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# Aperiodic Compositions and Classical Integer Sequences

Margherita Maria Ferrari and Norma Zagaglia Salvi*
Department of Mathematics
Politecnico di Milano
Milano, 20133
Italy
margheritamaria.ferrari@polimi.it
norma.zagaglia@polimi.it

**Abstract**

In this paper we define the notion of singular composition of a positive integer. We provide a characterization of these compositions, together with methods for decomposing and also extending a singular composition in terms of other singular compositions. Consecutive extensions of particular compositions determine sequences of integers which coincide with classical integer sequences involving Fibonacci and Lucas numbers.

## 1 Introduction

Let $k, n$ be integers where $1 \leq k \leq n$, and let $\alpha = (a_1, a_2, \ldots, a_k)$ denote a composition of $n$ into $k$ parts [3]. We call $\alpha$ $(h, i)$-*singular* if

$$(a_1, a_2, \ldots, a_i + a_{i+1}, \ldots, a_k) = (a_{1+h}, a_{2+h}, \ldots, a_{i+h} + a_{i+1+h}, \ldots, a_{k+h}), \tag{1}$$

where $1 \leq h \leq k-1$, $1 \leq i \leq k$ and the indices are modulo $k$. Note that shifting a $(h, i)$-singular composition of one position to the right, we obtain a $(h, i+1)$-singular composition. Consequently, the choice of a single index $i$ is sufficient for identifying such compositions.

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Thus we fix $i = 1$ and we call the composition $\alpha = (a_1, a_2, \ldots, a_k)$ *h-singular* if

$$(a_1 + a_2, a_3, \ldots, a_k) = (a_{1+h} + a_{2+h}, a_{3+h}, \ldots, a_{k+h}). \tag{2}$$

A $k$-composition of $n$ is *singular* when it is $h$-singular for a suitable value of $1 \leq h \leq k-1$.

**Example 1.** The 5-composition $(1, 2, 2, 1, 2)$ of $n = 8$ is 2-singular.

Kramer [2] used singular compositions in order to define the middle levels partition graph of $n$.

The *concatenation* of the compositions $\alpha = (a_1, a_2, \ldots, a_k)$ and $\beta = (b_1, b_2, \ldots, b_h)$ of the positive integers $n$ and $m$ respectively is the composition $\alpha\beta = (a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_h)$ of $n + m$. We let $\alpha^i$ denote the concatenation of $\alpha$ with itself $i$ times. A composition $\alpha$ is *periodic* if $\alpha = \pi^j$, where $1 < j \leq k$ and $\pi$ is a suitable composition.

Fibonacci and Lucas numbers will appear in some of our results. Recall that the Fibonacci sequence $(F_n)_{n\geq 0}$ is defined by setting $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$. The Lucas sequence $(L_n)_{n\geq 0}$ is defined by setting $L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$, for $n \geq 2$.

The paper is outlined as follows. In Section 2 we determine a characterization of aperiodic singular compositions which allows us to obtain a method for constructing such compositions (Theorem 11). In Section 3 we study decompositions (Theorem 14) and also extensions (Theorem 18) of a singular composition in terms of other singular compositions. In Section 4 we prove that consecutive extensions of particular compositions determine sequences of integers which coincide with classical sequences involving Fibonacci and Lucas numbers. We conclude the paper by posing a more general definition of singular composition together with an open problem.

## 2 A characterization of singular compositions

Let $\alpha$ be an $h$-singular $k$-composition of $n$; from (2) it follows that $(a_3, \ldots, a_k) = (a_{3+h}, \ldots, a_{k+h})$.

