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Integer Sequences Connected with Extensions of the Bell Polynomials

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Abstract

The *Encyclopedia of Integer Sequences* includes some sequences that are connected with the Bell numbers and that have a particular combinatorial meaning. In this article, we find a general meaning for framing sequences, including the above mentioned ones. Furthermore, by using Laguerre-type derivatives, we derive the Laguerre-type Bell numbers of higher order, showing, as a by-product, that it is possible to construct new integer sequences which are not included in the *Encyclopedia*.

1 Introduction

The Bell polynomials [3] are a mathematical tool for representing the nth derivative of a composite function. They are strictly related to partitions [1, 2, 24]. They are also applied in many different frameworks such as the Blissard problem [24, p. 46], the representation of Lucas polynomials of the first and second kind [10], the construction of representation formulas for the Newton sum rules for the zeros of polynomials [16, 17], the recurrence relations for a class of Freud-type polynomials [6], the representation formulas for the symmetric functions of a countable set of numbers generalizing the classical algebraic Newton-Girard formulas [8]. In particular, Cassisa and Ricci [8] found reduction formulas for the *orthogonal invariants* of a strictly positive compact operator, obtaining, as a by-product, the so-called Robert formulas [25].

Recently Kataria and Vellaisamy [14, 15] obtained connections of Bell polynomials with another important class of polynomials known as Adomian polynomials.

Some generalized forms of Bell polynomials can be found in the literature, (see e.g., [7, 12, 18, 19, 21, 22, 23]). For instance, some papers [7, 22] are concerned with the multi-dimensional case.

The authors [18] introduced the higher order Bell polynomials and their main properties and they [21] used some recursion formulas to compute, by means of the program Mathematica©, the rth order complete Bell polynomials $B_n^{[r]}$ (for $r = 2, 3, 4, 5$) and the relevant Bell numbers $b_n^{[r]}$.

In this article, after briefly recalling this theory, we show how to merge a family of integer sequences, already known in the *Encyclopedia of Integer Sequences* [28], into a general framework.

Furthermore, we construct new sequences of integer numbers, not included in the *Encyclopedia*, related to the Laguerre-type Bell polynomials. To this aim, we recall, in Section 6, a short introduction to the Laguerre derivative.

2 Recalling the Bell polynomials

By considering the composite function $\Phi(t) := f(g(t))$ of functions $x = g(t)$ and $y = f(x)$, defined in suitable intervals of the real axis and n times differentiable with respect to the relevant independent variables and by using the notation

$$\Phi_h := D_t^h \Phi(t), \qquad f_h := D_x^h f(x)|_{x=g(t)}, \qquad g_h := D_t^h g(t)\,, \tag{1}$$

and

$$([f,g]_n) := (f_1, g_1; f_2, g_2; \ldots ; f_n, g_n)\,, \tag{2}$$

the Bell polynomials are defined as follows:

$$Y_n\left([f,g]_n\right) := \Phi_n. \tag{3}$$

Inductively, by letting

$$[g]_n := (g_1, g_2, \ldots , g_n),$$

we can write

$$Y_n\left([f,g]_n\right)=\sum_{k=1}^{n}A_{n,k}([g]_n)f_k, \tag{4}$$

where the coefficient $A_{n,k}$, for any $k=1,\dots,n$, is a polynomial in $g_1,g_2,\dots,g_n$, homogeneous of degree k and *isobaric* of weight n (i.e., it is a linear combination of monomials $g_1^{k_1}g_2^{k_2}\cdots g_n^{k_n}$ whose weight is constantly given by $k_1+2k_2+\ldots+nk_n=n$).

Proposition 1. *The Bell polynomials satisfy the recurrence relation*

$$\begin{cases} Y_0\left([f,g]_0\right):=f_1 \\ Y_{n+1}\left([f,g]_{n+1}\right)=\displaystyle\sum_{k=0}^{n}\binom{n}{k}Y_{n-k}\left([f_1,g]_{n-k}\right)g_{k+1}, \end{cases} \tag{5}$$

where

$$\left([f_1,g]_{n-k}\right):=(f_2,g_1;f_3,g_2;\dots;f_{n-k+1},g_{n-k}).$$

The Faà di Bruno formula [11, 26, 27] gives an explicit expression for the Bell polynomials, however, as this formula makes use of partitions, it is not useful in practice for computing higher order Bell polynomials, whereas this can be done in a easy way by means of the following recursion formula for the coefficients $A_{n,k}$ in equation (4) which are known as partial Bell polynomials.

