{0\le m\le k\le n\\0\le s\le n}} \left(\binom{h}{k}\binom{m}{s}\begin{bmatrix}k\\m\end{bmatrix}(-1)^m\alpha^n\left(\frac{R}{\alpha}\right)_{n-k}\left(\frac{R}{\alpha}-1\right)^{m-s}\right)\times h^s \tag{51a}
$$

$$
= \sum_{\substack{0\le m\le k\le n\\0\le t\le s\le n}} \left(\binom{h}{k}\binom{m}{t}\begin{bmatrix}k\\m\end{bmatrix}\begin{bmatrix}n-k\\s-t\end{bmatrix}(-1)^m\alpha^{n-s}(h-1)^{m-t}\right)\times R^s \tag{51b}
$$

$$
= \sum_{\substack{0\le m\le k\le n\\0\le i\le s\le n}} \binom{h}{k}\binom{h}{i}\binom{m}{s}\begin{bmatrix}k\\m\end{bmatrix}\begin{Bmatrix}s\\i\end{Bmatrix}(-1)^m\alpha^n\left(\frac{R}{\alpha}\right)_{n-k}\left(\frac{R}{\alpha}-1\right)^{m-s}\times i! \tag{51c}
$$

$$
\begin{aligned}
&= \sum_{\substack{0\le m\le k\le n\\0\le v\le i\le s\le n}} \binom{h}{k}\binom{m}{s}\binom{i}{v}\binom{h+v}{v}\begin{bmatrix}k\\m\end{bmatrix}\begin{Bmatrix}s\\i\end{Bmatrix}(-1)^{m+i-v}\alpha^n\times \\
&\qquad \times\left(\frac{R}{\alpha}\right)_{n-k}\left(\frac{R}{\alpha}-1\right)^{m-s}\times i!.
\end{aligned} \tag{51d}
$$

Since the powers of R in the second identity are expanded by the Stirling numbers of the second kind as [12, §6.1]

$$
R^p = \alpha^p \times \sum_{i=0}^{p} \begin{Bmatrix}p\\i\end{Bmatrix}(-1)^{p-i}\left(\frac{R}{\alpha}\right)_i,
$$

for all natural numbers $p \geq 0$, the multiple sum identity in (51b) also implies the next finite multiple sum expansion for these auxiliary coefficient subsequences (see Table 9.5).

$$C_{h,n}(\alpha, R) = \sum_{i=0}^{n} \underbrace{\left(\sum_{\substack{0 \leq m \leq k \leq n \\ 0 \leq t \leq s \leq n}} \binom{h}{k} \binom{m}{t} \begin{bmatrix} k \\ m \end{bmatrix} \begin{bmatrix} n-k \\ s-t \end{bmatrix} \begin{Bmatrix} s \\ i \end{Bmatrix} (-1)^{m+s-i} (h-1)^{m-t} \right)}_{\text{polynomial function of } h \text{ only } := \frac{(-1)^n m_{n,h}}{n!} \times \binom{n}{i} p_{n,i}(h)} \times \alpha^n \left(\frac{R}{\alpha} \right)_i \tag{51e}$$

Similarly, for all $h \geq 1$ and $0 \leq k < h$, the sequences, $R_{h,k}(\alpha, R)$, are expanded as follows:

$$\begin{aligned} R_{h,k}(\alpha; z) &= \sum_{\substack{0 \leq m \leq i \leq n < h \\ 0 \leq t \leq k}} \left(\binom{h}{i} \binom{m}{t} \begin{bmatrix} i \\ m \end{bmatrix} \begin{bmatrix} n-i \\ k-t \end{bmatrix} (-1)^m \alpha^{n-k} (h-1)^{m-t} \right) \times z^n \qquad (52) \\ &= \sum_{\substack{0 \leq m \leq i \leq n < h \\ 0 \leq t \leq k \\ 0 \leq p \leq m-t}} \left(\binom{h}{i} \binom{m}{t} \binom{h-1}{p} \begin{bmatrix} i \\ m \end{bmatrix} \begin{bmatrix} n-i \\ k-t \end{bmatrix} \begin{Bmatrix} m-t \\ p \end{Bmatrix} (-1)^m \alpha^{n-k} \times p! \right) \times z^n. \end{aligned}$$

A more careful immediate treatment of the properties satisfied by these subsequences is omitted from this section for brevity. A number of the new congruence results cited in the next sections do, at any rate, have alternate expansions given by the more involved termwise structure implicit to these finite multiple sums modulo some application-specific prescribed functions of h.

6 Applications and motivating examples

6.1 Lemmas

We require the next lemma to formally enumerate the generalized products and factorial function sequences already stated without proof in the examples from Section 3.

Lemma 10 (Sequences Generated by the Generalized Convergent Functions)**.** *For fixed integers $\alpha \neq 0$, $0 \leq d < \alpha$, and each $n \geq 1$, the generalized α-factorial sequences defined in (2) satisfy the following expansions by the generalized products in (1):*

$$\begin{aligned} (\alpha n - d)!_{(\alpha)} &= p_n(-\alpha, \alpha n - d) && (53) \\ &= p_n(\alpha, \alpha - d) && (54) \\ n!_{(\alpha)} &= p_{\lfloor (n+\alpha-1)/\alpha \rfloor}(-\alpha, n). && (55) \end{aligned}$$

Proof. The related cases of each of these identities cited in the equations above correspond to proving to the equivalent expansions of the product-wise representations for the α-factorial functions given in each of the next equations:

$$(\alpha n-d)!_{(\alpha)} = \prod_{j=0}^{n-1} (\alpha n - d - j\alpha) \tag{a}$$

$$= \prod_{j=0}^{n-1} (\alpha - d + j\alpha) \tag{b}$$

$$n!_{(\alpha)} = \prod_{j=0}^{\lfloor (n+\alpha-1)/\alpha \rfloor - 1} (n - i\alpha). \tag{c}$$

The first product in (a) is easily obtained from (2) by induction on n, which then implies the second result in (b). Similarly, an inductive argument applied to the definition provided by (2) proves the last product representation given in (c). □

Corollaries. The proof of the lemma provides immediate corollaries to the special cases of the α-factorial functions, $(\alpha n-d)!_{(\alpha)}$, expanded by the results from (20). We explicitly state the following particular special cases of the lemma corresponding to $d := 0$ in (56) below for later use in Section 6.4 of the article:

$$\begin{aligned} (\alpha n)!_{(\alpha)} &= \alpha^n \cdot (1)_n = [z^n] \operatorname{Conv}_{n+n_0}(-\alpha, \alpha n; z), \ \forall n_0 \geq 0 \\ &= \alpha^n \cdot n! \quad = [z^n] \operatorname{Conv}_{n+n_0}(-1, n; \alpha z), \ \forall n_0 \geq 0. \end{aligned} \tag{56}$$

□

Remark 11 (Generating Floored Arithmetic Progressions of α-Factorial Functions). Lemma 10 provides proofs of the convergent-function-based generating function identities enumerating the α-factorial sequences given in (20) and (22) of the introduction. The last convergent-based generating function identity that enumerates the α-factorial functions, $n!_{(\alpha)}$, when $n > \alpha$ expanded in the form of (23) from Section 3 follows from the product function expansions provided in (55) of the lemma by applying a result proved in the exercises section of the reference [12, §7, Ex. 7.36; p. 569]. In particular, for any fixed $m \geq 1$ and some sequence, $(a_n)_{n\geq 0}$, a generating function for the modified sequences, $\left(a_{\lfloor n/m \rfloor}\right)_{n\geq 0}$, is given by

$$\widehat{A}_m(z) := \sum_{n\geq 0} a_{\lfloor \frac{n}{m} \rfloor} z^n = \frac{1-z^m}{1-z} \times \widetilde{A}(z^m) = \left(1 + z + \cdots + z^{m-2} + z^{m-1}\right) \times \widetilde{A}(z^m),$$

where $\widetilde{A}(z) := \sum_n a_n z^n$ denotes the ordinary power series generating function formally enumerating the prescribed sequence over $n \geq 0$ (compare to Remark 21 in Section 6.4.3).

Lemma 12 (Identities and Other Formulas Connecting Pochhammer Symbols)**.** *The next equations provide statements of several known identities from the references involving the falling factorial function, the Pochhammer symbol, or rising factorial function, and the binomial coefficients required by the applications given in the next sections of the article.*

- ***Relations between rising and falling factorial functions:*** *The following identities provide known relations between the rising and falling factorial functions for fixed $x \neq \pm 1$ and integers $m, n \geq 0$ where the coefficients, $L(n, k) = \binom{n-1}{k-1}\frac{n!}{k!}$, in the first connection formula denote the* Lah numbers *(A105278):*

$$\begin{aligned}
x^{\underline{n}} &= \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{n!}{k!} \times (x)_k && \text{(Connection Formulas)} \\
&= (-1)^n (-x)_n = (x-n+1)_n = \frac{1}{(x+1)^{\overline{-n}}} \\
(x)_n &= \sum_{k=0}^{n} \binom{n}{k} (n-1)^{\underline{n-k}} \times x^{\underline{k}} \\
&= (-1)^n (-x)^{\underline{n}} \\
&= (x+n-1)^{\underline{n}} = \frac{1}{(x-1)^{\underline{-n}}} \\
x^{\underline{m+n}} &= x^{\underline{m}} (x-m)^{\underline{n}} && \text{(Generalized Exponent Laws)} \\
x^{\overline{m+n}} &= x^{\overline{m}} (x+m)^{\overline{n}} \\
x^{\overline{-n}} &= \frac{1}{(x-n)_n} = \frac{1}{(x-1)^{\underline{n}}} && \text{(Negative Rising and Falling Powers)} \\
x^{\underline{-n}} &= \frac{1}{(x+1)_n} \\
&= \frac{1}{n! \cdot \binom{x+n}{n}} = \frac{1}{(x+1)(x+2)\cdots(x+n)}.
\end{aligned}$$

- ***Expansions of polynomial powers by the Stirling numbers of the second kind:*** *For any fixed $x \neq 0$ and integers $n \geq 0$, the polynomial powers of x^n are expanded as follows:*

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop n-k} \right\} x^{\underline{n-k}} = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} (-1)^{n-k} x^{\overline{n}}. \qquad \text{(Expansions of Polynomial Powers)}$$

- ***Binomial coefficient identities expanded by the Pochhammer symbol:*** *For fixed $x \neq 0$ and integers $n \geq 1$, the binomial coefficients are expanded by*

$$\begin{aligned}
(x)_n &= \binom{-x}{n} \times (-1)^n n! && \text{(Binomial Coefficient Identities)} \\
&= \binom{x+n-1}{n} \times n!.
\end{aligned}$$

- ***Connection formulas for products and ratios of Pochhammer symbols**: The next identities connecting products and ratios of the Pochhammer symbols are stated for integers $m, n, i \geq 0$ and fixed $x \neq 0$.*

$$(x)_n (x)_m = \sum_{k=0}^{\min(m,n)} \binom{m}{k}\binom{n}{k} k! \cdot (x)_{m+n-k}\, , n \neq m \quad \text{(Connection Formulas for Products)}$$

$$\frac{(x)_n}{(x)_i} = (x+i)_{n-i}\, ,\ n \geq i \quad \text{(Fractions of Pochhammer Symbols)}$$

Proof. The statements of the formulas given in the lemma are contained in the references, which in many cases also provide explicit proofs of these results. The first relations between the rising and falling factorial functions are given in the references [19, §4.1.2, §5; cf. §4.3.1] [12, §2, Ex. 2.17, 2.9, 2.16; §5.3; §6, Ex. 6.31, p. 552], the expansions of polynomial powers are given in the reference [12, §6.1], the binomial coefficient identities for the Pochhammer symbol are found in the references [14, 26], and the connection formulas for products and ratios of Pochhammer symbols are given in the references [5, Ex. 1.23, p. 83] [26]. □

6.2 New congruences for the α-factorial functions, the generalized Stirling number triangles, and Pochhammer k-symbols

The new results stated in this subsection follow immediately from the congruences properties modulo integer divisors of the M_h summarized by Section 4.1 considered as in the references [8, 9] [16, cf. §5.7]. The particular cases of the J-fraction representations enumerating the product sequences defined by (1) always yield a factor of $h := N_h \mid M_h$ in the statement of (29) (see Remark 13 below). One consequence of this property implicit to each of the generalized factorial-like sequences observed so far, is that it is straightforward to formulate new congruence relations for these sequences modulo any fixed integers $p \geq 2$.

Remark 13 (Congruences for Rational-Valued Parameters). The J-fraction parameters, $\lambda_h = \lambda_h(\alpha, R)$ and $M_h = M_h(\alpha, R)$, defined as in the summary of the enumerative properties from Section 4.1, corresponding to the expansions of the generalized convergents defined by the proof in Section 4.2 satisfy

$$\begin{aligned}
\lambda_k(\alpha, R) &:= \mathrm{as}_{k-1}(\alpha, R) \cdot \mathrm{bs}_k(\alpha, R) \\
&= \alpha(R + (k-1)\alpha) \cdot k \\
M_h(\alpha, R) &:= \lambda_1(\alpha, R) \cdot \lambda_2(\alpha, R) \times \cdots \times \lambda_h(\alpha, R) \\
&= \alpha^h \cdot h! \times p_h(\alpha, R) \\
&= \alpha^h \cdot h! \times (R)_{h,\alpha}\, ,
\end{aligned}$$

so that for integer divisors, $N_h(\alpha, R) \mid M_h(\alpha, R)$, we have that

$$p_n(\alpha, R) \equiv [z^n] \operatorname{Conv}_h(\alpha, R; z) \pmod{N_h(\alpha, R)}.$$

So far we have restricted ourselves to examples of the particular product sequence cases, $p_n(\alpha, R)$, where $\alpha \neq 0$ is integer-valued, i.e., so that $p, p\alpha^i \mid M_p(\alpha, R)$ for $1 \leq i \leq p$ whenever $p \geq 2$ is a fixed natural number. Explicit congruence identities arising in some other related applications when the choice of $\alpha \neq 0$ is strictly rational-valued are intentionally not treated in the examples cited in this section.

6.2.1 Congruences for the Stirling numbers of the first kind and the r-order harmonic number sequences

Generating the Stirling numbers as series coefficients of the generalized convergent functions.

For integers $h \geq 3$ and $n \geq m \geq 1$, the (unsigned) Stirling numbers of the first kind, $\left[{n \atop m}\right]$, are generated by the polynomial expansions of the rising factorial function, or Pochhammer symbol, $x^{\overline{n}} = (x)_n$, as follows ([12, §7.4; §6],A130534, A008275):

$$\begin{aligned} \begin{bmatrix} n \\ m \end{bmatrix} &= [R^m]R(R+1)\cdots(R+n-1) \\ &= [z^n][R^m]\,\mathrm{Conv}_h\left(1, R; z\right), \quad \text{for } 1 \leq m \leq n \leq h. \end{aligned} \tag{57}$$

Analogous formulations of new congruence results for the α-factorial triangles defined by (4), and the corresponding forms of the generalized harmonic number sequences, are expanded by noting that for all $n, m \geq 1$, and integers $p \geq 2$, we have the following expansions [20]:

$$\begin{aligned} \begin{bmatrix} n \\ m \end{bmatrix}_\alpha &= [s^{m-1}](s+1)(s+1+\alpha)\cdots(s+1+(n-2)\alpha) \\ &= [s^{m-1}]p_{n-1}\left(\alpha, s+1\right) \\ \begin{bmatrix} n \\ m \end{bmatrix}_\alpha &\equiv [z^{n-1}][R^{m-1}]\,\mathrm{Conv}_p\left(\alpha, R+1; z\right) \pmod{p}. \end{aligned} \tag{58}$$

The coefficients of R^m in the series expansions of the convergent functions, $\mathrm{Conv}_h\left(1, R; z\right)$, in the formal variable R are rational functions of z with denominators given by m^{th} powers of the reflected polynomials defined in the next equation.

Definitions of the quasi-polynomial expansions for the Stirling numbers of the first kind.

For a fixed $h \geq 3$, let the roots, $\omega_{h,i}$, be defined as follows:

$$(\omega_{h,i})_{i=1}^{h-1} := \left\{ \omega_j : \sum_{i=0}^{h-1} \binom{h-1}{i} \frac{h!}{(i+1)!} (-\omega_j)^i = 0,\ 1 \leq j < h \right\}. \tag{59}$$

The forms of both exact formulas and congruences for the Stirling numbers of the first kind modulo any prescribed integers $h \geq 3$ are then expanded as

$$\begin{bmatrix} n \\ m \end{bmatrix} = \left(\sum_{i=0}^{h-1} p_{h,i}^{[m]}(n) \times \omega_{h,i}^n \right) [n > m]_\delta + [n = m]_\delta \qquad\qquad m \leq n \leq h$$

$$\begin{bmatrix} n \\ m \end{bmatrix} \equiv \left(\sum_{i=0}^{h-1} p_{h,i}^{[m]}(n) \times \omega_{h,i}^n\right) [n > m]_\delta + [n = m]_\delta \pmod{h},\ \forall n \geq m,$$

where the functions, $p_{h,i}^{[m]}(n)$, denote fixed polynomials of degree m in n for each h, m, and i [12, §7.2]. For example, when $h := 2, 3$, the respective reflected roots defined by the previous equations in (59) are given exactly by

$$\{\omega_{2,1}\} := \{2\} \qquad \text{and} \qquad (\omega_{3,i})_{i=1}^2 := \left\{3 - \sqrt{3}, 3 + \sqrt{3}\right\}.$$

Comparisons to known congruences for the Stirling numbers.
The special case of (25) from the examples given in the introduction when $\alpha := 1$ corresponding to the single factorial function, $n!$, agrees with the known congruence for the Stirling numbers of the first kind derived in the reference ([25, §4.6], [5, cf. §5.8]). In particular, for all $n \geq 1$ we can prove that

$$n! \equiv \sum_{m=1}^{n} \binom{\lfloor n/2 \rfloor}{m - \lceil n/2 \rceil} (-1)^{n-m} n^m + [n = 0]_\delta \pmod{2}.$$