This equality determines the function $f_h : \{3, 4, \ldots, k\} \to \{3+h, 4+h, \ldots, k+h\}$ on the indices of the elements of previous sequences such that

$$f_h(i) = i + h,$$

where $3 \leq i \leq k$ and the integers are modulo $k$. We may represent $f_h$ in two-line notation

$$\begin{pmatrix} 3 & 4 & \cdots & k \\ 3+h & 4+h & \cdots & k+h \end{pmatrix}. \tag{3}$$

Note that the second line is obtained by shifting of $h$ positions to the left the elements of the sequence $(1, 2, 3, 4, \ldots, k)$ and ignoring the first two elements $1+h$ and $2+h$. A consequence is that the elements 1 and 2, which do not belong to the first line, belong to the second one, except for $h = 1$ and $h = k - 1$. Indeed for $h = 1$ the second line contains 1, but not 2; for $h = k - 1$ the second line contains 2, but not 1. In a similar way, the elements $1 + h$ and $2 + h$ do not belong to the second line while they belong to the first one, except for $h = 1$ and $h = k - 1$.

**Proposition 2.** *An $h$-singular $k$-composition of $n$, where $h$ and $k$ are not coprime, is periodic.*

*Proof.* Let $\gcd(k, h) = t > 1$, where $h = th', k = tk'$ and $\gcd(k', h') = 1$. Note that the sets $H_i = \{i, i+h, \ldots, i+(k'-1)h\}$, $1 \leq i \leq t$, determine a partition of the set $[k]$. Then, for an $h$-singular $k$-composition $\alpha = (a_1, a_2, \ldots, a_k)$, the elements of the sets $\{a_i, a_{i+h}, \ldots, a_{i+(k'-1)h}\}$, $1 \leq i \leq t$, coincide and $\alpha$ turns out to be the concatenation $(a_1, a_2, \ldots, a_t)^{k'}$. □

Throughout the paper we consider only aperiodic compositions.

Beggas et al. [1] proved that a particular bijection, called widened permutation, between two $n$-sets having $n - 1$ elements in common has a decomposition into a linear order and a possible permutation. In this case we have a similar function in which the two sets have $n - 2$ elements in common, but for $h = 1$ and $h = k - 1$.

**Lemma 3.** *Let $h, k$ be coprime integers, where $1 \leq h \leq k - 1$. The function $f_h$ does not contain cycles.*

*Proof.* By way of contradiction we assume there is a cycle

$$C = (d, d + h, \ldots, d + (r - 1)h),$$

where $1 \leq d \leq k$ and $d + rh \equiv d \pmod{k}$. This means that $rh \equiv 0 \pmod{k}$ and therefore $k$ divides $rh$. Then, because $\gcd(k, h) = 1$, $k$ divides $r$. The unique possibility is $r = k$; so the cycle contains all the elements. But this implies the impossible condition that also every line of (3) contains all the elements. □

**Theorem 4.** *Let $h, k$ be coprime integers, where $1 \leq h \leq k - 1$. The function $f_h$ is decomposed into the linear orders:*

1. $$E_h = (1 + h, 1 + 2h, \ldots, 1 + rh) \tag{4}$$

   *and*

   $$F_h = (2 + h, 2 + 2h, \ldots, 2 + sh), \tag{5}$$

   *where $r = h^{-1}$, $s = (k - 1)h^{-1}$ in $\mathbb{Z}_k$, for $h \neq 1, k - 1$;*

2. $E_1 = (2)$ *and* $F_1 = (3, 4, \ldots, k, 1)$, *for $h = 1$;*

3. $E_{k-1} = (k, k - 1, \ldots, 2)$ *and* $F_{k-1} = (1)$, *for $h = k - 1$.*

*Proof.* Let $h \neq 1, k-1$. Starting from $1+h$ we obtain the sequence $(1+h, 1+2h, \ldots, 1+rh = a)$, where $a$ is one of the two elements which are in the second but not in the first line. So we have either $a = 1$ or $a = 2$. If $a = 1$, we obtain the impossible relation $rh \equiv 0 \pmod{k}$. If $a = 2$, we obtain $rh \equiv 1 \pmod{k}$, which is satisfied for $r = h^{-1}$ in $\mathbb{Z}_k$. Now starting from $2+h$ we obtain the sequence $(2+h, 2+2h, \ldots, 2+sh = b)$, where either $b = 1$ or $b = 2$. The unique possibility is $b = 1$, which holds for $s = (k-1)h^{-1}$ in $\mathbb{Z}_k$. By Lemma 3 the function does not contain cycles; therefore it is decomposed into the previous linear orders.