Theorem 2. *We have, for all integers n,*

$$A_{n+1,1}=g_{n+1},\qquad A_{n+1,n+1}=g_1^{n+1}. \tag{6}$$

Furthermore, for all $k=1,2,\dots,n-1$, the $A_{n,k}$ coefficients can be computed by the recurrence relation

$$A_{n+1,k+1}([g]_{n+1})=\sum_{h=0}^{n-k}\binom{n}{h}A_{n-h,k}([g]_{n-h})g_{h+1}. \tag{7}$$

Definition 3. The complete Bell polynomials are defined by

$$B_n([g]_n)=Y_n(1,g_1;1,g_2;\dots 1,g_n)=\sum_{k=1}^{n}A_{n,k}([g]_n), \tag{8}$$

and the Bell numbers by

$$b_n=Y_n(1,1;1,1;\dots;1,1)=\sum_{k=1}^{n}A_{n,k}(1,1,\dots,1). \tag{9}$$

3 Bell polynomials of order r

Bernardini et al. [7] introduced the multi-dimensional Bell polynomials of higher order. Here, we briefly recall this extension of the classical Bell polynomials only in the one-dimensional case.

Consider $\Phi(t) := f(\varphi^{(1)}(\varphi^{(2)}(\cdots(\varphi^{(r)}(t)))))$, i.e., the composition of functions $x^{(r)} = \varphi^{(r)}(t)$, $\ldots$, $x^{(2)} = \varphi^{(2)}\left(x^{(3)}\right)$, $x^{(1)} = \varphi^{(1)}\left(x^{(2)}\right)$, $y = f\left(x^{(1)}\right)$ defined in suitable intervals of the real axis, and suppose that the functions $\varphi^{(r)}$, $\ldots$, $\varphi^{(2)}$, $\varphi^{(1)}$, f are n times differentiable with respect to the relevant independent variables so that, by using the chain rule, $\Phi(t)$ can be differentiated n times with respect to t.

We use the following notation

$$\begin{aligned}
\Phi_h &:= D_t^h \Phi(t), \\
f_h &:= D_{x^{(1)}}^h f|_{x^{(1)}=\varphi^{(1)}(\cdots(\varphi^{(r)}(t)))}, \\
\varphi_h^{(1)} &:= D_{x^{(2)}}^h \varphi^{(1)}|_{x^{(2)}=\varphi^{(2)}(\cdots(\varphi^{(r)}(t)))}, \\
&\vdots \\
\varphi_h^{(r)} &:= D_t^h \varphi^{(r)}(t),
\end{aligned} \tag{10}$$

and

$$\left(\left[f, \varphi^{(1)}, \ldots, \varphi^{(r)}\right]_n\right) := (f_1, \varphi_1^{(1)}, \ldots, \varphi_1^{(r)}; \ldots; f_n, \varphi_n^{(1)}, \ldots, \varphi_n^{(r)}).$$

Then the nth derivative of the function Φ allows us to define the one-dimensional Bell polynomials of order r, $Y_n^{[r]}$, as follows:

$$Y_n^{[r]}\left(\left[f, \varphi^{(1)}, \ldots, \varphi^{(r)}\right]_n\right) := \Phi_n. \tag{11}$$

For $r = 1$ we obtain the ordinary Bell polynomials $Y_n^{[1]}\left(\left[f, \varphi^{(1)}\right]_n\right) = Y_n\left(\left[f, \varphi^{(1)}\right]_n\right)$.

The first polynomials have the following explicit expressions

$$Y_1^{[r]}\left(\left[f, \varphi^{(1)}, \ldots, \varphi^{(r)}\right]_1\right) = f_1 \varphi_1^{(1)} \cdots \varphi_1^{(r)} \tag{12}$$

$$\begin{aligned}
Y_2^{[r]}\left(\left[f, \varphi^{(1)}, \ldots, \varphi^{(r)}\right]_2\right) &= f_2\left(\varphi_1^{(1)} \cdots \varphi_1^{(r)}\right)^2 + f_1 \varphi_2^{(1)}\left(\varphi_1^{(2)} \cdots \varphi_1^{(r)}\right)^2 \\
&\quad + f_1 \varphi_1^{(1)} \varphi_2^{(2)}\left(\varphi_1^{(3)} \cdots \varphi_1^{(r)}\right)^2 + f_1 \varphi_1^{(1)} \varphi_1^{(2)} \cdots \varphi_1^{(r-1)} \varphi_2^{(r)}.
\end{aligned}$$