For comparison with the known result for the Stirling numbers of the first kind modulo 2 expanded as in the result from the reference stated above, several particular cases of these congruences for the Stirling numbers, $\left[{n \atop m}\right]$, modulo 2 are given by

$$\begin{aligned}
\begin{bmatrix} n \\ 1 \end{bmatrix} &\equiv \frac{2^n}{4} [n \geq 2]_\delta + [n = 1]_\delta && \pmod{2} \\
\begin{bmatrix} n \\ 2 \end{bmatrix} &\equiv \frac{3 \cdot 2^n}{16} (n-1) [n \geq 3]_\delta + [n = 2]_\delta && \pmod{2} \\
\begin{bmatrix} n \\ 3 \end{bmatrix} &\equiv 2^{n-7} (9n - 20)(n-1) [n \geq 4]_\delta + [n = 3]_\delta && \pmod{2} \\
\begin{bmatrix} n \\ 4 \end{bmatrix} &\equiv 2^{n-9} (3n - 10)(3n - 7)(n-1) [n \geq 5]_\delta + [n = 4]_\delta && \pmod{2} \\
\begin{bmatrix} n \\ 5 \end{bmatrix} &\equiv 2^{n-13} (27n^3 - 279n^2 + 934n - 1008)(n-1) [n \geq 6]_\delta + [n = 5]_\delta && \pmod{2} \\
\begin{bmatrix} n \\ 6 \end{bmatrix} &\equiv \frac{2^{n-15}}{5} (9n^2 - 71n + 120)(3n - 14)(3n - 11)(n-1) [n \geq 7]_\delta + [n = 6]_\delta && \pmod{2},
\end{aligned}$$

where

$$\begin{bmatrix} n \\ m \end{bmatrix} \equiv \binom{\lfloor n/2 \rfloor}{m - \lceil n/2 \rceil} = [x^m] \left(x^{\lceil n/2 \rceil} (x+1)^{\lfloor n/2 \rfloor}\right) \pmod{2},$$

for all $n \geq m \geq 1$ ([25, §4.6], A087755). The termwise expansions of the row generating functions, $(x)_n$, for the Stirling number triangle considered modulo 3 with respect to the non-zero indeterminate x similarly imply the next properties of these coefficients for any $n \geq m > 0$.

$$\begin{bmatrix} n \\ m \end{bmatrix} \equiv [x^m] \left(x^{\lceil n/3 \rceil}(x+1)^{\lceil (n-1)/3 \rceil}(x+2)^{\lfloor n/3 \rfloor}\right) \pmod{3}$$

$$\equiv \sum_{k=0}^{m} \binom{\lceil (n-1)/3 \rceil}{k} \binom{\lfloor n/3 \rfloor}{m-k-\lceil n/3 \rceil} \times 2^{\lceil n/3 \rceil + \lfloor n/3 \rfloor - (m-k)} \pmod{3}$$

The next few particular examples of the special case congruences satisfied by the Stirling numbers of the first kind modulo 3 obtained from the results in (59) above are expanded in the following forms:

$$\begin{bmatrix} n \\ 1 \end{bmatrix} \equiv \sum_{j=\pm 1} \tfrac{1}{36}\left(9 - 5j\sqrt{3}\right) \times \left(3 + j\sqrt{3}\right)^n [n \geq 2]_\delta + [n = 1]_\delta \pmod{3}$$

$$\begin{bmatrix} n \\ 2 \end{bmatrix} \equiv \sum_{j=\pm 1} \tfrac{1}{216}\left((44n-41) - (25n-24) \cdot j\sqrt{3}\right) \times \left(3 + j\sqrt{3}\right)^n [n \geq 3]_\delta + [n = 2]_\delta \pmod{3}$$

$$\begin{bmatrix} n \\ 3 \end{bmatrix} \equiv \sum_{j=\pm 1} \tfrac{1}{15552}\left((1299n^2 - 3837n + 2412) - (745n^2 - 2217n + 1418) \cdot j\sqrt{3}\right) \times$$

$$\times \left(3 + j\sqrt{3}\right)^n [n \geq 4]_\delta + [n = 3]_\delta \pmod{3}$$

$$\begin{bmatrix} n \\ 4 \end{bmatrix} \equiv \sum_{j=\pm 1} \tfrac{1}{179936}\big((6409n^3 - 383778n^2 + 70901n - 37092)$$

$$- (3690n^3 - 22374n^2 + 41088n - 21708) \cdot j\sqrt{3}\big) \times$$

$$\times \left(3 + j\sqrt{3}\right)^n [n \geq 5]_\delta + [n = 4]_\delta \pmod{3}.$$

Additional congruences for the Stirling numbers of the first kind modulo 4 and modulo 5 are straightforward to expand by related formulas with exact algebraic expressions for the roots of the third-degree and fourth-degree equations defined as in (59).

Rational generating function expansions enumerating the first-order harmonic numbers.
The next several particular cases of the congruences for integers $p \geq 4$ satisfied by the first-order harmonic numbers, H_n or $H_n^{(1)}$, are stated exactly in terms the rational generating functions in z that lead to generalized forms of the congruences in the last equations modulo the integers $p := 2, 3$ ([12, §6.3], A001008, A002805).

$$n! \times H_n^{(1)} = \begin{bmatrix} n+1 \\ 2 \end{bmatrix}$$

$$n! \times H_n^{(1)} \equiv [z^{n+1}] \left(\tfrac{36z^2 - 48z + 325}{576} + \tfrac{17040z^2 + 1782z + 6467}{576(24z^3 - 36z^2 + 12z - 1)} + \tfrac{78828z^2 - 33987z + 3071}{288(24z^3 - 36z^2 + 12z - 1)^2}\right) \pmod{4}$$

$$\equiv [z^n] \left(\tfrac{3z-4}{48} + \tfrac{1300z^2 + 890z + 947}{96(24z^3 - 36z^2 + 12z - 1)} + \tfrac{24568z^2 - 10576z + 955}{96(24z^3 - 36z^2 + 12z - 1)^2}\right) \pmod{4}$$

$$\equiv [z^{n-1}]\left(\tfrac{1}{16}+\tfrac{-96z^2+794z+397}{48(24z^3-36z^2+12z-1)}+\tfrac{5730z^2-2453z+221}{24(24z^3-36z^2+12z-1)^2}\right) \pmod{4}$$

$$n!\times H_n^{(1)} \equiv [z^n]\Big(\tfrac{12z-29}{300}+\tfrac{80130z^3+54450z^2+79113z+108164}{900(120z^4-240z^3+120z^2-20z+1)} + \tfrac{17470170z^3-11428050z^2+2081551z-108077}{900(120z^4-240z^3+120z^2-20z+1)^2}\Big) \pmod{5}$$

$$n!\times H_n^{(1)} \equiv [z^n]\Big(\tfrac{10z-37}{360}+\tfrac{1419408z^4+903312z^3+1797924z^2+2950002z+4780681}{2160(720z^5-1800z^4+1200z^3-300z^2+30z-1)} + \tfrac{5581246248z^4-4906594848z^3+1347715644z^2-140481648z+4780903}{2160(720z^5-1800z^4+1200z^3-300z^2+30z-1)^2}\Big) \pmod{6}$$

Remark 14. For each $h \geq 1$, we have h-order finite difference equations for the Pochhammer symbol, $(R)_n = [z^n]\operatorname{Conv}_{h\geq n}(1,R;z)$, implied by the rationality of the convergent functions, $\operatorname{Conv}_h(1,R;z)$, in z for all h, and by the expansions of the coefficients of the component convergent function sequences in (42) and in (51). In particular, since $n!\cdot H_n$ is generated by the Pochhammer symbol as $\left[{n+1 \atop 2}\right] = [R^2](R)_{n+1}$, for all $n \geq 2$, we see that we also have the following identities following from the coefficients of the convergent-based generating functions, $\operatorname{Conv}_h(1,R;z)$:

$$(n-1)!\cdot H_{n-1} = [R^2]\left(-\sum_{k=0}^{\min(n,h)-1}\binom{h}{n-k}(1-h-R)_{n-k}(R)_k + C_{h,n}(1,R)\right),\ 0<n\leq h$$

$$(n-1)!\cdot H_{n-1} \equiv [R^2]\left(-\sum_{k=0}^{\min(n,h)-1}\binom{h}{n-k}(1-h-R)_{n-k}(R)_k\right) \pmod{h},\ n\geq h.$$

Rational generating functions enumerating congruences for the r-order harmonic numbers. The expansions of the integer-order harmonic number sequences cited in the reference [20, §4.3.3] also yield additional related expansions of congruences for the terms, $(n!)^r\times H_n^{(r)}$, provided by the noted identities for these functions involving the Stirling numbers of the first kind modulo any fixed integers $p\geq 2$. The *second-order* and *third-order harmonic numbers*, $H_n^{(2)}$ and $H_n^{(3)}$, respectively, are expanded exactly through the following formulas involving the Stirling numbers of the first kind modulo any fixed integers $p\geq 2$, and where the Stirling number sequences, $\left[{n \atop m}\right] \pmod{p}$ at each fixed $m := 1,2,3,4$, are generated by the predictably rational functions of z generated through the identities stated above ([20, §4.3.3], A007406, A007407, A007408, A007409):

$$\begin{aligned}(n!)^2\times H_n^{(2)} &= (n!)^2\times\sum_{k=1}^{n}\frac{1}{k^2} \\ &\equiv \left[{n+1 \atop 2}\right]^2 - 2\left[{n+1 \atop 1}\right]\left[{n+1 \atop 3}\right] \pmod{p}\end{aligned}$$

$$(n!)^3\times H_n^{(3)} = (n!)^3\times\sum_{k=1}^{n}\frac{1}{k^3}$$

$$\equiv \begin{bmatrix} n+1 \\ 2 \end{bmatrix}^3 - 3\begin{bmatrix} n+1 \\ 1 \end{bmatrix}\begin{bmatrix} n+1 \\ 2 \end{bmatrix}\begin{bmatrix} n+1 \\ 3 \end{bmatrix} + 3\begin{bmatrix} n+1 \\ 1 \end{bmatrix}^2\begin{bmatrix} n+1 \\ 4 \end{bmatrix}. \qquad \pmod{p}$$

The Hadamard product, or diagonal-coefficient, generating function constructions formulated in the examples introduced by Section 6.3 below give expansions of rational convergent-function-based generating functions in z that generate these corresponding r-order sequence cases modulo any fixed integers $p \geq 2$. The proof of Theorem 2.4 given in the reference [16, §2] suggests the direct method for obtaining the next rational generating functions for these sequences in the working source code documented in the reference [21], each of which generate series coefficients for these particular harmonic number sequence variants (modulo p) that always satisfy some finite-degree linear difference equation with constant coefficients over n when α and R are treated as constant parameters.

$$\begin{aligned}
(n!)^2 \times H_n^{(2)} &\equiv [z^n]\left(\frac{z\left(1-3z+9z^2-8z^3\right)}{(1-4z)^2}\right) && \pmod{2} \\
&= [z^n]\left(\frac{5z}{16} - \frac{z^2}{2} + \frac{11}{64(1-4z)^2} - \frac{11}{64(1-4z)}\right) \\
&= [z^n]\left(z + 5z^2 + 33z^3 + 176z^4 + 880z^5 + 4224z^6 + \cdots\right) \\
(n!)^2 \times H_n^{(2)} &\equiv [z^n]\left(\frac{z\left(1-61z+1339z^2-13106z^3+62284z^4-144264z^5+151776z^6-124416z^7+41472z^8\right)}{(1-6z)^3(1-24z+36z^2)^2}\right) && \pmod{3} \\
&= [z^n]\Bigg(-\frac{13}{324} + \frac{14z}{81} - \frac{4z^2}{27} + \frac{25}{1944(-1+6z)^3} + \frac{115}{1944(-1+6z)^2} + \frac{5}{162(-1+6z)} \\
&\qquad + \frac{-787+17624z}{216(1-24z+36z^2)^2} + \frac{2377+3754z}{648(1-24z+36z^2)}\Bigg) \\
&= [z^n]\left(z + 5z^2 + 49z^3 + 820z^4 + 18232z^5 + 437616z^6 + \cdots\right) \\
(n!)^3 \times H_n^{(3)} &\equiv [z^n]\left(\frac{z\left(1-7z+49z^2-144z^3+192z^4\right)}{(1-8z)^2}\right) && \pmod{2} \\
&= [z^n]\left(\frac{11z}{32} - \frac{3z^2}{2} + 3z^3 + \frac{21}{256(1-8z)^2} - \frac{21}{256(1-8z)}\right) \\
&= [z^n]\left(z + 9z^2 + 129z^3 + 1344z^4 + 13440z^5 + 129024z^6 + \cdots\right) \\
(n!)^3 \times H_n^{(3)} &\equiv [z^n]\Bigg(-\frac{143}{5832} + \frac{625z}{2916} - \frac{4z^2}{9} + \frac{4z^3}{3} + \frac{115(-6719+711956z)}{93312(1-108z+216z^2)^2} && \pmod{3} \\
&\qquad + \frac{774079+1459082z}{93312(1-108z+216z^2)} - \frac{125(-11+312z)}{11664(1-36z+216z^2)^4} \\
&\qquad - \frac{10(1+306z)}{729(1-36z+216z^2)^3} + \frac{-20677+269268z}{93312(1-36z+216z^2)^2} + \frac{11851+89478z}{93312(1-36z+216z^2)}\Bigg) \\
&= [z^n]\left(z + 9z^2 + 251z^3 + 16280z^4 + 1586800z^5 + 171547200z^6 + \cdots\right)
\end{aligned}$$

The next cases of the rational generating functions enumerating the terms of these two sequences modulo 4 and 5 lead to less compact formulas expanded in partial fractions over z, roughly approximated in form by the generating function expansions from the previous formulas. The factored denominators, denoted $\mathrm{Denom}_{r,\,\mathrm{mod}\,p}[\![z]\!]$ immediately below, of the rational generating functions over the respective second-order and third-order cases of the

r-order sequences modulo $p := 4, 5$ are provided for reference in the following equations:

$$\begin{aligned}
\operatorname{Denom}_{2,\text{ mod } 4}[\![z]\!] &= \left(-1+72z-720z^2+576z^3\right)^2\left(-1+36z-288z^2+576z^3\right)^3 \\
\operatorname{Denom}_{2,\text{ mod } 5}[\![z]\!] &= \left(1-160z+5040z^2-28800z^3+14400z^4\right)^2 \times \\
&\quad \times \left(1-120z+4680z^2-76800z^3+561600z^4-1728000z^5+1728000z^6\right)^3 \\
\operatorname{Denom}_{3,\text{ mod } 4}[\![z]\!] &= (1-24z)^4\left(-1+504z-17280z^2+13824z^3\right)^2 \times \\
&\quad \times \left(-1+144z-5184z^2+13824z^3\right)^4\left(-1+216z-3456z^2+13824z^3\right)^4 \\
\operatorname{Denom}_{3,\text{ mod } 5}[\![z]\!] &= \left(1-1520z+273600z^2-4320000z^3+1728000z^4\right)^2 \times \\
&\quad \times \left(1-240z+14400z^2-288000z^3+1728000z^4\right)^4 \times \\
&\quad \times \Big(1-1680z+1051200z^2-319776000z^3+51914304000z^4 \\
&\qquad -4764026880000z^5+251795865600000z^6-7537618944000000z^7 \\
&\qquad +121956544512000000z^8-998751928320000000z^9 \\
&\qquad +4084826112000000000z^{10}-7739670528000000000z^{11} \\
&\qquad +5159780352000000000z^{12}\Big)^4.
\end{aligned}$$

The summary notebook reference contains further complete expansions of the rational generating functions enumerating these r-order sequence cases for $r := 1, 2, 3, 4$ modulo the next few prescribed cases of the integers $p \geq 6$ ([20, cf. §4.3.3], [21]).

6.2.2 Generalized expansions of the new integer congruences for the α-factorial functions and the symbolic product sequences

Generalized forms of the special case results expanded in the introduction.

Example 15 (The Special Cases Modulo 2, 3, and 4)**.** The first congruences for the α-factorial functions, $n!_{(\alpha)}$, modulo the prescribed integer bases, 2 and 2α, cited in (25) from the introduction result by applying Lemma 10 to the series for the generalized convergent function, $\operatorname{Conv}_2(\alpha, R; z)$, expanded by following equations:

$$\begin{aligned}
p_n(\alpha, R) &\equiv [z^n]\left(\frac{1-z(2\alpha+R)}{R(\alpha+R)z^2-2(\alpha+R)z+1}\right) && \pmod{2, 2\alpha} \\
&= \sum_{b=\pm 1}\frac{\left(\sqrt{\alpha(\alpha+R)}-b\cdot\alpha\right)\left(\alpha+b\cdot\sqrt{\alpha(\alpha+R)}+R\right)^n}{2\sqrt{\alpha(\alpha+R)}} && \pmod{2, 2\alpha}.
\end{aligned}$$

The next congruences for the α-factorial function sequences modulo 3 (3α) and modulo 4 (4α) cited as particular examples in Section 3 are established similarly by applying the previous lemma to the series coefficients of the next cases of the convergent functions, $\operatorname{Conv}_p(\alpha, R; z)$, for $p := 3, 4$ and where $\alpha :\mapsto -\alpha$ and $R :\mapsto n$.

$$p_n(\alpha, R) \equiv [z^n]\left(\frac{1-2(3\alpha+R)z+z^2\left(R^2+4\alpha R+6\alpha^2\right)}{1-3(2\alpha+R)z+3(\alpha+R)(2\alpha+R)z^2-R(\alpha+R)(2\alpha+R)z^3}\right)$$

$$(\bmod\ 3, 3\alpha)$$

$$p_n(\alpha, R) \equiv [z^n] \left(\frac{1-3(R+4\alpha)z+z^2\left(3R^2+19R\alpha+36\alpha^2\right)-(R+4\alpha)\left(R^2+3R\alpha+6\alpha^2\right)z^3}{1-4(R+3\alpha)z+6(R+2\alpha)(R+3\alpha)z^2-4(R+\alpha)(R+2\alpha)(R+3\alpha)z^3+R(R+\alpha)(R+2\alpha)(R+3\alpha)z^4} \right)$$

$$(\bmod\ 4, 4\alpha)$$

The particular cases of the new congruence properties satisfied modulo 3 (3α) and 4 (4α) cited in (27) from Section 3 of the introduction also phrase results that are expanded through exact algebraic formulas involving the reciprocal zeros of the convergent denominator functions, $\mathrm{FQ}_3(-\alpha, n; z)$ and $\mathrm{FQ}_4(-\alpha, n; z)$, given in Table 9.1 (page 67) [17, cf. §1.11(iii), §4.43]. The congruences cited in the example cases from the first two introduction sections to the article then correspond to the respective special cases of the reflected numerator polynomial sequences provided in Table 9.2 (page 68).

Definitions related to the reflected convergent numerator and denominator function sequences.

Definition 16. For any $h \geq 1$, let the reflected convergent numerator and denominator function sequences be defined as follows:

$$\widetilde{\mathrm{FP}}_h(\alpha, R; z) := z^{h-1} \times \mathrm{FP}_h\left(\alpha, R; z^{-1}\right) \qquad (\textit{Reflected Numerator Polynomials})$$
$$\widetilde{\mathrm{FQ}}_h(\alpha, R; z) := z^{h} \times \mathrm{FQ}_h\left(\alpha, R; z^{-1}\right) \qquad (\textit{Reflected Denominator Polynomials})$$

The listings given in Table 9.2 (page 68) provide the first few simplified cases of the reflected numerator polynomial sequences which lead to the explicit formulations of the congruences modulo p (and $p\alpha$) at each of the particular cases of $p := 4, 5$ given in (27), and then for the next few small special cases for subsequent cases of the integers $p \geq 6$.