Now let $h = 1$. The function $f_1$ is decomposed into $F_1 = (3, 4, \ldots, k, 1)$ and $E_1 = (2)$. In the case $h = k-1$, $f_{k-1}$ is decomposed into $E_{k-1} = (k, k-1, \ldots, 2)$ and $F_{k-1} = (1)$. This completes the proof of the theorem. □

In the following we let $E_h$ and $F_h$ also denote the sets of the elements of the assigned linear orders.

**Corollary 5.** *For every* $1 \leq h \leq k-1$ *such that* $\gcd(k, h) = 1$, $E_h \cup F_h = [k]$ *and, for* $k > 2$, $| E_h | \neq | F_h |$.

*Proof.* If $k - h^{-1} = h^{-1}$, then $k = 2h^{-1}$ and $kh = 2$ in $\mathbb{Z}_k$. This implies that $2 \equiv 0 \pmod{k}$, a contradiction for $k > 2$. □

**Lemma 6.** *Let* $h_1, h_2$ *be two integers such that* $1 \leq h_1 < h_2 \leq k-1$ *and* $\gcd(k, h_1) = \gcd(k, h_2) = 1$. *Then* $E_{h_1} \neq E_{h_2}$ *and* $F_{h_1} \neq F_{h_2}$.

*Proof.* The cardinalities of $E_{h_1}$ and $E_{h_2}$ coincide with $h_1^{-1}$ and $h_2^{-1}$ in $\mathbb{Z}_k$ respectively. Because $h_1 < h_2$, their inverses are distinct; then also the sets $E_{h_1}$ and $E_{h_2}$ are distinct. The same argument applies for $F_{h_1}$ and $F_{h_2}$. □

The following result is straightforward.

**Corollary 7.** *If* $k$ *is a prime integer, then all the sets* $E_h$ *(respectively* $F_h$*),* $1 \leq h \leq k-1$, *are distinct.*

Note that when $k$ and $h$ are coprime, then also $k$ and $k-h$ are coprime. In the following result we establish a relation between $E_{k-h}$ (respectively $F_{k-h}$) and $F_h$ (respectively $E_h$).

**Proposition 8.** *For every* $1 \leq h \leq \lfloor \frac{k}{2} \rfloor$ *such that* $\gcd(k, h) = 1$, $E_{k-h} = (F_h \setminus \{1\}) \cup \{2\}$ *and* $F_{k-h} = (E_h \setminus \{2\}) \cup \{1\}$.

*Proof.* The result is easy to prove for $h = 1$. Let $h' = k - h$. Since $\gcd(k, h) = 1$, then $\gcd(k, h') = 1$, $E_{h'} = \{1 + h', 1 + 2h', \ldots, 1 + (r'-1)h', 2\}$ and $F_{h'} = \{2 + h', 2 + 2h', \ldots, 2 + (s'-1)h', 1\}$, where $r' = (h')^{-1}$ and $s' = k - (h')^{-1}$ in $\mathbb{Z}_k$.
Let $s$ denote $k - h^{-1}$ in $\mathbb{Z}_k$; it follows that

$$1 + k - h \equiv 2 + (s-1)h \pmod{k}.$$

Then $1+2(k-h) \equiv 2+(s-2)h$ and so on until $1+(s-1)(k-h) \equiv 2+h$ and $1+s(k-h) \equiv 2 \pmod{k}$. Thus $E_{k-h} = \{2 + (s-1)h, 2 + (s-2)h, \ldots, 2 + h, 2\} = (F_h \setminus \{1\}) \cup \{2\}$.