In general, we have

$$Y_n^{[r]}([f, \varphi^{(1)}, \ldots, \varphi^{(r)}]_n) = \sum_{k=1}^{n} A_{n,k}^{[r]}([\varphi^{(1)}, \ldots, \varphi^{(r)}]_n) f_k. \tag{13}$$

The authors [18] proved the following useful properties for the polynomials $Y_n^{[r]}$

Theorem 4. *For every integer n, the polynomials $Y_n^{[r]}$ are expressed in terms of the Bell polynomials of lower order, by means of the following equation*

$$Y_n^{[r]}\left(\left[f,\varphi^{(1)},\dots,\varphi^{(r)}\right]_n\right) = Y_n\left(\left[f, Y^{[r-1]}\left(\left[\varphi^{(1)},\dots,\varphi^{(r)}\right]\right)\right]_n\right), \tag{14}$$

where

$$\begin{aligned}&\left(\left[f, Y^{[r-1]}\left(\left[\varphi^{(1)},\dots,\varphi^{(r)}\right]\right)\right]_n\right)\\ &\quad := \Bigg(f_1, Y_1^{[r-1]}\left(\left[\varphi^{(1)},\dots,\varphi^{(r)}\right]_1\right);\dots;f_n, Y_n^{[r-1]}\left(\left[\varphi^{(1)},\dots,\varphi^{(r)}\right]_n\right)\Bigg).\end{aligned}$$

Theorem 5. *The following recurrence relation for the Bell polynomials $Y_n^{[r]}$ holds*

$$\begin{cases} Y_0^{[r]}\left(\left[f,\varphi^{(1)},\dots,\varphi^{(r)}\right]_0\right) = f_1 \\ Y_{n+1}^{[r]}\left(\left[f,\varphi^{(1)},\dots,\varphi^{(r)}\right]_{n+1}\right) = \displaystyle\sum_{k=0}^{n}\binom{n}{k} Y_{n-k}^{[r]}\left(\left[f_1,\varphi^{(1)},\dots,\varphi^{(r)}\right]_{n-k}\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad \times Y_{k+1}^{[r-1]}\left(\left[\varphi^{(1)},\dots,\varphi^{(r)}\right]_{k+1}\right), \end{cases} \tag{15}$$

where

$$\left(\left[f_1,\varphi^{(1)},\dots,\varphi^{(r)}\right]_{n-k}\right) := \left(f_2,\varphi_1^{(1)},\dots,\varphi_1^{(r)};\dots;f_{n-k+1},\varphi_{n-k}^{(1)},\dots,\varphi_{n-k}^{(r)}\right).$$

The recurrence relation (6)–(7) can be generalized as follows:

Theorem 6. *For all integer n, we have*

$$\begin{aligned} A_{n+1,1}^{[r]} &= Y_{n+1}^{[r-1]}\left(\left[\varphi^{(1)},\dots,\varphi^{(r)}\right]_{n+1}\right), \\ A_{n+1,n+1}^{[r]} &= \left(Y_1^{[r-1]}\left(\left[\varphi^{(1)},\dots,\varphi^{(r)}\right]_1\right)\right)^{n+1} = \left(\varphi_1^{(1)}\cdots\varphi_1^{(r)}\right)^{n+1}. \end{aligned} \tag{16}$$

Furthermore, for all $k = 1,2,\dots,n-1$, the rth *order partial Bell polynomials $A_{n,k}^{[r]}$ satisfy the recursion*

$$\begin{aligned} A_{n+1,k+1}^{[r]}\left(\left[\varphi^{(1)},\dots,\varphi^{(r)}\right]_{n+1}\right) &= \sum_{h=0}^{n-k}\binom{n}{h} A_{n-h,k}^{[r]}\left(\left[\varphi^{(1)},\dots,\varphi^{(r)}\right]_{n-h}\right) \\ &\quad \times Y_{h+1}^{[r-1]}\left(\left[\varphi^{(1)},\dots,\varphi^{(r)}\right]_{h+1}\right). \end{aligned} \tag{17}$$