Definition 17. More generally, let the respective sequences of p-order roots, and then the corresponding sequences of reflected partial fraction coefficients, be defined as ordered sequences over any $p \geq 2$ and each $1 \leq i \leq p$ through some fixed ordering of the special function zeros defined by the next equations.

$$\left(\ell_{p,i}^{(\alpha)}(R)\right)_{i=1}^{p} := \left\{z_i : \widetilde{\mathrm{FQ}}_h(\alpha, R; z_i) = 0,\ 1 \leq i \leq p\right\} \qquad (\textit{Sequences of Reflected Roots})$$
$$\left(C_{p,i}^{(\alpha)}(R)\right)_{i=1}^{p} := \left(\widetilde{\mathrm{FP}}_h\left(\alpha, R; \ell_{p,i}^{(\alpha)}(R)\right)\right)_{i=1}^{p}. \qquad (\textit{Sequences of Reflected Coefficients})$$

The sequences of reflected roots defined by the previous equations above in terms of the reflected denominator polynomials correspond to special zeros of the confluent hypergeometric function, $U(-h, b, w)$, and the associated Laguerre polynomials, $L_p^{(\beta)}(w)$, defined as in the special zero sets from Section 2.3 of the introduction [17, §18.2(vi), §18.16] [10, 3].

Generalized statements of the exact formulas and new congruence expansions.
The notation in the previous definition is then employed by the next restatements of the exact

formula expansions and congruence properties cited in the initial forms of the expansions given in (18) and (19) of Section 2.3. In particular, for any $p \geq 2$, $h \geq n \geq 1$, and where $\alpha, R \neq 0$ correspond to fixed integer-valued (or symbolic indeterminate) parameters, the expansions of the generalized product sequence cases defined by (1) satisfy the following relations (see Remark 13 above):

$$p_n(\alpha, R) = \sum_{1 \leq i \leq h} \frac{C_{h,i}^{(\alpha)}(R)}{\prod_{j \neq i} \left(\ell_{h,i}^{(\alpha)}(R) - \ell_{h,j}^{(\alpha)}(R)\right)} \times \left(\ell_{h,i}^{(\alpha)}(R)\right)^{n+1}, \qquad \forall h \geq n \geq 1 \tag{60}$$

$$p_n(\alpha, R) \equiv \sum_{1 \leq i \leq p} \frac{C_{p,i}^{(\alpha)}(R)}{\prod_{j \neq i} \left(\ell_{p,i}^{(\alpha)}(R) - \ell_{p,j}^{(\alpha)}(R)\right)} \times \left(\ell_{p,i}^{(\alpha)}(R)\right)^{n+1} \qquad \pmod{p, p\alpha, p\alpha^2, \ldots, p\alpha^p}.$$

The expansions in the next equations similarly state the desired results stating the generalized forms of the congruence formulas for the α-factorial functions, $n!_{(\alpha)}$, given by the particular special case expansions in (27) from Section 3.3 of the introduction to the article.

$$n!_{(\alpha)} = \sum_{1 \leq i \leq h} \frac{C_{h,i}^{(-\alpha)}(n)}{\prod_{j \neq i} \left(\ell_{h,i}^{(-\alpha)}(n) - \ell_{h,j}^{(-\alpha)}(n)\right)} \times \left(\ell_{h,i}^{(-\alpha)}(n)\right)^{\left\lfloor \frac{n-1}{\alpha} \right\rfloor + 1}, \qquad \forall h \geq n \geq 1 \tag{61}$$

$$n!_{(\alpha)} \equiv \underbrace{\sum_{1 \leq i \leq p} \frac{C_{p,i}^{(-\alpha)}(n)}{\prod_{j \neq i} \left(\ell_{p,i}^{(-\alpha)}(n) - \ell_{p,j}^{(-\alpha)}(n)\right)} \times \left(\ell_{p,i}^{(-\alpha)}(n)\right)^{\left\lfloor \frac{n-1}{\alpha} \right\rfloor + 1}}_{:= R_p^{(\alpha)}(n) \text{ in Section 3.3 and in Table 9.3 (page 71)}} \qquad \pmod{p, p\alpha, p\alpha^2, \ldots, p\alpha^p}.$$

The first pair of expansions given in (60) for the generalized product sequences, $p_n(\alpha, R)$, provide exact formulas and the corresponding new congruence properties for the Pochhammer symbol and Pochhammer k-symbol, in the respective special cases where $(x)_n :\mapsto \alpha^{-n} p_n(\alpha, \alpha x)$ and $(x)_{n,\alpha} :\mapsto p_n(\alpha, x)$ in the equations above.

6.3 Applications of rational diagonal-coefficient generating functions and Hadamard product sequences involving the generalized convergent functions

6.3.1 Generalized definitions and coefficient extraction formulas for sequences involving products of rational generating functions

We define the next extended notation for the *Hadamard product* generating functions, $(F_1 \odot F_2)(z)$ and $(F_1 \odot \cdots \odot F_k)(z)$, at some fixed, formal $z \in \mathbb{C}$. Phrased in slightly different wording, we define (62) as an alternate notation for the *diagonal generating functions* that enumerate the corresponding product sequences generated by the diagonal coefficients of the multiple-variable product series in k formal variables treated as in the reference [24, §6.3].

$$F_1 \odot F_2 \odot \cdots \odot F_k := \sum_{n \geq 0} f_{1,n} f_{2,n} \cdots f_{k,n} \times z^n \qquad \text{where} \qquad F_i(z) := \sum_{n \geq 0} f_{i,n} z^n \text{ for } 1 \leq i \leq k \tag{62}$$

When $F_i(z)$ is a rational function of z for each $1 \leq i \leq k$, we have particularly nice expansions of the coefficient extraction formulas of the rational diagonal generating functions from the references ([24, §6.3], [16, §2.4]). In particular, when $F_i(z)$ is rational in z at each respective i, these rational generating functions are expanded through the next few useful formulas:

$$\begin{aligned} F_1 \odot F_2 &= [x_1^0] \left(F_2\left(\frac{z}{x_1}\right) \cdot F_1(x_1) \right) \qquad \left(\underline{\textit{Diagonal Coefficient Extraction Formulas}}\right) \\ F_1 \odot F_2 \odot F_3 &= [x_2^0 x_1^0] \left(F_3\left(\frac{z}{x_2}\right) \cdot F_2\left(\frac{x_2}{x_1}\right) \cdot F_1(x_1) \right) \\ F_1 \odot F_2 \odot \cdots \odot F_k &= [x_{k-1}^0 \cdots x_2^0 x_1^0] \left(F_k\left(\frac{z}{x_{k-1}}\right) \cdot F_{k-1}\left(\frac{x_{k-1}}{x_{k-2}}\right) \times \cdots \times F_2\left(\frac{x_2}{x_1}\right) \cdot F_1(x_1) \right). \end{aligned} \tag{63}$$

Remark 18 (Integral Representations). Analytic formulas for the Hadamard products, $F_1 \odot F_2 = F_1(z) \odot F_2(z)$, when the component sequence generating functions are well enough behaved in some neighborhood of $z_0 = 0$ are given in the references ([5, §1.12(V); Ex. 1.30, p. 85], [7, §6.10]). In particular, we compare the first formula in (63) when $k := 2$ to the next known integral formula when the power series for both sequence generating functions, $F_1(z)$ and $F_2(z)$, are absolutely convergent in some $|z| \leq r < 1$ given by

$$\left(F_1 \odot F_2\right)\left(z^2\right) = \frac{1}{2\pi} \int_0^{2\pi} F_1\left(z e^{\imath t}\right) F_2\left(z e^{-\imath t}\right) dt.$$

The integral representation in the last equation is easily proved directly by noticing that

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} [z^n] F_1\left(z e^{\imath t}\right) F_2\left(z e^{-\imath t}\right) dt &= \sum_{0 \leq k \leq n} \frac{1}{2\pi} \int_0^{2\pi} e^{-(n-2k)\imath t} f_1(k) f_2(n-k) dt \\ &= \sum_{0 \leq k \leq n} f_1(k) f_2(n-k) \cdot [n = 2k]_\delta \\ &= \begin{cases} f_1\left(\frac{n}{2}\right) f_2\left(\frac{n}{2}\right) & \text{if } n \text{ is even;} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

We regard the rational convergents approximating the otherwise divergent ordinary generating functions for the generalized factorial function sequences strictly as formal power series in z whenever possible in this article. The remaining examples in this section illustrate this more formal approach using the generating functions enumerating the factorial-related product sequences considered here. The next several subsections of the article aim to provide concrete applications and some notable special cases illustrating the utility of this approach to the more general formal sequence products enumerated through the rational convergent functions, especially when combined with other generating function techniques discussed elsewhere and in the references.

6.3.2 Examples: Constructing hybrid rational generating function approximations from the convergent functions enumerating the generalized factorial product sequences

When one of the generating functions of an individual sequence from the Hadamard product representations in (63) is not rational in z, we still proceed, however slightly more carefully, to formally enumerate the terms of these sequences that arise in applications. For example, the *central binomial coefficients* are enumerated by the next convergent-based generating functions whenever $n \geq 1$ ([12, cf. §5.3], A000984).

$$\begin{aligned}\binom{2n}{n} &= \frac{2^{2n}}{n!} \times (1/2)_n \quad = [z^n][x^0]\left(e^{2x}\operatorname{Conv}_n\left(2, 1; \frac{z}{x}\right)\right) \qquad \text{(Central Binomial Coefficients)} \\ &= \frac{2^n}{n!} \times (2n-1)!! = [z^n][x^1]\left(e^{2x}\operatorname{Conv}_n\left(-2, 2n-1; \frac{z}{x}\right)\right).\end{aligned}$$

Since the reciprocal factorial squared terms, $(n!)^{-2}$, are generated by the power series for the *modified Bessel function of the first kind*, $I_0(2\sqrt{z}) = \sum_{n\geq 0} z^n/(n!)^2$, these central binomial coefficients are also enumerated as the diagonal coefficients of the following convergent function products ([17, §10.25(ii)], [12, §5.5], [5, cf. §1.13(II)]):

$$\begin{aligned}\binom{2n}{n} &= \frac{2^{2n}\,(1)_n\left(\frac{1}{2}\right)_n}{(n!)^2} \\ &= [x_1^0 x_2^0 z^n]\left(\operatorname{Conv}_n\left(2, 2; \frac{z}{x_2}\right)\operatorname{Conv}_n\left(2, 1; \frac{x_2}{x_1}\right) I_0\left(2\sqrt{x_1}\right)\right) \\ &= \frac{(2n)!!(2n-1)!!}{(n!)^2} \\ &= [x_1^0 x_2^0 z^n]\left(\operatorname{Conv}_n\left(-2, 2n; \frac{z}{x_2}\right)\operatorname{Conv}_n\left(-2, 2n-1; \frac{x_2}{x_1}\right) I_0\left(2\sqrt{x_1}\right)\right).\end{aligned}$$

The next binomial coefficient product sequence is enumerated through a similar construction of the convergent-based generating function identities expanded in the previous equations.

$$\binom{3n}{n}\binom{2n}{n} = \frac{3^{3n}\left(\frac{1}{3}\right)_n\left(\frac{2}{3}\right)_n}{(n!)^2} \qquad \text{(Paired Binomial Coefficient Products)}$$

$$
\begin{aligned}
&= [x_1^0 x_2^0 z^n]\left(\operatorname{Conv}_n\left(3, 2; \frac{3z}{x_2}\right)\operatorname{Conv}_n\left(3, 1; \frac{x_2}{x_1}\right) I_0\left(2\sqrt{x_1}\right)\right) \\
&= \frac{3^n}{(n!)^2} \times (3n-1)!_{(3)}\,(3n-2)!_{(3)} \\
&= [x_1^0 x_2^0 z^n]\left(\operatorname{Conv}_n\left(-3, 3n-1; \frac{3z}{x_2}\right)\operatorname{Conv}_n\left(-3, 3n-2; \frac{x_2}{x_1}\right) I_0\left(2\sqrt{x_1}\right)\right)
\end{aligned}
$$

Generating ratios of factorial functions and binomial coefficients.
The next few identities for the convergent generating function products over the binomial coefficient variants cited in (24) from the introduction are generated as the diagonal coefficients of the corresponding products of the convergent functions convolved with arithmetic progressions extracted from the exponential series in the form of the following equation, where $\omega_a := \exp(2\pi\imath/a)$ denotes the primitive a^{th} root of unity for integers $a \geq 2$ ([15, §1.2.9], [5, Ex. 1.26, p. 84]):

$$\widehat{E}_a(z) := \sum_{n\geq 0} \frac{z^n}{(an)!} = \frac{1}{a}\left(e^{z^{1/a}} + e^{\omega_a \cdot z^{1/a}} + e^{\omega_a^2 \cdot z^{1/a}} + \cdots + e^{\omega_a^{a-1}\cdot z^{1/a}}\right),\ a > 1. \qquad (64)$$

The modified generating functions, $\widehat{E}_a(z) = E_{a,1}(z)$, correspond to special cases of the *Mittag-Leffler function* defined as in the reference by the following series [17, §10.46]:

$$E_{a,b}(z) := \sum_{n\geq 0} \frac{z^n}{\Gamma(an+b)},\ a > 0.$$

These modified exponential series generating functions then denote the power series expansions of arithmetic progressions over the coefficients of the ordinary generating function for the exponential series sequences, $f_n := 1/n!$ and $f_{an} = 1/(an)!$. For $a := 2, 3, 4$, the particular cases of these exponential series generating functions are given by

$$
\begin{aligned}
\widehat{E}_2(z) &= \cosh\left(\sqrt{z}\right) \\
\widehat{E}_3(z) &= \frac{1}{3}\left(e^{z^{1/3}} + 2e^{-\frac{z^{1/3}}{2}}\cos\left(\tfrac{\sqrt{3}\cdot z^{1/3}}{2}\right)\right) \\
\widehat{E}_4(z) &= \frac{1}{2}\left(\cos\left(z^{1/4}\right) + \cosh\left(z^{1/4}\right)\right),
\end{aligned}
$$

where the powers of the a^{th} roots of unity in these special cases satisfy $\omega_2 = -1$, $\omega_3 = \frac{\imath}{2}\left(\imath + \sqrt{3}\right)$, $\omega_3^2 = -\frac{\imath}{2}\left(-\imath + \sqrt{3}\right)$, and $(\omega_4^m)_{1\leq m\leq 4} = (\imath, -1, -\imath, 1)$ (see the computations given in the reference [21]).

The next particular special cases of these diagonal-coefficient generating functions corresponding to the binomial coefficient sequence variants from (24) of Section 3.2 are then given through the following coefficient extraction identities provided by (63) (A166351, A066802):

$$\frac{(6n)!}{(3n)!} = \frac{6^{6n}\cancel{(1)_n}\cancel{\left(\frac{2}{6}\right)_n}\cancel{\left(\frac{3}{6}\right)_n} \times \left(\frac{1}{6}\right)_n\left(\frac{3}{6}\right)_n\left(\frac{5}{6}\right)_n}{3^{3n}\cancel{(1)_n}\cancel{\left(\frac{1}{3}\right)_n}\cancel{\left(\frac{2}{3}\right)_n}}$$

$$= 24^n \times 6^n \left(1/6\right)_n \times 2^n \left(1/2\right)_n \times 6^n \left(5/6\right)_n$$

$$= [x_2^0 x_1^0 z^n] \left(\mathrm{Conv}_n\left(6, 5; \frac{24z}{x_2}\right) \mathrm{Conv}_n\left(2, 1; \frac{x_2}{x_1}\right) \mathrm{Conv}_n\left(6, 1; x_1\right)\right)$$

$$= 8^n \times (6n-5)!_{(6)} \, (6n-3)!_{(6)} \, (6n-1)!_{(6)}$$

$$= [x_2^0 x_1^0 z^n] \Bigg(\mathrm{Conv}_n\left(-6, 6n-5; \frac{8z}{x_2}\right) \mathrm{Conv}_n\left(-6, 6n-3; \frac{x_2}{x_1}\right) \times$$

$$\times \mathrm{Conv}_n\left(-6, 6n-1; x_1\right)\Bigg)$$

$$\binom{6n}{3n} = [x_3^0 x_2^0 x_1^0 z^n] \Bigg(\mathrm{Conv}_n\left(6, 5; \frac{8z}{x_3}\right) \mathrm{Conv}_n\left(6, 3; \frac{x_3}{x_2}\right) \mathrm{Conv}_n\left(6, 1; \frac{x_2}{x_1}\right) \times$$

$$\times \underbrace{\frac{1}{3}\left(e^{x_1^{1/3}} + 2e^{-\frac{x_1^{1/3}}{2}} \cos\left(\frac{\sqrt{3} \cdot x_1^{1/3}}{2}\right)\right)}_{\widehat{E}_3(x_1) = E_{3,1}(x_1)}\Bigg)$$

$$= [x_3^0 x_2^0 x_1^0 z^n] \Bigg(\mathrm{Conv}_n\left(-6, 6n-5; \frac{8z}{x_3}\right) \mathrm{Conv}_n\left(-6, 6n-3; \frac{x_3}{x_2}\right) \times$$

$$\times \mathrm{Conv}_n\left(-6, 6n-1; \frac{x_2}{x_1}\right) \times$$

$$\times \underbrace{\frac{1}{3}\left(e^{x_1^{1/3}} + 2e^{-\frac{x_1^{1/3}}{2}} \cos\left(\frac{\sqrt{3} \cdot x_1^{1/3}}{2}\right)\right)}_{\widehat{E}_3(x_1)}\Bigg).$$

Similarly, the following related sequence cases forming particular expansions of these binomial coefficient variants are generated by

$$\binom{8n}{4n} = \frac{2^{16n}}{(4n)!} \times \left(\tfrac{1}{8}\right)_n \left(\tfrac{3}{8}\right)_n \left(\tfrac{5}{8}\right)_n \left(\tfrac{7}{8}\right)_n$$

$$= [x_1^0 x_2^0 x_3^0 x_4^0 z^n] \Bigg(\mathrm{Conv}_n\left(8, 7; \frac{16z}{x_4}\right) \mathrm{Conv}_n\left(8, 5; \frac{x_4}{x_3}\right) \mathrm{Conv}_n\left(8, 3; \frac{x_3}{x_2}\right) \times$$

$$\times \mathrm{Conv}_n\left(8, 1; \frac{x_2}{x_1}\right) \times \underbrace{\frac{1}{2}\left(\cos\left(x_1^{1/4}\right) + \cosh\left(x_1^{1/4}\right)\right)}_{\widehat{E}_4(x_1) = E_{4,1}(x_1)}\Bigg)$$

$$= \frac{2^{4n}}{(4n)!} \times (8n-7)!_{(8)} \, (8n-5)!_{(8)} \, (8n-3)!_{(8)} \, (8n-3)!_{(8)}$$

$$= [x_1^0 x_2^0 x_3^0 x_4^0 z^n] \Bigg(\mathrm{Conv}_n\left(-8, 8n-7; \frac{16z}{x_4}\right) \mathrm{Conv}_n\left(-8, 8n-5; \frac{x_4}{x_3}\right) \times \times \mathrm{Conv}_n\left(-8, 8n-3; \frac{x_3}{x_2}\right) \mathrm{Conv}_n\left(-8, 8n-1; \frac{x_2}{x_1}\right) \times \widehat{E}_4(x_1) \Bigg).$$

***Generating the subfactorial function (sequence of derangements)*.**
Another pair of convergent-based generating function identities enumerating the sequence of *subfactorials*, $(!n)_{n\geq 1}$, or *derangements*, $(n¡)_{n\geq 1}$, are expanded for $n \geq 1$ as follows (see Example 26 and the related examples cited in Section 6.6) ([12, §5.3], [17, cf. §8.4], A000166):

$$!n := n! \times \sum_{i=0}^{n} \frac{(-1)^i}{i!} \xrightarrow{\texttt{A000166}} (0, 1, 2, 9, 44, 265, 1854, 14833, \ldots) \quad (\textit{Subfactorial Function})$$

$$= [z^n x^0] \left(\frac{e^{-x}}{(1-x)} \times \mathrm{Conv}_n\left(-1, n; \frac{z}{x}\right) \right)$$

$$= [x^0 z^n] \left(\frac{e^{-x}}{(1-x)} \times \mathrm{Conv}_n\left(1, 1; \frac{z}{x}\right) \right).$$

Remark 19 (Laplace-Borel Transformations of Formal Power Series). The sequence of subfactorials is enumerated through the previous equations as the diagonals of generating function products where the rational convergent functions, $\mathrm{Conv}_n(\alpha, R; z)$, generate the sequence multiplier of $n!$ corresponding to the (formal) *Laplace transform*, $\mathcal{L}(f(t); z)$, defined by the integral transformations in the next equations ([8, cf. §2.2], [7, §B.4], [12, p. 566]). In this case the integral formulas are applied termwise to the power series given by the exponential generating function, $f(x) := e^{-x}/(1-x)$, for this sequence.

$$\mathcal{L}(\widehat{F}; z) = \int_0^\infty e^{-t} \widehat{F}(tz) dt \quad (\textit{Laplace-Borel Transformation Integrals})$$

$$\mathcal{L}^{-1}(\widetilde{F}; z) = \frac{1}{2\pi} \int_0^{2\pi} \widetilde{F}\left(-ze^{-\imath s}\right) e^{-e^{\imath s}} ds.$$

The applications cited in Section 6.6 and Section 6.7 in this article below employ this particular generating function technique to enumerate the factorial function multipliers provided by these rational convergent functions in several particular cases of sequences involving finite sums over factorial functions, sums of powers sequences, and new forms of approximate generating functions for the binomial coefficients and sequences of binomials.