Moreover, $2 + k - h \equiv 1 + (r-1)h \pmod{k}$, where $r = h^{-1}$ in $\mathbb{Z}_k$; thus $F_{k-h} = \{1 + (r-1)h, 1 + (r-2)h, \ldots, 1 + h, 1\} = (E_h \setminus \{2\}) \cup \{1\}$. □

**Corollary 9.** *If $\alpha = (a_1, a_2, \ldots, a_k)$ is an aperiodic $h$-singular $k$-composition of $n$, then every $a_i$ is equal to $a_1$ or $a_2$, $1 \le i \le k$, as long as $i \in F_h$ or $i \in E_h$ respectively. Then $a_1$ and $a_2$ are distinct, and they satisfy the relation*

$$(k - h^{-1})a_1 + h^{-1}a_2 = n. \tag{6}$$

**Corollary 10.** *If an aperiodic composition contains more than two distinct elements, then it is not singular.*

Previous results allow us to give a characterization of singular compositions, which turns out to be a method for their construction.

**Theorem 11.** *Let $h, k, n$ be positive integers such that $1 \le h < k \le n$ and $\gcd(k, h) = 1$. An aperiodic $k$-composition $\alpha = (a_1, a_2, \ldots, a_k)$ is $h$-singular if and only if $a_1 \ne a_2$ and the pair of elements $(a_1, a_2)$ is a solution of the equation*

$$(k - h^{-1})x_1 + h^{-1}x_2 = n, \tag{7}$$

*where $h^{-1}, k - h^{-1} \in \mathbb{Z}_k$, and each $a_i$ coincides with $a_1$ or $a_2$ for $i \in F_h$ or $i \in E_h$ respectively.*

*Proof.* If $\alpha$ is $h$-singular, then by Corollary 9 the property holds.
Now let us assume that the pair of distinct integers $(a_1, a_2)$ is solution of the equation (7) and each $a_i$ coincides with $a_1$ or $a_2$ for $i \in F_h$ or $i \in E_h$ respectively. Hence, for $h \ne 1, k - 1$, the composition $\alpha = (a_1, a_2, \ldots, a_k)$ which has the elements $a_1$ and $a_2$ in the positions given by (5) and (4) respectively, is $h$-singular. Lastly, if $h = 1$, $\alpha = (a_1, a_2, a_1, \ldots, a_1)$ is 1-singular, while if $h = k - 1$, $\alpha = (a_1, a_2, \ldots, a_2)$ is $(k - 1)$-singular. □

**Example 12.** The list of $h$-singular 9-compositions with $a_1 = 1$ and $a_2 = 2$ is

1. for $h = 1$, $\alpha_1 = (1, 2, 1, 1, 1, 1, 1, 1, 1)$;
2. for $h = 2$, $\alpha_2 = (1, 2, 2, 1, 2, 1, 2, 1, 2)$;
3. for $h = 4$, $\alpha_4 = (1, 2, 2, 2, 2, 1, 2, 2, 2)$;
4. for $h = 5$, $\alpha_5 = (1, 2, 1, 1, 1, 2, 1, 1, 1)$;
5. for $h = 7$, $\alpha_7 = (1, 2, 1, 2, 1, 2, 1, 2, 1)$;
6. for $h = 8$, $\alpha_8 = (1, 2, 2, 2, 2, 2, 2, 2, 2)$

where the corresponding integers are $n_1 = 10$, $n_2 = 14$, $n_4 = 16$, $n_5 = 11$, $n_7 = 13$ and $n_8 = 17$. Note that the compositions $\alpha_5$, $\alpha_7$ and $\alpha_8$ are obtained from $\alpha_4$, $\alpha_2$ and $\alpha_1$ respectively, by exchanging 1 with 2 after the first two positions.

Let $\alpha = (a_1, a_2, \ldots, a_k)$ be an $h$-singular composition. By Proposition 8, it follows that by exchanging $a_1$ and $a_2$ after the first two positions, we obtain a $(k-h)$-singular composition. We now prove that by exchanging only the first two elements we obtain again a $(k-h)$-singular composition.

**Proposition 13.** *Let $\alpha = (a_1, a_2, \ldots, a_k)$ be an aperiodic $h$-singular composition of $n$, where $1 \leq h \leq k-1$. Then $\alpha^* = (a_2, a_1, a_3, \ldots, a_k)$ is a $(k-h)$-singular composition of $n$, obtained from $\alpha$ by rotation.*

*Proof.* Consider the composition $\alpha^* = (a_1^*, a_2^*, \ldots, a_k^*) = (a_2, a_1, a_3, \ldots, a_k)$ of $n$. The set $E^*$ of indices of the elements equal to $a_2^*$ in $\alpha^*$ satisfies $E^* = (F_h \setminus \{1\}) \cup \{2\} = E_{k-h}$ (Proposition 8). The same relation holds for $F^* = F_{k-h}$, where $F^*$ is the set of indices of the elements equal to $a_1^*$ in $\alpha^*$. Then $\alpha^*$ is a $(k-h)$-singular composition of $n$. Note that the composition $\alpha' = (a_{1+h}, a_{2+h}, \ldots, a_k, a_1, \ldots, a_h)$ is $(k-h)$-singular and is obtained from $\alpha$ by rotation. Moreover $a_2 = a_{1+h}$ and $a_1 = a_{2+h}$. Since the first two elements of $\alpha^*$ coincide with the first two of $\alpha'$ and both the compositions are $(k-h)$-singular, $E^* = E'$ and $F^* = F'$. Thus $\alpha^* = \alpha'$, and the result follows. □

# 3 Decompositions and extensions

In this section we investigate two decompositions and some extensions of an aperiodic singular composition.

**Theorem 14.** *Let $\alpha = (a_1, a_2, \ldots, a_k)$ be an aperiodic $h$-singular $k$-composition of $n$, where $k = hq + r$ and $1 \leq r < h$. Then $\alpha = \lambda\mu\lambda\cdots\lambda$, where $\lambda = (a_1, a_2, \ldots, a_h)$, $\mu$ is the sequence of the last $r$ elements of $\lambda$ and $q$ is the multiplicity of $\lambda$. Moreover $\lambda$ is a $(h-r)$-singular $h$-composition of $a_1 + \cdots + a_h$.*

*Proof.* Since $\alpha$ is $h$-singular, the sequences $\beta = (a_1 + a_2, a_3, \ldots, a_k)$ and $\gamma = (a_{1+h} + a_{2+h}, a_{3+h}, \ldots, a_{k+h})$ coincide. In particular this holds for the subsequences $\beta'$ and $\gamma'$ obtained by deleting the first $h-1$ elements of $\beta$ and $\gamma$ respectively. If $1 \leq h \leq \lfloor \frac{k}{2} \rfloor$, by comparing $\beta' = (a_{1+h}, a_{2+h}, \ldots, a_k)$ and $\gamma' = (a_{1+2h}, a_{2+2h}, \ldots, a_k, a_1, \ldots, a_h) = (a_{1+2h}, \ldots, a_k)\lambda$, where $\lambda = (a_1, a_2, \ldots, a_h)$, we obtain that the sequence $(a_{k-(h-1)}, \ldots, a_k)$ formed by the last $h$ elements of $\beta'$ coincides with $\lambda$. Then the sequence of length $h$ in $\gamma'$ which precedes the last subsequence $\lambda$ coincides again with $\lambda$. We continue until we find a subsequence $\mu$ of length less than $h$ in $\beta'$, which is formed by the last $r$ elements of $\lambda$. Thus $\mu = (a_{h-(r-1)}, a_{h-(r-2)}, \ldots, a_h)$. If $\lfloor \frac{k}{2} \rfloor < h \leq k-1$, by comparing $\beta'$ and $\gamma' = \mu$ we obtain $\alpha = \lambda\mu$. In both cases $\alpha = \lambda\mu\lambda\cdots\lambda$, where $\lambda$ occurs $q$ times.