6.4 Examples: Expanding arithmetic progressions of the single factorial function

One application suggested by the results in the previous subsection provides a-fold reductions of the h-order series approximations otherwise required to exactly enumerate arithmetic progressions of the single factorial function according to the next result.

$$(an+r)! = [z^{an+r}]\,\mathrm{Conv}_h(-1, an+r; z), \ \forall n \geq 1, a \in \mathbb{Z}^+, 0 \leq r < a, \forall h \geq an+r \tag{65}$$

The arithmetic progression sequences of the single factorial function formed in the particular special cases when $a := 2, 3$ (and then for particular cases of $a := 4, 5$) expanded in the examples cited below illustrate the utility to these convergent-based formal generating function approximations.

Proposition 20 (Factorial Function Multiplication Formulas)**.** *The statement of* Gauss's multiplication formula *for the gamma function yields the following decompositions of the single factorial functions,* $(an+r)!$*, into a finite product over* a *of the integer-valued multiple factorial sequences defined by* (2) *for* $n \geq 1$ *and whenever* $a \geq 2$ *and* $0 \leq r < a$ *are fixed natural numbers ([17, §5.5(iii)], [23, §2], [26]):*

$$(an+r)! = \underbrace{(an+r)!_{(a)} \times (an+r-1)!_{(a)} \times \cdots \times (an+r-a+1)!_{(a)}}_{(an+r)! = \prod_{i=0}^{a-1} (an+r-i)!_{(a)} = \prod_{i=0}^{a-1} p_n(-a, an+r-i)} \tag{66}$$

$$(an+r)! = \underbrace{r! \cdot a^{an} \left(\frac{1+r}{a}\right)_n \left(\frac{2+r}{a}\right)_n \times \cdots \times \left(\frac{a-1+r}{a}\right)_n \left(\frac{a+r}{a}\right)_n}_{(an+r)! = \prod_{i=1}^{a} a^n \times \left(\frac{r+i}{a}\right)_n = \prod_{i=1}^{a} p_n(a, r+i)}$$

$$= r! \cdot a^{an} \left(1 + \frac{r}{a}\right)_n \left(1 + \frac{r-1}{a}\right)_n \times \cdots \times \left(1 + \frac{r-a+1}{a}\right)_n, \ \forall a, n \in \mathbb{Z}^+, r \geq 0.$$

Proof. The first identity corresponds to the expansions of the single factorial function by a product of α distinct α-factorial functions for any fixed integer $\alpha \geq 2$ in the following forms:

$$\begin{aligned} n! &= n!! \cdot (n-1)!! \\ &= n!!! \cdot (n-1)!!! \cdot (n-2)!!! \\ &= \prod_{i=0}^{\alpha-1} (n-i)!_{(\alpha)}, \ \alpha \in \mathbb{Z}^+. \end{aligned}$$

The expansions of the last two identities stated above also follow from the known *multiplication formula* for the Pochhammer symbol expanded by the next equation [26] for any fixed integers $a \geq 1$ and $r \geq 0$, and where $(an+r)! = (1)_{an+r}$ by Lemma 10.

$$(x)_{an+r} = (x)_r \times a^{an} \times \prod_{j=0}^{a-1} \left(\frac{x+j+r}{a}\right)_n \qquad (\underline{\textit{Pochhammer Symbol Multiplication Formula}})$$

The two identities involving the corresponding products of the sequences from (1) provided by the braced formulas in (66) follow similarly from the lemma, and moreover, lead to several direct expansions of the convergent-function-based product sequences identities expanded in the next examples. □

6.4.1 Expansions of arithmetic progression sequences involving the double factorial function ($a := 2$)

In the particular cases where $a := 2$ (with $r := 0, 1$), we obtain the following forms of the corresponding alternate expansions of (65) enumerated by the diagonal coefficients of the next convergent-based product generating functions for all $n \geq 1$ ([23, cf. §2], A010050, A009445):

$$\begin{aligned}
(2n)! &= 2^n n! \times (2n-1)!! \qquad \left(\underline{\textit{Double Factorial Function Expansions}}\right) \\
&= [z^n][x^0]\left(\mathrm{Conv}_n\left(-1, n; \frac{2z}{x}\right)\mathrm{Conv}_n\left(-2, 2n-1; x\right)\right) \\
&= 2^n n! \times 2^n \left(1/2\right)_n \\
&= [x^0 z^n]\left(\mathrm{Conv}_n\left(1, 1; \frac{2z}{x}\right)\mathrm{Conv}_n\left(2, 1; x\right)\right) \\
(2n+1)! &= 2^n n! \times (2n+1)!! \\
&= [z^n][x^0]\left(\mathrm{Conv}_n\left(-1, n; \frac{2z}{x}\right)\mathrm{Conv}_n\left(-2, 2n+1; x\right)\right) \\
&= 2^n n! \times 2^n \left(3/2\right)_n \\
&= [x^1 z^n]\left(\mathrm{Conv}_n\left(1, 1; \frac{2z}{x}\right)\mathrm{Conv}_n\left(2, 1; x\right)\right) \\
&= [x^0 z^n]\left(\mathrm{Conv}_n\left(1, 1; \frac{2z}{x}\right)\mathrm{Conv}_n\left(2, 3; x\right)\right).
\end{aligned}$$

6.4.2 Expansions of arithmetic progression sequences involving the triple factorial function ($a := 3$)

When $a := 3$ we similarly obtain the next few alternate expansions generating the triple factorial products for the arithmetic progression sequences in (65) stated in the following equations for any $n \geq 2$ by extending the constructions of the identities for the expansions of the double factorial products in the previous equations ([23, §2], A100732, A100089, A100043):

$$\begin{aligned}
(3n)! &= (3n)!!! \times (3n-1)!!! \times (3n-2)!!! \qquad \left(\underline{\textit{Triple Factorial Function Expansions}}\right) \\
&= [z^n][x_2^0 x_1^0]\left(\mathrm{Conv}_n\left(-1, n; \frac{3z}{x_2}\right)\mathrm{Conv}_n\left(-3, 3n-1; \frac{x_2}{x_1}\right)\mathrm{Conv}_n\left(-3, 3n-2; x_1\right)\right) \\
&= 3^n n! \times 3^n \left(2/3\right)_n \times 3^n \left(1/3\right)_n \\
&= [x_1^0 x_2^0 z^n]\left(\mathrm{Conv}_n\left(1, 1; \frac{3z}{x_2}\right)\mathrm{Conv}_n\left(3, 1; \frac{x_2}{x_1}\right)\mathrm{Conv}_n\left(3, 2; x_1\right)\right) \\
(3n+1)! &= (3n)!!! \times (3n-1)!!! \times (3(n+1)-2)!!! \\
&= [z^n][x_2^0 x_1^{-1}]\Bigg(\mathrm{Conv}_n\left(-1, n; \frac{3z}{x_2}\right)\mathrm{Conv}_n\left(-3, 3n-1; \frac{x_2}{x_1}\right) \times
\end{aligned}$$

$$\left. \times \operatorname{Conv}_n(-3, 3n+1; x_1) \right)$$

$$= 3^n n! \times 3^n (2/3)_n \times 3^n (4/3)_n$$

$$= [x_1^0 x_2^0 z^n] \left(\operatorname{Conv}_n\left(1, 1; \frac{3z}{x_2}\right) \operatorname{Conv}_n\left(3, 4; \frac{x_2}{x_1}\right) \operatorname{Conv}_n(3, 2; x_1) \right)$$

$$(3n+2)! = (3n)!!! \times (3(n+1)-1)!!! \times (3(n+1)-2)!!!$$

$$= [z^n][x_2^{-1} x_1^0] \left(\operatorname{Conv}_n\left(-1, n; \frac{3z}{x_2}\right) \operatorname{Conv}_n\left(-3, 3n+2; \frac{x_2}{x_1}\right) \times \right.$$

$$\left. \times \operatorname{Conv}_n(-3, 3n+1; x_1) \right)$$

$$= 2 \times 3^n n! \times 3^n (5/3)_n \times 3^n (4/3)_n$$

$$= [x_1^1 x_2^0 z^n] \left(\operatorname{Conv}_n\left(1, 1; \frac{3z}{x_2}\right) \operatorname{Conv}_n\left(3, 4; \frac{x_2}{x_1}\right) \operatorname{Conv}_n(3, 2; x_1) \right).$$

6.4.3 Other special cases involving the quadruple and quintuple factorial functions ($a := 4, 5$)

The additional forms of the diagonal-coefficient generating functions corresponding to the special cases of the sequences in (65) where $(a, r) := (4, 2)$ and $(a, r) := (5, 3)$, respectively involving the *quadruple* and *quintuple factorial* functions are also cited in the next equations to further illustrate the procedure outlined by the previous two example cases.

$$(4n+2)! = (4n)!!!! \times (4n-1)!!!! \times (4(n+1)-2)!!!! \times (4(n+1)-3)!!!!$$

$$= [z^n][x_3^0 x_2^{-1} x_1^0] \left(\operatorname{Conv}_n\left(-1, n; \frac{4z}{x_3}\right) \operatorname{Conv}_n\left(-4, 4n-1; \frac{x_3}{x_2}\right) \times \right.$$

$$\left. \times \operatorname{Conv}_n\left(-4, 4n+2; \frac{x_2}{x_1}\right) \operatorname{Conv}_n(-4, 4n+1; x_1) \right)$$

$$= 2 \times 4^{4n} \times (1)_n (3/4)_n (3/2)_n (5/4)_n$$

$$= [x_1^0 x_2^1 x_3^0 z^n] \left(\operatorname{Conv}_n\left(1, 1; \frac{4z}{x_3}\right) \operatorname{Conv}_n\left(4, 3; \frac{x_3}{x_2}\right) \operatorname{Conv}_n\left(4, 2; \frac{x_2}{x_1}\right) \times \right.$$

$$\left. \times \operatorname{Conv}_n(4, 1; x_1) \right), \quad n \geq 2$$

$$(5n+3)! = (5n)!_{(5)} \times (5n-1)!_{(5)} \times (5(n+1)-2)!_{(5)} \times (5(n+1)-3)!_{(5)} \times (5n+1)!_{(5)}$$

$$= [z^n][x_4^0 x_3^{-1} x_2^0 x_1^0] \left(\operatorname{Conv}_n\left(-1, n; \frac{5z}{x_4}\right) \operatorname{Conv}_n\left(-5, 5n-1; \frac{x_4}{x_3}\right) \times \right.$$

$$\times \operatorname{Conv}_n\left(-5, 5n+3; \frac{x_3}{x_2}\right) \operatorname{Conv}_n\left(-5, 5n+2; \frac{x_2}{x_1}\right) \times$$
$$\left. \times \operatorname{Conv}_n\left(-5, 5n+1; x_1\right)\right)$$
$$= 6 \times 5^{5n} \times (1)_n\, (4/5)_n\, (8/5)_n\, (7/5)_n\, (6/5)_n$$
$$= [x_1^0 x_2^0 x_3^1 x_4^0 z^n]\left(\operatorname{Conv}_n\left(1, 1; \frac{5z}{x_4}\right) \operatorname{Conv}_n\left(5, 4; \frac{x_4}{x_3}\right) \operatorname{Conv}_n\left(5, 3; \frac{x_3}{x_2}\right) \times\right.$$
$$\left. \times \operatorname{Conv}_n\left(5, 2; \frac{x_2}{x_1}\right) \operatorname{Conv}_n\left(5, 1; x_1\right)\right), \ n \geq 2$$

Convergent-based generating function identities enumerating specific expansions corresponding to other cases of (65) when $a := 4, 5$ are given in [21].

Remark 21 (Generating Arithmetic Progressions of a Sequence)*.* The truncated power series approximations generating the single factorial functions formulated in the last few special case examples expanded in this section are also compared to the known results for extracting arithmetic progressions from any formal ordinary power series generating function, $F(z) = \sum_n f_n z^n$, of an arbitrary sequence, $\langle f_n \rangle_{n=0}^{\infty}$, through the primitive a^{th} roots of unity, $\omega_a := \exp(2\pi\imath/a)$, stated in the references as ([15, §1.2.9], [5, Ex. 1.26, p. 84])

$$F_{an+b}(z) := \sum_{n \geq 0} f_{an+b} z^{an+b} = \frac{1}{a} \sum_{0 \leq m < a} \omega_a^{-mb} F\left(\omega_a^m\right),$$

for integers $a \geq 2$ and $0 \leq b < a$ (compare to Remark 11 in Section 6.1). This formula is also employed in the special cases of the exponential series generating functions defined in (64) of the previous subsection.

6.5 Examples: Generalized superfactorial function products and relations to the Barnes G-function

The *superfactorial function*, $S_1(n)$, also denoted by $S_{1,0}(n)$ in the notation of (68) below, is defined for integers $n \geq 1$ by the factorial products (A000178):

$$S_1(n) := \prod_{k \leq n} k! \quad \xrightarrow{\texttt{A000178}} \quad (1, 2, 12, 288, 34560, 24883200, \ldots). \qquad (\underline{\textit{Superfactorial Function}})$$

These superfactorial functions are given in terms of the *Barnes G-function*, $G(z)$, for $z \in \mathbb{Z}^+$ through the relation $S_1(n) = G(n+2)$. The Barnes G-function, $G(z)$, corresponds to a so-termed "*double gamma function*" satisfying a functional equation of the following form for natural numbers $n \geq 1$ [17, §5.17] [1]:

$$G(n+2) = \Gamma(n+1) G(n+1) \, [n > 1]_\delta + [n = 1]_\delta \, . \qquad (\underline{\textit{Barnes G-Function}})$$

We can similarly expand the superfactorial function, $S_1(n)$, by unfolding the factorial products in the previous definition recursively according to the formulas given in the next equation.

$$S_1(n) = n! \cdot (n-1)! \times \cdots \times (n-k+1)! \cdot S_1(n-k),\ 0 \leq k < n$$

The product sequences over the single factorial functions formed by the last equations then lead to another application of the diagonal-coefficient product generating functions involving the rational convergent functions that enumerate the functions, $(n-k)!$, when $n-k \geq 1$. In particular, these particular cases of the diagonal coefficient, Hadamard-product-like sequences involving the single factorial function are generated as the coefficients

$$\begin{aligned}
S_1(n) = ([z^n]\operatorname{Conv}_n(-1, n; z)) &\times ([z^n]z \cdot \operatorname{Conv}_n(-1, n-1; z)) \times \\
&\times \left([z^n]z^2 \cdot \operatorname{Conv}_n(-1, n-2; z)\right) \times \cdots \times \\
&\times ([z^n]z^n \cdot \operatorname{Conv}_n(-1, 1; z)) \\
S_1(n) = ([z^n]\operatorname{Conv}_n(1, 1; z)) \qquad &\times ([z^n]z \cdot \operatorname{Conv}_n(1, 1; z)) \times \\
&\times \left([z^n]z^2 \cdot \operatorname{Conv}_n(1, 1; z)\right) \times \cdots \times ([z^n]z^n \cdot \operatorname{Conv}_n(1, 1; z)).
\end{aligned}$$

Stated more precisely, the superfactorial sequence is generated by the following finite, rational products of the generalized convergent functions for any $n \geq 2$:

$$\begin{aligned}
S_1(n) &= [x_1^{-1}x_2^{-1}\cdots x_{n-1}^{-1}x_n^n]\left(\prod_{i=0}^{n-2}\operatorname{Conv}_n\left(-1, n-i; \frac{x_{n-i}}{x_{n-i-1}}\right) \times \operatorname{Conv}_n(-1, 1; x_1)\right) \qquad (67) \\
&= [x_1^{-1}x_2^{-1}\cdots x_{n-1}^{-1}x_n^n]\left(\prod_{i=0}^{n-2}\operatorname{Conv}_n\left(1, 1; \frac{x_{n-i}}{x_{n-i-1}}\right) \times \operatorname{Conv}_n(1, 1; x_1)\right).
\end{aligned}$$

6.5.1 Generating generalized superfactorial product sequences

Let the more general superfactorial functions, $S_{\alpha,d}(n)$, forming the analogous products of the integer-valued multiple, α-factorial function cases from (2) correspond to the expansions defined by the next equation.

$$S_{\alpha,d}(n) := \prod_{j=1}^{n}(\alpha j - d)!_{(\alpha)},\ n \geq 1, \alpha \in \mathbb{Z}^+, 0 \leq d < \alpha \qquad (68)$$

Observe that the corollary of Lemma 10 cited in (56) implies that whenever $n \geq 1$, and for any fixed $\alpha \in \mathbb{Z}^+$, we immediately obtain the next identity corresponding to the so-termed "ordinary" case of these superfactorial functions, $S_1(n) = S_{1,0}(n)$, in the notation for these sequences defined above.

$$S_1(n) = \alpha^{-\binom{n+1}{2}}\prod_{j=1}^{n}(\alpha j)!_{(\alpha)} = \alpha^{-\binom{n+1}{2}}S_{\alpha,0}(n),\ \forall \alpha \in \mathbb{Z}^+,\ n \geq 1.$$