Let us assume that $r > 1$. Since $\alpha$ is $h$-singular, the sequence

$$(a_1 + a_2, a_3, \ldots, a_h, a_{h-(r-1)}, a_{h-(r-2)}, \ldots, a_h)\lambda^{q-1}$$

coincides with

$$(a_{h-(r-1)} + a_{h-(r-2)}, a_{h-(r-3)}, \ldots, a_h)\lambda^q.$$

Therefore the sequences of the first $h-1$ elements coincide

$$(a_1+a_2, a_3, \dots, a_h) = (a_{h-(r-1)} + a_{h-(r-2)}, a_{h-(r-3)}, \dots, a_h, a_1, \dots, a_{h-r}).$$

Thus the composition $\lambda$ is $(h-r)$-singular. A similar argument applies in the case $r=1$. □

**Proposition 15.** *Let $\alpha = (a_1, a_2, \dots, a_k)$ be an aperiodic $h$-singular $k$-composition of $n$, where $k = hq + r$ and $1 < r < h$. Then $\alpha = \sigma\lambda\cdots\lambda$, where $\lambda = (a_1, a_2, \dots, a_h)$, $\sigma = (a_1, a_2, \dots, a_r)$ and the multiplicity of $\lambda$ is $q$. Moreover $\lambda$ is a $(h-r)$-singular $h$-composition of $a_1 + \cdots + a_h$.*

*Proof.* Let $\lambda = (a_1, a_2, \dots, a_h)$ and $\sigma = (a_1, a_2, \dots, a_r)$. By applying the same argument used in the proof of Theorem 14 to the subsequences obtained by deleting the first $r-1$ elements of $\beta$ and $\gamma$, the result follows. □

**Corollary 16.** *In the case of $r = 1$, there is not a decomposition $\alpha = \sigma\lambda\lambda\cdots\lambda$.*

*Proof.* In the case of $r = 1$, $\sigma$ is reduced to the element $a_1$. This implies the relation $a_1 + a_2 = 2a_1$; then $a_2 = a_1$, a contradiction to the assumption that $\alpha$ is aperiodic. □

**Corollary 17.** *If $k = hq + r$ and $1 < r < h$, then $\sigma\lambda = \lambda\mu$.*

Now we investigate an operation which can be considered the inverse of the decomposition; namely we want to determine an extension of a singular composition which turns out to be again a singular composition.

**Theorem 18.** *Let $\alpha$ be an aperiodic $h$-singular $k$-composition of $n$, and let $\nu$ denote the sequence formed by the last $k-h$ elements of $\alpha$. The $k'$-composition $\beta = \alpha\nu\alpha\cdots\alpha$, where $k' = kq' + k - h$ and $q'$ is the multiplicity of $\alpha$, is $k$-singular.*

*Proof.* Let $\alpha = (a_1, a_2, \dots, a_k)$ be an aperiodic $h$-singular $k$-composition of $n$, where $k > 2$ and $1 \le h < k-1$. The composition $\beta = \alpha\nu\alpha\cdots\alpha$, where $\nu$ denotes the sequence formed by the last $k-h$ elements of $\alpha$, is $k$-singular if

$$(a_1+a_2, \dots, a_k, a_{1+h}, \dots, a_k)\alpha^{q'-1} = (a_{1+h} + a_{2+h}, \dots, a_k)\alpha^{q'}.$$

In order to prove the equality, it is sufficient to show that

$$(a_1+a_2, a_3, \dots, a_k, a_{1+h}, \dots, a_k) = (a_{1+h} + a_{2+h}, \dots, a_k, a_1, \dots, a_k). \tag{8}$$

Since $\alpha$ is $h$-singular, $(a_1+a_2, a_3, \dots, a_k) = (a_{1+h}+a_{2+h}, \dots, a_k, a_1, \dots, a_h)$. Thus the left side of (8) coincides with $(a_{1+h}+a_{2+h}, \dots, a_k, a_1, \dots, a_h, a_{1+h}, \dots, a_k)$ and the result follows. A similar argument applies in the cases $k=2$ and $h = k-1$. □

# 4 Classical integer sequences

Let $\alpha$ be an $h$-singular $k$-composition of $n$. The composition $\beta = \alpha\nu\alpha\cdots\alpha$, where $\nu$ is the sequence formed by the last $k-h$ elements of $\alpha$ and $\alpha$ is repeated $q$ times, is called a *$q$-extension* of $\alpha$. By consecutive extensions, we determine a sequence of singular compositions and therefore a sequence of integers corresponding to the numbers of their parts.