For other cases of the parameter $d > 0$, the generalized superfactorial function products defined by (68) are enumerated in a similar fashion to the previous constructions of the convergent-based generating function identities expanded by (67) in the following forms for $n \geq 1$, $\alpha \in \mathbb{Z}^+$, and any fixed $0 \leq d < \alpha$:

$$\begin{aligned} S_{\alpha,d}(n) &= [x_1^{-1}x_2^{-1}\cdots x_{n-1}^{-1}x_n^n]\left(\prod_{i=0}^{n-2}\operatorname{Conv}_n\left(-\alpha, \alpha(n-i)-d; \frac{x_{n-i}}{x_{n-i-1}}\right)\operatorname{Conv}_n\left(-\alpha,\alpha-d;x_1\right)\right) \\ &= [x_1^{-1}x_2^{-1}\cdots x_{n-1}^{-1}x_n^n]\left(\prod_{i=0}^{n-2}\operatorname{Conv}_n\left(\alpha, \alpha-d; \frac{x_{n-i}}{x_{n-i-1}}\right)\times\operatorname{Conv}_n\left(\alpha,\alpha-d;x_1\right)\right). \end{aligned} \tag{69}$$

6.5.2 Special cases of the generalized superfactorial products and their relations to the Barnes G-function at rational z

The special case sequences formed by the double factorial products, $S_{2,1}(n)$, and the quadruple factorial products, $S_{4,2}(n)$, are simplified by *Mathematica* to obtain the next closed-form expressions given by [21]

$$\begin{aligned} S_{2,1}(n) &:= \prod_{j=1}^{n}(2j-1)!! = \frac{A^{3/2}}{2^{1/24}e^{1/8}\pi^{1/4}}\cdot\frac{2^{n(n+1)/2}}{\pi^{n/2}}\times G\left(n+\frac{3}{2}\right) \\ S_{4,2}(n) &:= \prod_{j=1}^{n}(4j-2)!!!! = \frac{A^{3/2}}{2^{1/24}e^{1/8}\pi^{1/4}}\cdot\frac{4^{n(n+1)/2}}{\pi^{n/2}}\times G\left(n+\frac{3}{2}\right), \end{aligned}$$

where $A \approx 1.2824271$ denotes *Glaisher's constant* [17, §5.17], and where the particular constant multiples in the previous equation correspond to the special case values, $\Gamma(1/2) = \sqrt{\pi}$ and $G(3/2) = A^{-3/2}2^{1/24}e^{1/8}\pi^{1/4}$ [1]. In addition, since the sequences defined by (68) are also expanded as the products

$$S_{\alpha,d}(n) = \prod_{j=1}^{n}\underbrace{\left(\alpha^j\times\left(1-\frac{d}{\alpha}\right)_j\right)}_{=\alpha^j\times\frac{\Gamma\left(j+1-\frac{d}{\alpha}\right)}{\Gamma\left(1-\frac{d}{\alpha}\right)}} = \alpha^{\binom{n+1}{2}}G(n+2)\times\prod_{j=1}^{n}\binom{j-\frac{d}{\alpha}}{j},$$

further computations with *Mathematica* yield the next few representative special cases of these generalized superfactorial functions when $\alpha := 3, 4, 5$ [1, cf. §2] [21]:

$$\begin{aligned} S_{3,1}(n) &:= \prod_{j=1}^{n}(3j-1)!!! = 3^{n(n-1)/2}\left(\frac{2\cdot G\left(\frac{5}{3}\right)}{G\left(\frac{8}{3}\right)}\right)^n\times\frac{G\left(n+\frac{5}{3}\right)}{G\left(\frac{5}{3}\right)} \qquad \text{(\underline{Special Case Products})} \\ S_{4,1}(n) &:= \prod_{j=1}^{n}(4j-1)!!!! = 4^{n(n-1)/2}\left(\frac{3\cdot G\left(\frac{7}{4}\right)}{G\left(\frac{11}{4}\right)}\right)^n\times\frac{G\left(n+\frac{7}{4}\right)}{G\left(\frac{7}{4}\right)} \end{aligned}$$

$$S_{5,1}(n) := \prod_{j=1}^{n} (5j-1)!_{(5)} = 5^{n(n-1)/2} \left(\frac{4 \cdot G\left(\frac{9}{5}\right)}{G\left(\frac{14}{5}\right)}\right)^n \times \frac{G\left(n+\frac{9}{5}\right)}{G\left(\frac{9}{5}\right)}$$

$$S_{5,2}(n) := \prod_{j=1}^{n} (5j-2)!_{(5)} = 5^{n(n-1)/2} \left(\frac{3 \cdot G\left(\frac{8}{5}\right)}{G\left(\frac{13}{5}\right)}\right)^n \times \frac{G\left(n+\frac{8}{5}\right)}{G\left(\frac{8}{5}\right)}.$$

We are then led to conjecture inductively, without proof given in this example, that these sequences satisfy the form of the next equation involving the Barnes G-function over the rational-valued inputs prescribed according to the next formula for $n \geq 1$.

$$S_{\alpha,d}(n) = \frac{\alpha^{\binom{n}{2}} (\alpha - d)^n}{\Gamma\left(2 - \frac{d}{\alpha}\right)^n} \times \frac{G\left(n+2-\frac{d}{\alpha}\right)}{G\left(2-\frac{d}{\alpha}\right)}. \qquad \text{(\underline{Generalized Superfactorial Function Identity})}$$

Remark 22 (Generating Rational-Valued Cases of the Barnes G-Function). The identities for the α-factorial function products given in the previous examples suggest further avenues to generating other particular forms of the Barnes G-function formed by these generalized integer-parameter product sequence cases. These functions are generated by extending the constructions of the rational generating function methods outlined in this section [1, 6], which then suggest additional identities for the Barnes G-functions, $G(z+2)$, over rational-valued $z > 0$ involving the special function zeros already defined by Section 2.3 and in Section 6.2. The convergent-based generating function identities stated in the previous equations also suggest further applications to enumerating specific new identities corresponding to the special case constant formulas expanded in [1, §2].

Remark 23 (Expansions of Hyperfactorial Function Products). The generalized superfactorial sequences defined by (68) in the previous example are also related to the *hyperfactorial function*, $H_1(n)$, defined for $n \geq 1$ by the products (A002109)

$$H_1(n) := \prod_{1 \leq j \leq n} j^j \xrightarrow{\texttt{A002109}} (1, 4, 108, 27648, 86400000, \ldots). \qquad \text{(\underline{Hyperfactorial Products})}$$

The exercises in the reference state additional known formulas establishing relations between these expansions of the hyperfactorial function defined above, and products of the binomial coefficients, including the following identities ([12, §5; Ex. 5.13, p. 527], A001142):

$$B_1(n) := \prod_{k=0}^{n} \binom{n}{k} = \frac{(n!)^{n+1}}{S_1(n)^2} = \frac{H_1(n)}{S_1(n)} = \frac{H_1(n)^2}{(n!)^{n+1}}. \qquad \text{(\underline{Binomial Coefficient Products})}$$

Statements of congruence properties and other relations connecting these sequences are considered in [2, 1, 6].

6.6 Examples: Enumerating sequences involving sums of factorial-related functions, sums of factorial powers, and more challenging combinatorial sums involving factorial functions

The coefficients of the convergent-based generating function constructions for the factorial product sequences given in the previous subsection are compared to the next several identities expanding the corresponding sequences of finite sums involving factorial functions ([5, cf. §3; Ex. 3.30 p. 168], A003422, A005165, A033312, A001044, A104344, A061062).

$$\begin{aligned}
L!n &:= \sum_{k=0}^{n-1} k! = [z^n]\left(\frac{z}{(1-z)} \cdot \mathrm{Conv}_n(1,1;z)\right) && \text{(\underline{Left Factorials})} \\
\mathrm{af}(n) &:= \sum_{k=1}^{n} (-1)^{n-k} \cdot k! = [z^n]\left(\frac{1}{(1+z)} \cdot (\mathrm{Conv}_n(1,1;z) - 1)\right) && \text{(\underline{Alternating Factorials})} \\
\mathrm{sf}_2(n) &:= \sum_{k=1}^{n} k \cdot k! = (n+1)! - 1 && (70) \\
&= [x^0 z^n]\left(\frac{1}{(1-z)} \frac{x}{(1-x)^2} \mathrm{Conv}_n\left(1,1;\frac{z}{x}\right)\right) \\
\mathrm{sf}_3(n) &:= \sum_{k=1}^{n} (k!)^2 && \text{(\underline{Sums of Single Factorial Powers})} \\
&= [x^0 z^n]\left(\frac{1}{(1-z)} \times \left(\mathrm{Conv}_n(1,1;x)\, \mathrm{Conv}_n\left(1,1;\frac{z}{x}\right) - 1\right)\right) \\
\mathrm{sf}_4(n) &:= \sum_{k=0}^{n} (k!)^3 \\
&= [x_1^0 x_2^0 z^n]\left(\frac{1}{(1-z)} \times \mathrm{Conv}_n\left(1,1;\frac{z}{x_2}\right) \mathrm{Conv}_n\left(1,1;\frac{x_2}{x_1}\right) \mathrm{Conv}_n(1,1;x_1) + 1\right)
\end{aligned}$$

One generalization of the second identity given in (70) due to Gould is stated in [5, p. 168] as the formula

$$\sum_{k=0}^{n} \binom{x}{k}^p \left(\frac{k!}{x^{k+1}}\right)^p ((x-k)^p - x^p) = \binom{x}{n+1}^p \left(\frac{(n+1)!}{x^{n+1}}\right)^p - 1.$$

The MathWorld site providing an overview of results for factorial-related sums contains references to definitions of several other factorial-related finite sums and series in addition to those special cases defined in the first few of the labeled equations in (70).

6.6.1 Generating sums of double and triple factorial powers

The expansion of the second to last sum, denoted $\mathrm{sf}_3(n)$ in (70), is generalized to form the following variants of sums over the squares of the α-factorial functions, $n!!$ and $n!!!$, through the generating function identities given in (22) of the introduction (A184877):

$$\begin{aligned}
\mathrm{sf}_{3,2}(n) &:= \sum_{k=0}^{n} (k!!)^2 \qquad \left(\underline{\textit{Sums of Double Factorial Squares}}\right) \\
&= [x^0 z^n]\Biggl(\frac{1}{(1-z)} \times \Biggl(\mathrm{Conv}_n\left(2,2;x\right)\mathrm{Conv}_n\left(2,2;\frac{z^2}{x}\right) \\
&\qquad + z \cdot \mathrm{Conv}_n\left(2,3;x\right)\mathrm{Conv}_n\left(2,3;\frac{z^2}{x}\right)\Biggr)\Biggr) \\
&= [x^0 z^n]\Biggl(\frac{1}{(1-z)} \times \Biggl(\mathrm{Conv}_n\left(2,2;x\right)\mathrm{Conv}_n\left(2,2;\frac{z^2}{x}\right) \\
&\qquad + z^{-1} \cdot \mathrm{Conv}_n\left(2,1;x\right)\mathrm{Conv}_n\left(2,1;\frac{z^2}{x}\right) - 1\Biggr)\Biggr) \\
\mathrm{sf}_{3,3}(n) &:= \sum_{k=0}^{n} (k!!!)^2 \qquad \left(\underline{\textit{Sums of Triple Factorial Squares}}\right) \\
&= [x^0 z^n]\Biggl(\frac{1}{(1-z)} \times \Biggl(\mathrm{Conv}_n\left(3,3;x\right)\mathrm{Conv}_n\left(3,3;\frac{z^3}{x}\right) \\
&\qquad + z^{-1} \cdot \mathrm{Conv}_n\left(3,2;x\right)\mathrm{Conv}_n\left(3,2;\frac{z^3}{x}\right) \\
&\qquad + z^{-2} \cdot \mathrm{Conv}_n\left(3,1;x\right)\mathrm{Conv}_n\left(3,1;\frac{z^3}{x}\right) - 1 - \frac{1}{z}\Biggr)\Biggr).
\end{aligned}$$

The next form of the cube-factorial-power sequences, $\mathrm{sf}_4(n)$, defined in (70) corresponding to the next sums taken over powers of the double factorial function are similarly generated by

$$\begin{aligned}
\mathrm{sf}_{4,2}(n) &:= \sum_{k=0}^{n} (k!!)^3 \qquad \left(\underline{\textit{Sums of Double Factorial Cubes}}\right) \\
&= [x_1^0 x_2^0 z^n]\Biggl(\frac{1}{(1-z)} \times \Biggl(\mathrm{Conv}_n\left(2,2;\frac{z^2}{x_2}\right)\mathrm{Conv}_n\left(2,2;\frac{x_2}{x_1}\right)\mathrm{Conv}_n\left(2,2;x_1\right) \\
&\qquad + z^{-1} \cdot \mathrm{Conv}_n\left(2,1;\frac{z^2}{x_2}\right)\mathrm{Conv}_n\left(2,1;\frac{x_2}{x_1}\right)\mathrm{Conv}_n\left(2,1;x_1\right) - \frac{1}{z}\Biggr) \\
&\qquad + 2\Biggr).
\end{aligned}$$

6.6.2 Another convergent-based generating function identity

The second variant of the factorial sums, denoted $\mathrm{sf}_2(n)$ in (70), is enumerated through an alternate approach provided by the more interesting summation identities cited in [5, §3, p. 168]. In particular, we have another pair of identities generating these sums expanded as

$$\begin{aligned}(n+1)!-1 &= (n+1)! \times \sum_{k=0}^{n} \frac{k}{(k+1)!} \\ &= [z^n x^0]\left(\left(\frac{1}{x\cdot(1-x)} - \frac{e^x}{x}\right) \times \mathrm{Conv}_{n+2}\left(-1, n+1; \frac{z}{x}\right)\right) \\ &= [x^0 z^{n+1}]\left(\left(\frac{1}{(1-x)} - e^x\right) \times \mathrm{Conv}_{n+1}\left(1, 1; \frac{z}{x}\right)\right).\end{aligned}$$

Remark 24 (Sums of Squares of the Binomial Coefficients). We can form related convergent-based generating functions enumerating the following polynomial sum which is also expanded in terms of the *Legendre polynomials*, $P_n(x) = [z^n](z^2-2xz+1)^{-1/2}$, as follows [17, §18.3] [12, p. 543]:

$$\begin{aligned}(x-1)^n \cdot P_n\left(\frac{x+1}{x-1}\right) &= \sum_{0\leq k\leq n} \binom{n}{k}^2 x^k \\ &= (n!)^2 \times [z^n] I_0\left(2\sqrt{xz}\right) I_0\left(2\sqrt{z}\right) \\ &= [x_1^0 x_2^0][z^n]\Bigg(\mathrm{Conv}_n\left(1, 1; \frac{z}{x_1}\right) \mathrm{Conv}_n\left(1, 1; \frac{x_1}{x_2}\right) \times \\ &\qquad \times I_0\left(2\sqrt{x\cdot x_2}\right) I_0\left(2\sqrt{x_2}\right)\Bigg) \cdot [n\geq 1]_\delta + [n=0]_\delta .\end{aligned}$$

Other related sums over the alternating squares and cubes of the binomial coefficients, $(-1)^k\binom{2n}{k}^2$ and $(-1)^k\binom{2n}{k}^3$, are given through products of α-factorial functions and reciprocals of the single factorial function, and so may also be enumerated similarly to the sums in this and in the previous section.

6.6.3 Enumerating more challenging combinatorial sums involving double factorials

The convergent-based generating function identities enumerating the sequences stated next in Example 25 and Example 26 below provide additional examples of the termwise formal Laplace-Borel-like transform provided by coefficient extractions involving these rational convergent functions outlined by Remark 19.

Example 25. Since we know that $(2k-1)!! = [z^k]\,\mathrm{Conv}_n(2, 1; z)$ for all $0 \leq k < n$, the terms of the next modified product sequences are generated through the following related forms obtained from the formal series expansions of the convergent generating functions:

$$\frac{(k+1)}{k!} \cdot (2k-1)!! = [x^0][z^k]\left(\mathrm{Conv}_k\left(-1, 2k-1; \frac{z}{x}\right) \cdot (x+1)e^x\right)$$

$$= [x^0][z^k]\left(\mathrm{Conv}_k\left(2,1;\frac{z}{x}\right)\cdot(x+1)e^x\right).$$

The convergent-based expansions of the next "round number" identity generating the double factorial function given cited in the reference are then easily obtained from the previous equations in the following forms [4, §4.3]:

$$\begin{aligned}(2n-1)!! &= \sum_{k=1}^{n}\frac{(n-1)!}{(k-1)!}\cdot k\cdot(2k-3)!! \\ &= (n-1)!\times[x_2^n][x_1^0]\left(\frac{x_2}{(1-x_2)}\times\mathrm{Conv}_n\left(2,1;\frac{x_2}{x_1}\right)\times(x_1+1)e^{x_1}\right) \\ &= [x_1^0x_2^0x_3^{n-1}]\left(\mathrm{Conv}_n\left(1,1;\frac{x_3}{x_2}\right)\mathrm{Conv}_n\left(2,1;\frac{x_2}{x_1}\right)\times\frac{(x_1+1)}{(1-x_2)}\cdot e^{x_1}\right).\end{aligned}$$

Related challenges are posed in the statements of several other finite sum identities involving the double factorial function cited in the references [11, 4].

6.6.4 Other examples of convergent-based generating function identities enumerating the subfactorial function

The first convergent-based generating function expansions approximating the formal ordinary power series over the subfactorial sequence given in Section 6.3.2 are expanded as (A000166)

$$\begin{aligned}!n &= n!\times\sum_{i=0}^{n}\frac{(-1)^i}{i!} = [x^0z^n]\left(\frac{e^{-x}}{(1-x)}\times\mathrm{Conv}_n\left(1,1;\frac{z}{x}\right)\right) \qquad (\underline{\textit{Subfactorial OGF Identities}}) \\ &= \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}k! = [z^nx^n]\left(\frac{(x+z)^n}{(1+z)}\times\mathrm{Conv}_n(1,1;x)\right).\end{aligned}$$

The constructions of the convergent-based formal power series for the ordinary generating functions of the subfactorial function, $!n$, outlined in the previous section are extended in the next example.

Example 26. The next pair of alternate, factorial-function-like auxiliary recurrence relations from ([12, §5.3-§5.4], [5, §4.2]) exactly define the subfactorial function when $n \geq 2$. The following generating function identities then correspond to the respective expansions of the generating functions in previous equation involving the first-order partial derivatives of the convergent functions, $\mathrm{Conv}_n(\alpha, R; t)$, with respect to t [12, cf. §7.2] [25, cf. §2.2]:

$$\begin{aligned}!n &= (n-1)\,(!(n-1)+!(n-2)) \qquad (\underline{\textit{Factorial-Related Recurrence Relations}}) \\ &= (n-1)\times!(n-1)+(n-2)\times!(n-2)+!(n-2) \\ &= [x_1^0z^n]\left(\frac{(z^2+z^3)\cdot e^{-x_1}}{x_1\cdot(1-x_1)}\times\mathrm{Conv}_n^{(1)}\left(1,1;\frac{z}{x_1}\right)+\frac{z^2\cdot e^{-x_1}}{(1-x_1)}\times\mathrm{Conv}_n\left(1,1;\frac{z}{x_1}\right)+1\right)\end{aligned}$$

$$
\begin{aligned}
!n &= n\times !(n-1) + (-1)^n \\
&= (n-1)\times !(n-1) + !(n-1) + (-1)^n \\
&= [x_1^0 z^n]\left(\frac{z^2\cdot e^{-x_1}}{x_1\cdot(1-x_1)} \times \mathrm{Conv}_n^{(1)}\left(1,1;\frac{z}{x_1}\right) + \frac{z\cdot e^{-x_1}}{(1-x_1)} \times \mathrm{Conv}_n\left(1,1;\frac{z}{x_1}\right) + \frac{1}{(1+z)}\right).
\end{aligned}
$$

The next sums provide another summation-based recursive formula for the subfactorial function derived from the known exponential generating function, $\widehat{D}_{n¡}(z) = e^{-z}\cdot(1-z)^{-1}$, for this sequence ([12, §5.4], [5, §4.2]).

$$
\begin{aligned}
!n &= n! - \sum_{i=1}^{n}\binom{n}{i}!(n-i) \qquad \text{(\underline{\textit{Basic Subfactorial Recurrence}})} \\
&= n! \times \left(1 - \sum_{i=1}^{n}\frac{1}{i!}\cdot\frac{!(n-i)}{(n-i)!}\right) \\
&= n! \times \left(1 - [x_1^0 x_2^0 x_3^n]\left((e^{x_3}-1)\,\mathrm{Conv}_n\left(1,1;\frac{x_3}{x_2 x_1}\right)\frac{e^{x_2-x_1}}{(1-x_1)}\right)\right) \\
&= [x_x^0 x_2^0 x_3^0 z^n]\left(\mathrm{Conv}_n\left(1,1;\frac{z}{x_3}\right)\left(\frac{1}{(1-x_3)} - \mathrm{Conv}_n\left(1,1;\frac{x_3}{x_2 x_1}\right)\frac{e^{x_2-x_1}\cdot(e^{x_3}-1)}{(1-x_1)}\right)\right)
\end{aligned}
$$

The rational convergent-based expansions that generate the last equation immediately above then correspond to the effect of performing a termwise Laplace-Borel transformation approximating the complete integral transform, $\mathcal{L}(\widehat{D}_{n¡}(t);z)$, defined by Remark 19, which is related to the regularized sums involving the incomplete gamma function given in the examples from Section 2.2 of the introduction.

6.7 Examples: Generating sums of powers of natural numbers, binomial coefficient sums, and sequences of binomials

6.7.1 Generating variants of sums of powers sequences

As a starting point for the next generating function identities that provide expansions of the sums of powers sequences defined by (74) in this section below, let $p \geq 2$ be fixed, and suppose that $m \in \mathbb{Z}^+$. The convergent-based generating function series over the integer powers, m^p, are generated by an application of the binomial theorem to form the next sums:

$$
m^p - 1 = (p-1)!\cdot\left(p \times \sum_{k=0}^{p-1}\frac{(m-1)^{p-k}}{k!(p-k)!}\right) \tag{71}
$$

$$
m^p - 1 = [z^{p-1}][x^0]\left(\mathrm{Conv}_p\left(-1,p-1;\frac{z}{x}\right)\times(me^{mx}-e^x)\right). \tag{72}
$$

Next, consider the generating function expansions enumerating the finite sums of the p^{th} power sequences in (71) summed over $0 \leq m \leq n$ as follows [12, cf. §7.6]:

$$\begin{aligned}
\widetilde{B}_u(w,x) &:= \sum_{n\geq 0}\left(\sum_{m=0}^{n}(me^{mx}-e^x)u^m\right)w^n \qquad (73)\\
&= \sum_{n\geq 0}\Bigg(\left(\frac{1}{1-u}+\frac{ne^{nx}}{ue^x-1}-\frac{e^{nx}}{(ue^x-1)^2}\right)e^x u^{n+1}\\
&\qquad\qquad + \left(\frac{u}{(ue^x-1)^2}-\frac{1}{1-u}\right)e^x\Bigg)w^n\\
&= -\frac{u^2w^2e^{3x}-2uwe^{2x}+(u^2w^2-uw+1)e^x}{(1-w)(1-uw)(uwe^x-1)^2}\\
&= \frac{e^x}{(1-w)(1-uw)}-\frac{1}{(1-w)(e^xuw-1)^2}+\frac{1}{(1-w)(e^xuw-1)}\\
\widetilde{B}_{a,b,u}(w,x) &:= \sum_{n\geq 0}\left(\sum_{m=0}^{n}((am+b)e^{(am+b)x}-e^x)u^m\right)w^n\\
&= \frac{e^x-be^{bx}+\left((b-a)e^{(a+b)x}-2e^{(a+1)x}+be^{bx}\right)uw+\left((a-b)e^{(a+b)x}+e^{(2a+1)x}\right)(uw)^2}{(1-w)(1-uw)(uwe^{ax}-1)^2}\\
&= \frac{be^{bx}+(a-b)e^{(a+b)x}uw}{(1-w)(uwe^{ax}-1)^2}-\frac{e^x-2e^{(a+1)x}uw+e^{(2a+1)x}(uw)^2}{(1-w)(1-uw)(uwe^{ax}-1)^2}.
\end{aligned}$$

We then obtain the next cases of the convergent-based generating function identities exactly enumerating the corresponding first variants of the sums of powers sequences obtained from (71) stated in the following forms [12, §6.5, §7.6]:

$$\begin{aligned}
S_p(n) &:= \sum_{0\leq m<n} m^p \qquad (74)\\
&= n + [w^{n-1}][z^{p-1}x^0]\left(\operatorname{Conv}_p\left(-1,p-1;\frac{z}{x}\right)\widetilde{B}_1(w,x)\right) \qquad (75)\\
&= n + [w^{n-1}][x^0z^{p-1}]\left(\operatorname{Conv}_p\left(1,1;\frac{z}{x}\right)\widetilde{B}_1(w,x)\right).
\end{aligned}$$

A somewhat related set of results for variations of more general cases of the power sums expanded above is expanded similarly for $p \geq 1$, fixed scalars $a, b \neq 0$, and any non-zero indeterminate u according to the next convergent function identities given by

$$S_p(u,n) := \sum_{0\leq m<n} m^p u^m \qquad \left(\underline{\textit{Generalized Sums of Powers Sequences}}\right)$$

$$= \frac{u^n-1}{u-1} + [w^{n-1}][z^{p-1}x^0]\left(\operatorname{Conv}_p\left(-1, p-1; \frac{z}{x}\right) \widetilde{B}_u(w, x)\right)$$

$$= \frac{u^n-1}{u-1} + [w^{n-1}][x^0 z^{p-1}]\left(\operatorname{Conv}_p\left(1, 1; \frac{z}{x}\right) \widetilde{B}_u(w, x)\right) \tag{76}$$

$$S_p(a, b; u, n) := \sum_{0 \leq m < n} (am+b)^p u^m$$

$$= \frac{u^n-1}{u-1} + [w^{n-1}][z^{p-1}x^0]\left(\operatorname{Conv}_p\left(-1, p-1; \frac{z}{x}\right) \widetilde{B}_{a,b,u}(w, x)\right)$$

$$= \frac{u^n-1}{u-1} + [w^{n-1}][x^0 z^{p-1}]\left(\operatorname{Conv}_p\left(1, 1; \frac{z}{x}\right) \widetilde{B}_{a,b,u}(w, x)\right).$$

Remark 27 (Relations to the Bernoulli and Euler Polynomials). For fixed $n \geq 0$, integers a, b, and some $u \neq 0$, exponential generating functions for the generalized sums, $S_p(a, b; u, n+1)$, with respect to p are given by the following sums [12, §7.6]:

$$\frac{S_p(a, b; u, n+1)}{p!} = [z^p]\left(\sum_{0 \leq k \leq n} e^{(ak+b)z} u^k\right) = [z^p]\left(e^{bz} \times \frac{e^{a(n+1)z}u^{n+1} - 1}{ue^{az} - 1}\right).$$

The bivariate, two-variable exponential generating functions, $\widetilde{B}_u(w, x)$ and $\widetilde{B}_{a,b,u}(w, x)$, involved in enumerating the respective sequences in each of (75) and (76) are related to the generating functions for the *Bernoulli* and *Euler polynomials*, $B_n(x)$ and $E_n(x)$, defined in the references when the parameter $u :\mapsto \pm 1$ ([17, §24.2], [19, §4.2.2, §4.2.3]). For $u := \pm 1$, the sums defined by the left-hand-sides of the previous two equations in (76) also correspond to special cases of the following known identities involving these polynomial sequences [12, §6.5] [17, §24.4(iii)]:

$$\sum_{m=0}^{n} (am+b)^p = \frac{a^p}{p+1}\left(B_{p+1}\left(n+1+\frac{b}{a}\right) - B_{p+1}\left(\frac{b}{a}\right)\right) \qquad (\underline{\textit{Sums of Powers Formulas}})$$

$$\sum_{m=0}^{n} (-1)^m (am+b)^p = \frac{a^p}{2}\left((-1)^n \cdot E_p\left(n+1+\frac{b}{a}\right) + E_p\left(\frac{b}{a}\right)\right).$$

The results in the previous equations are also compared to the forms of other well-known sequence generating functions involving the *Bernoulli numbers*, B_n, the *first-order Eulerian numbers*, $\left\langle {n \atop m} \right\rangle$, and the *Stirling numbers of the second kind*, $\left\{ {n \atop k} \right\}$, in the next few cases of the established identities for these sequences expanded in Remark 28 ([12, cf. §6], A027641, A027642, A008292, A008277).

Remark 28 (Comparisons to Other Formulas and Special Generating Functions). The sequences generated by (75) are first compared to the following known expansions that exactly generate these finite sums over $n \geq 0$ and $p \geq 1$ ([17, §24.4(iii), §24.2], [12, §6.5, §7.4]):

$$\begin{aligned} S_p(n+1) &= \frac{B_{p+1}(n+1) - B_{p+1}(0)}{p+1} && \left(\underline{\textit{Relations to the Bernoulli Numbers}}\right) \\ &= \sum_{s=0}^{p} \binom{p+1}{s} \frac{B_s \cdot (n+1)^{p+1-s}}{(p+1)} \\ S_p(n+1) &= \sum_{j=0}^{p} \left\{ {p \atop j} \right\} \frac{(n+1)^{\underline{j+1}}}{(j+1)} && \left(\underline{\textit{Expansions by the Stirling Numbers}}\right) \\ &= \sum_{0 \leq j,k \leq p+1} \left\{ {p \atop j} \right\} \left[{j+1 \atop k} \right] \frac{(-1)^{j+1-k}(n+1)^k}{j+1} \\ S_p(n+1) &= [z^n] \left(\sum_{j=0}^{p} \left\{ {p \atop j} \right\} \frac{z^j \cdot j!}{(1-z)^{j+2}} \right) && \left(\underline{\textit{Relations to Special Generating Functions}}\right) \\ &= [z^n] \left(\sum_{i \geq 0} \left\langle {p \atop i} \right\rangle \frac{z^{i+1}}{(1-z)^{p+2}} \right) \\ &= p! \cdot [w^n z^p] \left(\frac{w \cdot e^z}{(1-w)(1-we^z)} \right). \end{aligned}$$

Two bivariate "super" generating function for the *first-order Eulerian numbers*, $\left\langle {n \atop m} \right\rangle$, employed in formulating variants of the last identity are given by the following equations where $\left\langle {n \atop m} \right\rangle = \left\langle {n \atop n-1-m} \right\rangle$ for all $n \geq 1$ and $0 \leq m < n$ by the row-wise symmetry in the triangle [12, §7.4, §6.2] [17, §26.14(ii)]:

$$\begin{aligned} \sum_{m,n \geq 0} \left\langle {n \atop m} \right\rangle \frac{w^m z^n}{n!} &= \frac{1-w}{e^{(w-1)z} - w} && \left(\underline{\textit{First-Order Eulerian Number EGFs}}\right) \\ \sum_{m,n \geq 0} \left\langle {m+n+1 \atop m} \right\rangle \frac{w^m z^n}{(m+n+1)!} &= \frac{e^w - e^z}{we^z - ze^w}. \end{aligned}$$

Similarly, the generalized forms of the sums generated by (76) are related to the more well-known combinatorial sequence identities expanded as follows ([17, §26.8], [12, §7.4]):

$$\begin{aligned} S_p(u, n+1) &= \sum_{j=0}^{p} \left\{ {p \atop j} \right\} u^j \times \frac{\partial^{(j)}}{\partial u^{(j)}} \left(\frac{1}{1-u} - \frac{u^{n+1}}{1-u} \right) \\ &= [w^n] \left(\sum_{j=0}^{p} \left\{ {p \atop j} \right\} \frac{(uw)^j \cdot j!}{(1-w)(1-uw)^{j+1}} \right) \\ &= p! \cdot [w^n z^p] \left(\frac{uw \cdot e^z}{(1-w)(1-uwe^z)} \right). \end{aligned}$$

As in the examples of termwise applications of the formal Laplace-Borel transformations noted above, the role of the parameter p corresponding to the forms of the special sequence triangles in the identities given above is phrased through the implicit dependence of the convergent functions on the fixed $p \geq 1$ in each of (72), (75), and (76).

6.7.2 Semi-rational generating function constructions enumerating sequences of binomials

A second motivating application highlighting the procedure outlined in the examples above expands the binomial power sequences, $2^p - 1$, for $p \geq 1$ through an extension of the first result given in (71) when $m := 2$. The finite sums for the integer powers provided by the binomial theorem in these cases correspond to removing, or selectively peeling off, the r uppermost-indexed terms from the first sum for subsequent choices of the $p \geq r \geq 1$ in the following forms [18, cf. §2.2, §2.4]:

$$\begin{aligned} m^p - 1 &= \sum_{i=0}^{r} \binom{p}{p+1-i} (m-1)^i \\ &\quad + (p-r-1)! \cdot \left(p(p-1)\cdots(p-r) \times \sum_{k=0}^{p-r-1} \frac{(m-1)^{k+1}}{(k+1)!(p-1-k)!} \right). \end{aligned}$$

The generating function identities phrased in terms of (71) in the previous examples are then modified slightly according to this equation for the next few special cases of $r \geq 1$. For example, these sums are employed to obtain the next forms of the convergent-based generating function expansions generalizing the result in (72) above (A000225).

$$\begin{aligned} 2^p - 1 &= [z^{p-2}][x^0]\left(\frac{1}{(1-z)} + \left(4e^{2x} - 2e^x\right) \times \mathrm{Conv}_p\left(-1, p-2; \frac{z}{x}\right)\right), \quad p \geq 2 \\ &= [z^{p-3}][x^0]\left(\frac{4-3z}{(1-z)^2} + \left(8e^{2x} - e^x \cdot (x+5)\right) \times \mathrm{Conv}_p\left(-1, p-3; \frac{z}{x}\right)\right), \quad p \geq 3 \\ &= [z^{p-4}][x^0]\Bigg(\frac{11 - 17z + 7z^2}{(1-z)^3} \\ &\qquad + \left(16e^{2x} - \frac{e^x}{2} \cdot (x^2 + 10x + 24)\right) \times \mathrm{Conv}_p\left(-1, p-4; \frac{z}{x}\right)\Bigg), \quad p \geq 4 \\ &= [z^{p-5}][x^0]\Bigg(\frac{26 - 62z + 52z^2 - 15z^3}{(1-z)^4} \\ &\qquad + \frac{x^5 e^x}{6}(192e^x - (x^3 + 18x^2 + 96x + 162)) \times \mathrm{Conv}_p\left(-1, p-5; \frac{z}{x}\right)\Bigg), \end{aligned}$$

$p \geq 5.$

The special cases of these generating functions for the p^{th} powers defined above are also expanded in the next more general forms of these convergent function identities for $p > m \geq 1$.

$$\begin{aligned} 2^p - 1 &= [z^{p-m-1}x^0]\left(\frac{\widetilde{\ell}_{m,2}(z)}{(1-z)^m} + \left(2^{m+1} \cdot e^{2x} - \frac{e^x}{(m-1)!} \cdot \widetilde{p}_{m,2}(x)\right) \times \right. \\ &\qquad \left. \times \operatorname{Conv}_p\left(-1, p-m-1; \frac{z}{x}\right)\right) \\ &= [x^0 z^{p-m-1}]\left(\frac{\widetilde{\ell}_{m,2}(z)}{(1-z)^m} + \left(2^{m+1} \cdot e^{2x} - \frac{e^x}{(m-1)!} \cdot \widetilde{p}_{m,2}(x)\right) \times \operatorname{Conv}_p\left(1, 1; \frac{z}{x}\right)\right) \end{aligned}$$

The listings provided in Table 9.4 (page 72) cite the particular special cases of the polynomials, $\ell_{m,2}(z)$ and $p_{m,2}(x)$, that provide the generalizations of the first cases expanded in the previous equations. The constructions of these new identities, including the variations for the sequences formed by the binomial coefficient sums for the powers, $2^p - 2$, are motivated in the context of divisibility modulo p by the reference ([13, §8], A000918).

Further cases of the more general p^{th} power sequences of the form $(s+1)^p - 1$ for any fixed $s > 0$ are enumerated similarly through the next formulas.

$$\begin{aligned} (s+1)^p - 1 &= [z^{p-m-1}x^0]\left(\frac{s^2 \ell_{m,s+1}(z)}{(1-sz)^m} - \left(e^x - (s+1)^{m+1}e^{(s+1)x} + \frac{s^2 e^{sx}}{(m-1)!} p_{m,s+1}(sx)\right) \times \right. \\ &\qquad \left. \times \operatorname{Conv}_p\left(-1, p-m-1; \frac{z}{x}\right)\right), \ p > m \geq 1 \\ &= [x^0 z^{p-m-1}]\left(\frac{s^2 \ell_{m,s+1}(z)}{(1-sz)^m} - \left(e^x - (s+1)^{m+1}e^{(s+1)x} + \frac{s^2 e^{sx}}{(m-1)!} p_{m,s+1}(sx)\right) \times \right. \\ &\qquad \left. \times \operatorname{Conv}_p\left(1, 1; \frac{z}{x}\right)\right), \ p > m \geq 1 \end{aligned}$$

The second set of listings provided in Table 9.4 (page 72) expand several additional special cases corresponding to the polynomial sequences, $\ell_{m,s+1}(z)$ and $p_{m,s+1}(x)$, required to generate the more general cases of these particular p^{th} power sequences when $p > m \geq 1$ (A000225, A024023, A024036, A024049). Related expansions of the *sequences of binomials* of the forms $a^n \pm 1$ and $a^n \pm b^n$ are considered in the references [18, cf. §2.2, §2.4].

7 Conclusions

We have defined several new forms of ordinary power series approximations to the typically divergent ordinary generating functions of generalized multiple, or α-factorial, function sequences. The generalized forms of these convergent functions provide partial truncated approximations to the sequence generating functions which enumerate the factorial products generated by these divergent formal power series. The exponential generating functions for the special case product sequences, $p_n(\alpha, s-1)$, are studied in the reference [20, §5]. The exponential generating functions that enumerate the cases corresponding to the more general factorial-like sequences, $p_n(\alpha, \beta n+\gamma)$, are less obvious in form. We have also suggested a number of new, alternate approaches to generating the factorial function sequences that arise in applications, including classical identities involving the single and double factorial functions, and in the forms of several other noteworthy special cases.