## 4.1 Fibonacci sequences

Let us consider the $h_0$-singular $k_0$-composition $\alpha_0 = (a, b)$, with $a \neq b$, $k_0 = 2$ and $h_0 = 1$. The 2-extension of $\alpha_0$ is the $h_1$-singular $k_1$-composition $\alpha_1 = \alpha_0\nu_0\alpha_0 = (a, b, b, a, b)$, where $k_1 = k_0 \cdot 2 + 1$, $h_1 = k_0 = 2$ and $\nu_0$ is the composition formed by last $(k_0 - h_0) = 1$ element of $\alpha_0$. The consecutive 2-extension is the $h_2$-singular $k_2$-composition $\alpha_2 = \alpha_1\nu_1\alpha_1 = (a, b, b, a, b, b, a, b, a, b, b, a, b)$, where $k_2 = k_1 \cdot 2 + 3$, $h_2 = k_1$ and $\nu_1$ is the composition formed by last $(k_1 - h_1) = 3$ elements of $\alpha_1$ and so on.

The first values of the sequence of the numbers $(k_n)_{n\geq 0}$ of parts of the 2-extensions of $\alpha_0$ are

$$2, 5, 13, 34, 89, 233, \ldots$$

These numbers appear as the first integers, but the first two, in the sequence A001519 [4], which is obtained from the recursive relation

$$a_n = 3a_{n-1} - a_{n-2}, \tag{9}$$

with the initial conditions $a_0 = 1$, $a_1 = 1$. We prove that the integers $k_n$ satisfy the same recursive relation.

**Lemma 19.** *The integers $k_n$ of the parts of the 2-extensions of the 1-singular 2-composition $(a, b)$, with $a \neq b$, satisfy the recursive relation:*

$$k_n = 3k_{n-1} - k_{n-2}$$

*with the initial conditions $k_0 = 2$, $k_1 = 5$.*

*Proof.* Recall that, by Theorem 18,

$$k_n = 2k_{n-1} + k_{n-1} - h_{n-1}.$$

Because $h_{n-1} = k_{n-2}$, the result follows. □

The following corollary is straightforward.

**Corollary 20.** *The integers $h_n$ associated to the 2-extensions of the 1-singular 2-composition $(a, b)$, with $a \neq b$, satisfy the recursive relation:*

$$h_n = 3h_{n-1} - h_{n-2}$$

*with the initial conditions $h_0 = 1$, $h_1 = 2$.*

It is easy to prove that the generating function of the sequence of the integers $k_n$ is

$$\frac{2-x}{1-3x+x^2},$$

and

$$k_n = \frac{2+\sqrt{5}}{\sqrt{5}}(\frac{3+\sqrt{5}}{2})^n + \frac{-2+\sqrt{5}}{\sqrt{5}}(\frac{3-\sqrt{5}}{2})^n.$$

**Proposition 21.** *The sequence*

$$k_0, k_1-h_1, k_1, k_2-h_2, k_2, k_3-h_3, \ldots \tag{10}$$

*coincides with the sequence of Fibonacci numbers $F_n$, with initial conditions $F_2 = 2, F_3 = 3$.*

*Proof.* We have to prove that every element of (10) is the sum of the preceding two elements and the initial conditions coincide. For $i \geq 1$, $k_i = k_i - h_i + k_{i-1}$, because $h_i = k_{i-1}$. Moreover, for $i \geq 2$, $k_i - h_i = k_{i-1} + k_{i-1} - h_{i-1}$ by Lemma 19. Because $k_0 = 2$, $k_1 = 5$ and $h_1 = 2$, the initial conditions are 2 and 3, which coincide with $F_2$ and $F_3$ of the Fibonacci sequence A000045. □

Another consequence of Proposition 21 is that the elements $k_i$, $i \geq 0$, form a bisection of the Fibonacci sequence; this result turns out to be one of the comments to A001519.