The key ingredient to the short proof given in Section 4 employs known characterizations of the Pochhammer symbols, $(x)_n$, by generalized Stirling number triangles as polynomial expansions in the indeterminate, x, each with predictably small finite-integral-degree at any fixed n. The more combinatorial proof in the spirit of Flajolet's articles suggested by the discussions in Section 4.2 may lead to further interesting interpretations of the α-factorial functions, $(s-1)!_{(\alpha)}$, which motivate the investigations of the coefficient-wise symbolic polynomial expansions of the functions first considered in the article [20]. A separate proof of the expansions of these new continued fractions formulated in terms of the generalized α-factorial function coefficients defined by (4), and by their strikingly Stirling-number-like combinatorial properties motivated in the introduction, is notably missing from this article.

The rationality of these convergent functions for all h suggests new insight to generating numeric sequences of interest, including several specific new congruence properties, derivations of finite difference equations that hold for these exact sequences modulo any integers p, and perhaps more interestingly, exact expansions of the classical single and double factorial functions by the special zeros of the generalized Laguerre polynomials and confluent hypergeometric functions. The techniques behind the specific identities given here are easily generalized and extended to other specific applications. The particular examples cited within this article are intended as suggestions at new starting points to tackling the expansions that arise in many other practical situations, both implicitly and explicitly involving the generalized variants of the factorial-function-like product sequences, $p_n(\alpha, R)$.

8 Acknowledgments

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9 Appendix: Tables

h	$\mathrm{FP}_h(\alpha, R; z)$
1	1
2	$1-(2\alpha+R)z$
3	$1-(6\alpha+2R)z+(6\alpha^2+4\alpha R+R^2)z^2$
4	$1-(12\alpha+3R)z+(36\alpha^2+19\alpha R+3R^2)z^2-(24\alpha^3+18\alpha^2R+7\alpha R^2+R^3)z^3$
5	$1-(20\alpha+4R)z+(120\alpha^2+51\alpha R+6R^2)z^2-(240\alpha^3+158\alpha^2R+42\alpha R^2+4R^3)z^3$ $+(120\alpha^4+96\alpha^3R+46\alpha^2R^2+11\alpha R^3+R^4)z^4$

Table 9.1.1. The convergent numerator functions, $\mathrm{FP}_h(\alpha, R; z)$

h	$\mathrm{FQ}_h(\alpha, R; z)$
0	1
1	$1-Rz$
2	$1-2(\alpha+R)z+R(\alpha+R)z^2$
3	$1-3(2\alpha+R)z+3(\alpha+R)(2\alpha+R)z^2-R(\alpha+R)(2\alpha+R)z^3$
4	$1-4(3\alpha+R)z+6(2\alpha+R)(3\alpha+R)z^2-4(\alpha+R)(2\alpha+R)(3\alpha+R)z^3$ $+R(\alpha+R)(2\alpha+R)(3\alpha+R)z^4$
5	$1-5(4\alpha+R)z+10(3\alpha+R)(4\alpha+R)z^2-10(2\alpha+R)(3\alpha+R)(4\alpha+R)z^3$ $+5(\alpha+R)(2\alpha+R)(3\alpha+R)(4\alpha+R)z^4-R(\alpha+R)(2\alpha+R)(3\alpha+R)(4\alpha+R)z^5$

Table 9.1.2. The convergent denominator functions, $\mathrm{FQ}_h(\alpha, R; z)$

h	$\mathrm{FP}_h(1, 1; z)$	$\mathrm{FQ}_h(1, 1; z)$
1	1	$1-z$
2	$1-3z$	$1-4z+2z^2$
3	$1-8z+11z^2$	$1-9z+18z^2-6z^3$
4	$1-15z+58z^2-50z^3$	$1-16z+72z^2-96z^3+24z^4$
5	$1-24z+177z^2-444z^3+274z^4$	$1-25z+200z^2-600z^3+600z^4-120z^5$
6	$1-35z+416z^2-2016z^3$ $+3708z^4-1764z^5$	$1-36z+450z^2-2400z^3+5400z^4$ $-4320z^5+720z^6$

Table 9.1.3. The h^{th}convergent generating functions, $\mathrm{Conv}_h(1, 1; z)$, generating the single factorial function, $n! = (1)_n$

h	$\mathrm{FP}_h(2, 1; z)$	$\mathrm{FQ}_h(2, 1; z)$
1	1	$1-z$
2	$1-5z$	$1-6z+3z^2$
3	$1-14z+33z^2$	$1-15z+45z^2-15z^3$
4	$1-27z+185z^2-279z^3$	$1-28z+210z^2-420z^3+105z^4$
5	$1-44z+588z^2-2640z^3+2895z^4$	$1-45z+630z^2-3150z^3+4725z^4-945z^5$
6	$1-65z+1422z^2-12558z^3$ $+41685z^4-35685z^5$	$1-66z+1485z^2-13860z^3+51975z^4$ $-62370z^5+10395z^6$

Table 9.1.4. The h^{th} convergent generating functions, $\mathrm{Conv}_h(2, 1; z)$, generating the double factorial function, $(2n-1)!! = 2^n \times \left(\frac{1}{2}\right)_n$

Table 9.1: The generalized convergent numerator and denominator function sequences

h	$z^{h-1} \cdot \mathrm{FP}_h\left(\alpha, R; z^{-1}\right)$
1	1
2	$-(2\alpha+R)+z$
3	$6\alpha^2+\alpha(4R-6z)+(R-z)^2$
4	$-24\alpha^3-18\alpha^2(R-2z)-\alpha(7R-12z)(R-z)-(R-z)^3$
5	$120\alpha^4+2\alpha^2\left(23R^2-79Rz+60z^2\right)+48\alpha^3(2R-5z)+\alpha(11R-20z)(R-z)^2+(R-z)^4$
6	$-720\alpha^5-2\alpha^3\left(163R^2-678Rz+600z^2\right)-\alpha^2\left(101R^2-368Rz+300z^2\right)(R-z)$
	$-600\alpha^4(R-3z)-2\alpha(8R-15z)(R-z)^3-(R-z)^5$
7	$5040\alpha^6+36\alpha^4\left(71R^2-347Rz+350z^2\right)+\alpha^2\left(197R^2-740Rz+630z^2\right)(R-z)^2$
	$+\alpha^3\left(932R^3-5102R^2z+8322Rz^2-4200z^3\right)+2160\alpha^5(2R-7z)$
	$+2\alpha(11R-21z)(R-z)^4+(R-z)^6$
8	$-40320\alpha^7-36\alpha^5\left(617R^2-3466Rz+3920z^2\right)-\alpha^2\left(351R^2-1342Rz+1176z^2\right)(R-z)^3$
	$+\alpha^4\left(-9080R^3+57286R^2z-105144Rz^2+58800z^3\right)$
	$-\alpha^3\left(2311R^3-13040R^2z+22210Rz^2-11760z^3\right)(R-z)-35280\alpha^6(R-4z)$
	$-\alpha(29R-56z)(R-z)^5-(R-z)^7$

Table 9.2.1. The reflected numerator polynomials, $\widetilde{\mathrm{FP}}_h(\alpha, R; z) := z^{h-1} \cdot \mathrm{FP}_h\left(\alpha, R; z^{-1}\right)$

h	$z^{h-1} \cdot \mathrm{FP}_h\left(\alpha, z-w; z^{-1}\right)$
2	$w-2\alpha$
3	$6\alpha^2+w^2-4\alpha w-2\alpha z$
4	$-24\alpha^3+w^3-7\alpha w^2+w\left(18\alpha^2-5\alpha z\right)+18\alpha^2 z$
5	$120\alpha^4+w^4-11\alpha w^3+w^2\left(46\alpha^2-9\alpha z\right)+w\left(66\alpha^2 z-96\alpha^3\right)+8\alpha^2z^2-144\alpha^3 z$
6	$-720\alpha^5+w^5-16\alpha w^4+w^3\left(101\alpha^2-14\alpha z\right)+w^2\left(166\alpha^2 z-326\alpha^3\right)$
	$+w\left(600\alpha^4+33\alpha^2z^2-704\alpha^3 z\right)-170\alpha^3z^2+1200\alpha^4 z$
7	$5040\alpha^6+w^6-22\alpha w^5+w^4\left(197\alpha^2-20\alpha z\right)+w^3\left(346\alpha^2 z-932\alpha^3\right)$
	$+w^2\left(2556\alpha^4+87\alpha^2z^2-2306\alpha^3 z\right)$
	$+w\left(-4320\alpha^5-914\alpha^3z^2+7380\alpha^4 z\right)-48\alpha^3z^3+2664\alpha^4z^2-10800\alpha^5 z$
8	$-40320\alpha^7+w^7-29\alpha w^6+w^5\left(351\alpha^2-27\alpha z\right)+w^4\left(640\alpha^2 z-2311\alpha^3\right)$
	$+w^3\left(9080\alpha^4+185\alpha^2z^2-6107\alpha^3 z\right)+w^2\left(-22212\alpha^5-3063\alpha^3z^2+30046\alpha^4 z\right)$
	$+w\left(35280\alpha^6-279\alpha^3z^3+17812\alpha^4z^2-80352\alpha^5 z\right)$
	$+1862\alpha^4z^3-38556\alpha^5z^2+105840\alpha^6 z$

h	$z^{h-1} \cdot \mathrm{FP}_h\left(-\alpha, z-w; z^{-1}\right)$
3	$6\alpha^2+w^2+\alpha(4w+2z)$
4	$24\alpha^3+w^3+\alpha\left(7w^2+5wz\right)+\alpha^2(18w+18z)$
5	$120\alpha^4+w^4+\alpha^2\left(46w^2+66wz+8z^2\right)+\alpha\left(11w^3+9w^2z\right)+\alpha^3(96w+144z)$
6	$720\alpha^5+w^5+\alpha^3\left(326w^2+704wz+170z^2\right)+\alpha\left(16w^4+14w^3z\right)$
	$+\alpha^2\left(101w^3+166w^2z+33wz^2\right)+\alpha^4(600w+1200z)$
7	$5040\alpha^6+w^6+\alpha^4\left(2556w^2+7380wz+2664z^2\right)+\alpha\left(22w^5+20w^4z\right)$
	$+\alpha^3\left(932w^3+2306w^2z+914wz^2+48z^3\right)+\alpha^2\left(197w^4+346w^3z+87w^2z^2\right)$
	$+\alpha^5(4320w+10800z)$
8	$40320\alpha^7+w^7+\alpha^5\left(22212w^2+80352wz+38556z^2\right)+\alpha\left(29w^6+27w^5z\right)$
	$+\alpha^4\left(9080w^3+30046w^2z+17812wz^2+1862z^3\right)+\alpha^2\left(351w^5+640w^4z+185w^3z^2\right)$
	$+\alpha^3\left(2311w^4+6107w^3z+3063w^2z^2+279wz^3\right)+\alpha^6(35280w+105840z)$

Table 9.2.2. Modified forms of the reflected numerator polynomials, $\widetilde{\mathrm{FP}}_h(\pm\alpha, z-w; z)$

Table 9.2: The reflected convergent numerator function sequences

n	$n!_{(2)}$	$\widetilde{R}_2^{(2)}(n)$	(2)	(4)	$\widetilde{R}_3^{(2)}(n)$	(3)	(6)	$\widetilde{R}_4^{(2)}(n)$	(4)	(8)	$\widetilde{R}_5^{(2)}(n)$	(5)	(10)
0	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	0	2	2	2	2	2	2	2	2	2	2
3	3	3	1	3	3	0	3	3	3	3	3	3	3
4	8	8	0	0	8	2	2	8	0	0	8	3	8
5	15	15	1	3	15	0	3	15	3	7	15	0	5
6	48	48	0	0	48	0	0	48	0	0	48	3	8
7	105	-175	1	1	105	0	3	105	1	1	105	0	5
8	384	0	0	0	384	0	0	384	0	0	384	4	4
9	945	-13671	1	1	945	0	3	945	1	1	945	0	5
10	3840	-17920	0	0	3840	0	0	3840	0	0	3840	0	0
11	10395	-633501	1	3	43659	0	3	10395	3	3	10395	0	5
12	46080	-960000	0	0	92160	0	0	46080	0	0	46080	0	0
13	135135	-28498041	1	3	3532815	0	3	135135	3	7	135135	0	5
14	645120	-45480960	0	0	5644800	0	0	645120	0	0	645120	0	0
15	2027025	-1343937855	1	1	257161905	0	3	-5386095	1	1	2027025	0	5
16	10321920	-2202927104	0	0	401522688	0	0	0	0	0	10321920	0	0
17	34459425	-67747539375	1	1	17642360385	0	3	-1211768415	1	1	34459425	0	5
18	185794560	-112925343744	0	0	27994595328	0	0	-1634992128	0	0	185794560	0	0
19	654729075	-3664567145437	1	3	1200706189875	0	3	-141536175885	3	3	3315215475	0	5
20	3715891200	-6182061834240	0	0	1941606236160	0	0	-211558072320	0	0	7431782400	0	0
21	13749310575	-212363430514977	1	3	83236453970607	0	3	-14054409745425	3	7	679112772975	0	5

Table 9.3.1. Congruences for the double factorial function, $n!! = n!_{(2)}$, modulo h (and $2h$) for $h := 2, 3, 4, 5$. Supplementary listings containing computational data for the congruences, $n!! \equiv R_h^{(2)}(n) \pmod{2^i h}$, for $2 \leq i \leq h \leq 5$ are tabulated in the summary notebook reference.

n	$n!_{(3)}$	$\widetilde{R}_2^{(3)}(n)$	(2)	(6)	$\widetilde{R}_3^{(3)}(n)$	(3)	(9)	$\widetilde{R}_4^{(3)}(n)$	(4)	(12)	$\widetilde{R}_5^{(3)}(n)$	(5)	(15)
0	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	0	2	2	2	2	2	2	2	2	2	2
3	3	3	1	3	3	0	3	3	3	3	3	3	3
4	4	4	0	4	4	1	4	4	0	4	4	4	4
5	10	10	0	4	10	1	1	10	2	10	10	0	10
6	18	18	0	0	18	0	0	18	2	6	18	3	3
7	28	28	0	4	28	1	1	28	0	4	28	3	13
8	80	80	0	2	80	2	8	80	0	8	80	0	5
9	162	162	0	0	162	0	0	162	2	6	162	2	12
10	280	-980	0	4	280	1	1	280	0	4	280	0	10
11	880	-704	0	4	880	1	7	880	0	4	880	0	10
12	1944	0	0	0	1944	0	0	1944	0	0	1944	4	9
13	3640	-92300	0	4	3640	1	4	3640	0	4	3640	0	10
14	12320	-115192	0	2	12320	2	8	12320	0	8	12320	0	5
15	29160	-136080	0	0	29160	0	0	29160	0	0	29160	0	0
16	58240	-6186752	0	4	395200	1	1	58240	0	4	58240	0	10
17	209440	-8349992	0	4	633556	1	1	209440	0	4	209440	0	10
18	524880	-10935000	0	0	1049760	0	0	524880	0	0	524880	0	0
19	1106560	-411766784	0	4	51684256	1	1	1106560	0	4	1106560	0	10
20	4188800	-572266240	0	2	70505120	2	2	4188800	0	8	4188800	0	5
21	11022480	-777084840	0	0	96446700	0	0	11022480	0	0	11022480	0	0
22	24344320	-28922921456	0	4	5645314048	1	4	-144674816	0	4	24344320	0	10
23	96342400	-40807520000	0	4	7668245080	1	1	-116486720	0	4	96342400	0	10
24	264539520	-56458612224	0	0	10290587328	0	0	0	0	0	264539520	0	0
25	608608000	-2177450514800	0	4	577086766300	1	1	-41321139200	0	4	608608000	0	10
26	2504902400	-3101148709984	0	2	793943072000	2	8	-52040160640	0	8	2504902400	0	5
27	7142567040	-4341229572096	0	0	1076206288752	0	0	-62854589952	0	0	7142567040	0	0
28	17041024000	-176120000000000	0	4	58548072721600	1	1	-7074936915200	0	4	153556480000	0	10

Table 9.3.2. Congruences for the triple factorial function, $n!!! = n!_{(3)}$, modulo h (and $3h$) for $h := 2, 3, 4, 5$. Supplementary listings containing computational data for the congruences, $n!!! \equiv R_h^{(3)}(n) \pmod{3^i h}$, for $2 \leq i \leq h \leq 5$ are tabulated in the summary notebook reference.

n	$n!_{(4)}$	$\widetilde{R}_2^{(4)}(n)$	(2)	(8)	$\widetilde{R}_3^{(4)}(n)$	(3)	(12)	$\widetilde{R}_4^{(4)}(n)$	(4)	(16)	$\widetilde{R}_5^{(4)}(n)$	(5)	(20)
0	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	0	2	2	2	2	2	2	2	2	2	2
3	3	3	1	3	3	0	3	3	3	3	3	3	3
4	4	4	0	4	4	1	4	4	0	4	4	4	4
5	5	5	1	5	5	2	5	5	1	5	5	0	5
6	12	12	0	4	12	0	0	12	0	12	12	2	12
7	21	21	1	5	21	0	9	21	1	5	21	1	1
8	32	32	0	0	32	2	8	32	0	0	32	2	12
9	45	45	1	5	45	0	9	45	1	13	45	0	5
10	120	120	0	0	120	0	0	120	0	8	120	0	0
11	231	231	1	7	231	0	3	231	3	7	231	1	11
12	384	384	0	0	384	0	0	384	0	0	384	4	4
13	585	-3159	1	1	585	0	9	585	1	9	585	0	5
14	1680	-2800	0	0	1680	0	0	1680	0	0	1680	0	0
15	3465	-1815	1	1	3465	0	9	3465	1	9	3465	0	5
16	6144	0	0	0	6144	0	0	6144	0	0	6144	4	4
17	9945	-364871	1	1	9945	0	9	9945	1	9	9945	0	5
18	30240	-437472	0	0	30240	0	0	30240	0	0	30240	0	0
19	65835	-508725	1	3	65835	0	3	65835	3	11	65835	0	15
20	122880	-573440	0	0	122880	0	0	122880	0	0	122880	0	0
21	208845	-32086803	1	5	1990989	0	9	208845	1	13	208845	0	5
22	665280	-40544064	0	0	2794176	0	0	665280	0	0	665280	0	0
23	1514205	-50324483	1	5	4031325	0	9	1514205	1	13	1514205	0	5
24	2949120	-61440000	0	0	5898240	0	0	2949120	0	0	2949120	0	0
25	5221125	-2829930075	1	5	358222725	0	9	5221125	1	5	5221125	0	5
26	17297280	-3647749248	0	0	452200320	0	0	17297280	0	0	17297280	0	0
27	40883535	-4637561553	1	7	570989007	0	3	40883535	3	15	40883535	0	15
28	82575360	-5821562880	0	0	722534400	0	0	82575360	0	0	82575360	0	0
29	151412625	-264205859375	1	1	52114215825	0	9	-1438808175	1	1	151412625	0	5
30	518918400	-344048090880	0	0	65833447680	0	0	-1378840320	0	0	518918400	0	0
31	1267389585	-442855631151	1	1	82524474513	0	9	-979895151	1	1	1267389585	0	5
32	2642411520	-563949338624	0	0	102789808128	0	0	0	0	0	2642411520	0	0
33	4996616625	-26469713463567	1	1	7078405640625	0	9	-516689348175	1	1	4996616625	0	5
34	17643225600	-34686740160000	0	0	9032888517120	0	0	-620425428480	0	0	17643225600	0	0

Table 9.3.3. Congruences for the quadruple factorial (4-factorial) function, $n!!!! = n!_{(4)}$, modulo h (and $4h$) for $h := 2, 3, 4, 5$. Supplementary listings containing computational data for the congruences, $n!!!! \equiv R_h^{(4)}(n) \pmod{4^i h}$, for $2 \leq i \leq h \leq 5$ are tabulated in the summary notebook reference.

Table 9.3. The α-factorial functions modulo h (and $h\alpha$) for $h := 2, 3, 4, 5$ defined by the special case expansions from Section 3.3 of the introduction and in Section 6.2 where $\widetilde{R}_p^{(\alpha)}(n) := \left[z^{\lfloor (n+\alpha-1)/\alpha \rfloor}\right] \operatorname{Conv}_p(-\alpha, n; z)$.

m	$\ell_{m,2}(z)$	$p_{m,2}(x)$
1	1	1
2	$4-3z$	$x+4$
3	$11-17z+7z^2$	$x^2+10x+22$
4	$26-62z+52z^2-15z^3$	$x^3+18x^2+96x+156$
5	$57-186z+238z^2-139z^3+31z^4$	$x^4+28x^3+264x^2+1008x+1368$
m	$\ell_{m,3}(z)$	$p_{m,3}(x)$
1	1	1
2	$5-8z$	$x+5$
3	$18-60z+52z^2$	$x^2+12x+36$
4	$58-300z+532z^2-320z^3$	$x^3+21x^2+144x+348$
5	$179-1268z+3436z^2-4192z^3+1936z^4$	$x^4+32x^3+372x^2+1968x+4296$
m	$\ell_{m,4}(z)$	$p_{m,4}(x)$
1	1	1
2	$6-15z$	$x+6$
3	$27-141z+189z^2$	$x^2+14x+54$
4	$112-906z+2484z^2-2295z^3$	$x^3+24x^2+204x+672$
5	$453-4998z+20898z^2-39123z^3+27621z^4$	$x^4+36x^3+504x^2+3504x+10872$
m	$\ell_{m,5}(z)$	$p_{m,5}(x)$
1	1	1
2	$7-24z$	$x+7$
3	$38-272z+496z^2$	$x^2+16x+76$
4	$194-2144z+7984z^2-9984z^3$	$x^3+27x^2+276x+1164$
5	$975-14640z+82960z^2-209920z^3+199936z^4$	$x^4+40x^3+660x^2+5760x+23400$

Table 9.4.1. Generating the sequences of binomials, 2^p-1, 3^p-1, 4^p-1, and 5^p-1

m	$\ell_{m,s+1}(z)$	$p_{m,s+1}(x)$
1	1	1
2	$3+s(1-2z)-s^2z$	$3+s(1+x)$
3	$6+s^4z^2-4s(-1+2z)+s^3z(-2+3z)$ $+s^2(1-7z+3z^2)$	$12+8s(1+x)+s^2(2+2x+x^2)$
4	$10-s^6z^3-10s(-1+2z)-s^5z^2(-3+4z)$ $+5s^2(1-5z+3z^2)-s^4z(3-13z+6z^2)$ $+s^3(1-14z+21z^2-4z^3)$	$60+60s(1+x)+15s^2(2+2x+x^2)$ $+s^3(6+6x+3x^2+x^3)$
5	$15+s^8z^4-20s(-1+2z)+s^7z^3(-4+5z)$ $+5s^2(3-13z+9z^2)+s^6z^2(6-21z+10z^2)$ $-3s^3(-2+18z-27z^2+8z^3)$ $+s^5z(-4+33z-44z^2+10z^3)$ $+s^4(1-23z+73z^2-46z^3+5z^4)$	$360+480s(1+x)+180s^2(2+2x+x^2)$ $+24s^3(6+6x+3x^2+x^3)$ $+s^4(24+24x+12x^2+4x^3+x^4)$

Table 9.4.2. Generating the p^{th} power sequences, $(s+1)^p-1$

Table 9.4: Convergent-based generating function identities for the binomial p^{th} power sequences generated by the examples in Section 6.7

n	$p_{n,0}(h)$	$p_{n,1}(h)$	$p_{n,2}(h)$
0	1	0	0
1	h	1	0
2	$h(h-1)^2$	$h(h-2)$	$h-1$
3	$h(h-1)^2(h-2)$	$h(h-1)(h-3)$	$h(h-3)$
4	$h(h-1)^2(h-2)^2(h-3)$	$h(h-1)(h-2)^2(h-4)$	$h(h-1)(h-3)(h-4)$
5	$h(h-1)^2(h-2)^2(h-3)(h-4)$	$h(h-1)(h-2)^2(h-3)(h-5)$	$h(h-1)(h-2)(h-4)(h-5)$
6	$h(h-1)^2(h-2)^2(h-3)^2(h-4)(h-5)$	$h(h-1)(h-2)^2(h-3)^2(h-4)(h-6)$	$h(h-1)(h-2)(h-3)^2(h-5)(h-6)$
7	$h(h-1)^2(h-2)^2(h-3)^2(h-4)(h-5)(h-6)$	$h(h-1)(h-2)^2(h-3)^2(h-4)(h-5)(h-7)$	$h(h-1)(h-2)(h-3)^2(h-4)(h-6)(h-7)$

n	$p_{n,3}(h)$	$p_{n,4}(h)$	$p_{n,5}(h)$
0	0	0	0
1	0	0	0
2	0	0	0
3	$h-1$	0	0
4	$h(h-2)(h-4)$	$(h-1)(h-2)$	0
5	$h(h-1)(h-2)(h-4)(h-5)$	$h(h-2)(h-5)$	$(h-1)(h-2)$
6	$h(h-1)(h-2)(h-4)(h-5)(h-6)$	$h(h-1)(h-3)(h-5)(h-6)$	$h(h-2)(h-3)(h-6)$
7	$h(h-1)(h-2)(h-3)(h-5)(h-6)(h-7)$	$h(h-1)(h-2)(h-5)(h-6)(h-7)$	$h(h-1)(h-3)(h-6)(h-7)$

n	$p_{n,6}(h)$	$p_{n,7}(h)$	$m_{n,h}$
0	0	0	1
1	0	0	$h-1$
2	0	0	$h-2$
3	0	0	$(h-2)(h-3)$
4	0	0	$(h-3)(h-4)$
5	0	0	$(h-3)(h-4)(h-5)$
6	$(h-1)(h-2)(h-3)$	0	$(h-4)(h-5)(h-6)$
7	$h(h-2)(h-3)(h-7)$	$(h-1)(h-2)(h-3)$	$(h-4)(h-5)(h-6)(h-7)$

Table 9.5.1. The auxiliary numerator subsequences, $C_{h,n}(\alpha, R) :\mapsto \frac{(-\alpha)^n m_{n,h}}{n!} \times \sum_{i=0}^{n} \binom{n}{i} p_{n,i}(h) \left(R/\alpha\right)_i$, expanded by the finite-degree polynomial sequence terms defined by the Stirling number sums in (51e) of Section 5.2.1.

n	m_h	$(-1)^n n! \cdot m_h^{-1} \cdot C_{h,n}(\alpha, R)$
1	1	$-(h-1)(R+h\alpha)$
2	$(h-2)$	$\alpha\left(2h^2-3h-1\right)R+\alpha^2(h-1)^2h+(h-1)R^2$
3	$(h-3)$	$3\alpha(h-2)\left(h^2-2h-1\right)R^2+\alpha^2(h-2)\left(3h^3-9h^2+2h-2\right)R$
		$+\alpha^3(h-2)^2(h-1)^2h+(h-2)(h-1)R^3$
4	$(h-3)(h-4)$	$\alpha^2\left(6h^4-36h^3+53h^2-9h+22\right)R^2+2\alpha^3\left(2h^5-15h^4+36h^3-36h^2+19h+6\right)R$
		$+\alpha^4(h-3)(h-2)^2(h-1)^2h+(h-2)(h-1)R^4+2\alpha(h-3)(h-2)(2h+1)R^3$
5	$(h-3)(h-4)(h-5)$	$5\alpha(h-2)\left(h^2-3h-2\right)R^4+5\alpha^2\left(2h^4-14h^3+23h^2-h+14\right)R^3$
		$+5\alpha^3\left(2h^5-18h^4+49h^3-49h^2+40h+20\right)R^2$
		$+\alpha^4\left(5h^6-55h^5+215h^4-395h^3+374h^2-72h+48\right)R$
		$+\alpha^5(h-4)(h-3)(h-2)^2(h-1)^2h+(h-2)(h-1)R^5$

n	m_h	$(-1)^n n! \cdot m_h^{-1} \cdot C_{h,n}(\alpha, R)$
1	1	$\alpha h^2+(R-\alpha)h-R$
2	$(h-2)$	$\alpha^2h^3+2\alpha h^2(R-\alpha)+h\left(\alpha^2+R^2-3\alpha R\right)-R(\alpha+R)$
3	$(h-3)$	$\alpha^3h^5+3\alpha^2h^4(R-2\alpha)+\alpha h^3\left(13\alpha^2+3R^2-15\alpha R\right)$
		$+h^2\left(-12\alpha^3+R^3-12\alpha R^2+20\alpha^2R\right)+h\left(4\alpha^3-3R^3+9\alpha R^2-6\alpha^2R\right)$
		$+2R(\alpha+R)(2\alpha+R)$
4	$(h-3)(h-4)$	$\alpha^4h^6+\alpha^3h^5(4R-9\alpha)+\alpha^2h^4\left(31\alpha^2+6R^2-30\alpha R\right)$
		$+\alpha h^3\left(-51\alpha^3+4R^3-36\alpha R^2+72\alpha^2R\right)$
		$+h^2\left(40\alpha^4+R^4-18\alpha R^3+53\alpha^2R^2-72\alpha^3R\right)$
		$+h\left(-12\alpha^4-3R^4+14\alpha R^3-9\alpha^2R^2+38\alpha^3R\right)+2R(\alpha+R)(2\alpha+R)(3\alpha+R)$
5	$(h-3)(h-4)(h-5)$	$\alpha^5h^7+\alpha^4h^6(5R-13\alpha)+\alpha^3h^5\left(67\alpha^2+10R^2-55\alpha R\right)$
		$+5\alpha^2h^4\left(-35\alpha^3+2R^3-18\alpha R^2+43\alpha^2R\right)$
		$+\alpha h^3\left(244\alpha^4+5R^4-70\alpha R^3+245\alpha^2R^2-395\alpha^3R\right)$
		$+h^2\left(-172\alpha^5+R^5-25\alpha R^4+115\alpha^2R^3-245\alpha^3R^2+374\alpha^4R\right)$
		$+h\left(48\alpha^5-3R^5+20\alpha R^4-5\alpha^2R^3+200\alpha^3R^2-72\alpha^4R\right)$
		$+2R(\alpha+R)(2\alpha+R)(3\alpha+R)(4\alpha+R)$

Table 9.5.2. Alternate factored forms of the convergent function subsequences, $C_{h,n}(\alpha, R) := [z^n]\,\mathrm{FP}_h(\alpha, R; z)$, gathered with respect to powers of R and h.

n	$n! \cdot C_{h,n}(\alpha, R)$
2	$\alpha^2h^4+2\alpha h^3(R-2\alpha)+h^2\left(5\alpha^2+R^2-7\alpha R\right)-h(3R-2\alpha)(R-\alpha)+2R(\alpha+R)$
3	$-\alpha^3h^6-3\alpha^2h^5(R-3\alpha)-\alpha h^4\left(31\alpha^2+3R^2-24\alpha R\right)+h^3\left(51\alpha^3-R^3+21\alpha R^2-65\alpha^2R\right)$
	$+h^2\left(-40\alpha^3+6R^3-45\alpha R^2+66\alpha^2R\right)-h(R-\alpha)\left(12\alpha^2+11R^2-10\alpha R\right)+6R(\alpha+R)(2\alpha+R)$
4	$\alpha^4h^8+4\alpha^3h^7(R-4\alpha)+2\alpha^2h^6\left(53\alpha^2+3R^2-29\alpha R\right)+2\alpha h^5\left(-188\alpha^3+2R^3-39\alpha R^2+165\alpha^2R\right)$
	$+h^4\left(769\alpha^4+R^4-46\alpha R^3+377\alpha^2R^2-936\alpha^3R\right)-2h^3\left(452\alpha^4+5R^4-94\alpha R^3+406\alpha^2R^2-703\alpha^3R\right)$
	$+h^2\left(564\alpha^4+35R^4-302\alpha R^3+721\alpha^2R^2-1118\alpha^3R\right)-2h(R-\alpha)\left(-72\alpha^3+25R^3-17\alpha R^2+114\alpha^2R\right)$
	$+24R(\alpha+R)(2\alpha+R)(3\alpha+R)$
5	$-\alpha^5h^{10}-5\alpha^4h^9(R-5\alpha)-5\alpha^3h^8\left(54\alpha^2+2R^2-23\alpha R\right)-10\alpha^2h^7\left(-165\alpha^3+R^3-21\alpha R^2+111\alpha^2R\right)$
	$-\alpha h^6\left(6273\alpha^4+5R^4-190\alpha R^3+1795\alpha^2R^2-5860\alpha^3R\right)$
	$+h^5\left(15345\alpha^5-R^5+85\alpha R^4-1425\alpha^2R^3+8015\alpha^3R^2-18519\alpha^4R\right)$
	$+5h^4\left(-4816\alpha^5+3R^5-111\alpha R^4+1055\alpha^2R^3-4011\alpha^3R^2+7205\alpha^4R\right)$
	$-5h^3\left(-4660\alpha^5+17R^5-339\alpha R^4+1947\alpha^2R^3-5703\alpha^3R^2+8438\alpha^4R\right)$
	$+h^2\left(-12576\alpha^5+225R^5-2200\alpha R^4+7975\alpha^2R^3-22900\alpha^3R^2+26400\alpha^4R\right)$
	$-2h(R-\alpha)\left(1440\alpha^4+137R^4+7\alpha R^3+1802\alpha^2R^2-1848\alpha^3R\right)+120R(\alpha+R)(2\alpha+R)(3\alpha+R)(4\alpha+R)$

Table 9.5: The auxiliary convergent function subsequences, $C_{h,n}(\alpha, R) := [z^n]\,\mathrm{FP}_h(\alpha, R; z)$, defined in Section 5.2.

k	$(-1)^{h-k} z^{-(h-k)} \cdot R_{h,h-k}(\alpha; z)$
1	1
2	$-\frac{1}{2}\alpha\left(h^2-h+2\right)z+h-1$
3	$\frac{1}{2}(h-2)(h-1)-\frac{1}{2}\alpha(h-2)\left(h^2+3\right)z+\frac{1}{24}\alpha^2\left(3h^4-10h^3+21h^2-14h+24\right)z^2$
4	$-\frac{1}{4}\alpha(h-3)(h-2)\left(h^2+h+4\right)z+\frac{1}{24}\alpha^2(h-3)\left(3h^4-4h^3+19h^2-2h+56\right)z^2$
	$-\frac{1}{48}\alpha^3\left(h^6-7h^5+23h^4-37h^3+48h^2-28h+48\right)z^3$
	$+\frac{1}{6}(h-3)(h-2)(h-1)$
5	$-\frac{1}{12}\alpha(h-4)(h-3)(h-2)\left(h^2+2h+5\right)z+\frac{1}{48}\alpha^2(h-4)(h-3)\left(3h^4+2h^3+23h^2+16h+100\right)z^2$
	$-\frac{1}{48}\alpha^3(h-4)\left(h^6-4h^5+14h^4-16h^3+61h^2-12h+180\right)z^3$
	$+\frac{\alpha^4}{5760}\left(15h^8-180h^7+950h^6-2688h^5+4775h^4-5340h^3+5780h^2-3312h+5760\right)z^4$
	$+\frac{1}{24}(h-4)(h-3)(h-2)(h-1)$

k	$k!(-1)^{h-k} z^{-(h-k)} \cdot R_{h,h-k}(\alpha; z)$
1	1
2	$-\alpha h^2 z+h(\alpha z+2)-2(\alpha z+1)$
3	$\frac{3}{4}\alpha^2h^4z^2-\frac{1}{2}\alpha h^3z(5\alpha z+6)+\frac{3}{4}h^2\left(7\alpha^2z^2+8\alpha z+4\right)$
	$+\frac{1}{2}h\left(-7\alpha^2z^2-18\alpha z-18\right)+6\left(\alpha^2z^2+3\alpha z+1\right)$
4	$-\frac{1}{2}\alpha^3h^6z^3+\frac{1}{2}\alpha^2h^5z^2(7\alpha z+6)-\frac{1}{2}\alpha h^4z\left(23\alpha^2z^2+26\alpha z+12\right)$
	$+\frac{1}{2}h^3\left(37\alpha^3z^3+62\alpha^2z^2+48\alpha z+8\right)$
	$+h^2\left(-24\alpha^3z^3-59\alpha^2z^2-30\alpha z-24\right)$
	$+2h\left(7\alpha^3z^3+31\alpha^2z^2+42\alpha z+22\right)-24\left(\alpha^3z^3+7\alpha^2z^2+6\alpha z+1\right)$
5	$\frac{5}{16}\alpha^4h^8z^4-\frac{5}{4}\alpha^3h^7z^3(3\alpha z+2)+\frac{5}{24}\alpha^2h^6z^2\left(95\alpha^2z^2+96\alpha z+36\right)$
	$-\frac{1}{2}\alpha h^5z\left(112\alpha^3z^3+150\alpha^2z^2+95\alpha z+20\right)$
	$+\frac{5}{48}h^4\left(955\alpha^4z^4+1728\alpha^3z^3+1080\alpha^2z^2+672\alpha z+48\right)$
	$-\frac{5}{4}h^3\left(89\alpha^4z^4+250\alpha^3z^3+242\alpha^2z^2+104\alpha z+40\right)$
	$+\frac{5}{12}h^2\left(289\alpha^4z^4+1536\alpha^3z^3+1584\alpha^2z^2+408\alpha z+420\right)$
	$+h\left(-69\alpha^4z^4-570\alpha^3z^3-1270\alpha^2z^2-820\alpha z-250\right)$
	$+120\left(\alpha^4z^4+15\alpha^3z^3+25\alpha^2z^2+10\alpha z+1\right)$

Table 9.6: The auxiliary convergent numerator function subsequences, $R_{h,k}(\alpha; z) := [R^k]\,\mathrm{FP}_h(\alpha, R; z)$, defined by Section 5.2.1.

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