By repeating the previous procedure for $q > 2$, we easily obtain a sequence satisfying the recursive relation

$$a_n = (q+1)a_{n-1} - a_{n-2},$$

with the initial conditions $a_0 = 2$, $a_1 = 2q+1$.

In the particular case of $q = 3$, we obtain the sequence whose first elements are

$$2, 7, 26, 97, \ldots$$

which coincides with A001075, but the first element.

Again, for $q = 4$ we obtain a sequence whose first elements are

$$2, 9, 43, 206, \ldots$$

which coincides with A002310, but the first element.

## 4.2 Lucas sequences

The first values of the sequence of the numbers $(p_n)_{n\geq 0}$ of parts of the 2-extensions of the 2-singular 3-composition $(a, b, b)$, with $a \neq b$, are

$$3, 7, 18, 47, 123, \ldots$$

These integers coincide with the first integers, but the first one, of A005248, which is obtained from the recursive relation (9), with the initial conditions $a_0 = 2$, $a_1 = 3$.

Using the same procedure of Lemma 19, the numbers $p_n$ satisfy the same recursive relation with initial conditions $p_0 = 3$ and $p_1 = 7$. Moreover the generating function of the sequence of the integers $p_n$ is

$$\frac{3-2x}{1-3x+x^2},$$

and

$$p_n = (\frac{3+\sqrt{5}}{2})^{n+1} + (\frac{3-\sqrt{5}}{2})^{n+1}.$$

**Proposition 22.** *The sequence*

$$h_0, p_0 - h_0, p_0, p_1 - h_1, p_1, p_2 - h_2, p_2, p_3 - h_3, \ldots \tag{11}$$

*coincides with the sequence of Lucas numbers $L_n$, with initial conditions $L_0 = 2, L_1 = 1$.*

Another consequence of the previous result is that the elements $p_i$, $i \geq 0$, form a bisection of the Lucas sequence A000032, as noted in a comment to A005248.

### 4.3 Other integer sequences

We now consider the sequence of the numbers $(t_n)_{n\geq 0}$ of parts of 2-extensions of the 3-singular 4-compositions $(a, b, b, b)$, with $a \neq b$, that is

$$4, 9, 23, 60, 157, \ldots$$

This sequence, which is not contained in [4], satisfies the recursive relation (9), with initial conditions $t_0 = 4$ and $t_1 = 9$. The corresponding generating function is

$$\frac{4-3x}{1-3x+x^2},$$

and

$$t_n = \frac{3+2\sqrt{5}}{\sqrt{5}}(\frac{3+\sqrt{5}}{2})^n + \frac{-3+2\sqrt{5}}{\sqrt{5}}(\frac{3-\sqrt{5}}{2})^n.$$

By continuing, we may obtain other integer sequences by $q$-extension, with $q \geq 2$, of the singular composition $(a, b, \ldots, b)$, where $b$ occurs more than three times.

## 5 Conclusion

The notion of singular composition can be generalized as follows. We call the composition $\alpha = (a_1, a_2, \ldots, a_k)$ $(h, i, j)$-*singular*, if

$$\begin{aligned}&(a_1, a_2, \ldots, a_{i-1}, a_i + a_j, a_{i+1}, \ldots, a_{j-1}, a_{j+1}, \ldots, a_k) = \\ &= (a_{1+h}, a_{2+h}, \ldots, a_{i-1+h}, a_{i+h} + a_{j+h}, a_{i+1+h}, \ldots, a_{j-1+h}, a_{j+1+h}, \ldots, a_{k+h}),\end{aligned} \tag{12}$$

where $1 \leq h \leq k-1$, $1 \leq i < j \leq k$ and the indices are modulo $k$.

This definition leads to compositions which can not be obtained from equation (1). In fact, $(1,1,2,2,2)$ satisfies $(a_1 + a_3, a_2, a_4, a_5) = (a_{1+h} + a_{3+h}, a_{2+h}, a_{4+h}, a_{5+h})$ for $h = 4$, but it does not satisfy any equation (1).

Thus this definition poses the problem to find necessary and sufficient conditions based on which a given aperiodic sequence with two distinct elements satisfies (12).

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