Definition 7. The complete Bell polynomials of order r, $B_n^{[r]}$, are defined by the equation

$$\begin{aligned} B_n^{[r]}\left(\left[\varphi^{(1)},\dots,\varphi^{(r)}\right]_n\right) &= Y_n^{[r]}(1,\varphi_1^{(1)},\dots,\varphi_1^{(r)};\dots;1,\varphi_n^{(1)},\dots,\varphi_n^{(r)}) \\ &= \sum_{k=1}^{n} A_{n,k}^{[r]}\left(\left[\varphi^{(1)},\dots,\varphi^{(r)}\right]_n\right), \end{aligned}$$

and the rth order Bell numbers by

$$b_n^{[r]} = Y_n^{[r]}(1,1,1;\dots;1,1,1) = \sum_{k=1}^{n} A_{n,k}^{[r]}(1,1;\dots;1,1).$$

4 Higher order Bell numbers, for $r = 2, 3, 4, 5$

The sequences of higher order Bell numbers, which are presented here, appear in the *Encyclopedia of Integer Sequences* [28] under A144150, arising from a problem of combinatorial analysis and even [4, 13] as the McLaurin coefficients of the functions

$$\begin{aligned}
&\exp(\exp(\exp(x)-1)-1),\\
&\exp(\exp(\exp(\exp(x)-1)-1)-1),\\
&\exp(\exp(\exp(\exp(\exp(x)-1)-1)-1)-1),\\
&\exp(\exp(\exp(\exp(\exp(\exp(x)-1)-1)-1)-1)-1),
\end{aligned}$$

for the cases $r = 2$, $r = 3$, $r = 4$, $r = 5$, respectively, and so on for the subsequent values of r. In our approach they assume a more general meaning, as they are independent of the functions $f, \varphi^{(1)}, \dots, \varphi^{(r)}$.

By using the recurrence relation (16)–(17) and by means of the computer algebra program Mathematica©, we find the following sequences for the higher order Bell numbers $b_n^{[2]}$, $b_n^{[3]}$, $b_n^{[4]}$, $b_n^{[5]}$, $(n = 1, 2, \dots, 10)$:

n	$b_n^{[2]}$	$b_n^{[3]}$	$b_n^{[4]}$	$b_n^{[5]}$
1	1	1	1	1
2	3	4	5	6
3	12	22	35	51
4	60	154	315	561
5	358	1304	3455	7556
6	2471	12915	44590	120196
7	19302	146115	660665	2201856
8	167894	1855570	11035095	45592666
9	1606137	26097835	204904830	1051951026
10	16733779	402215465	4183174520	26740775306

Table 1: Higher order Bell numbers for $n = 1, 2, \dots, 10$.

The above table can be extended up to the desired order, as the Mathematica© program runs efficiently.

5 Sequences of integer numbers connected with the above schemes

Putting, for all integers n, $b_n^{[0]} := 1, b_n^{[1]} := b_n$, and reading Table 1 by row, we find the new array

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$b_1^{[k]}$	1	1	1	1	1	1
$b_2^{[k]}$	1	2	3	4	5	6
$b_3^{[k]}$	1	5	12	22	35	51
$b_4^{[k]}$	1	15	60	154	315	561
$b_5^{[k]}$	1	52	358	1304	3455	7556
$b_6^{[k]}$	1	203	2471	12915	44590	120196

Table 2: The above sequences to be read by row.

The construction law of the sequence $\left(b_3^{[k]}\right)_{k\geq 0}$ (third row) is

$$\begin{aligned} b_3^{[1]} &= b_2^{[1]} + b_2^{[2]} \\ b_3^{[2]} &= b_2^{[2]} + b_2^{[3]} + b_2^{[4]} \\ &\vdots \\ b_3^{[k]} &= b_2^{[k]} + b_2^{[k+1]} + \cdots + b_2^{[2k]}. \end{aligned} \tag{18}$$

Deriving a similar law for the subsequent rows is an open problem, since there is no connection between the meaning of the sequences $\left(b_3^{[k]}\right)_{k\geq 0}$ and $\left(b_4^{[k]}\right)_{k\geq 0}$. In the *Encyclopedia of Integer Sequences*, the third row $(1, 5, 12, 22, 35, 51, \ldots)$ appears under A000326, and $b_3^{[n]}$ is identified, according to the above recursion (18), as the sum of n integers starting from n, or as the pentagonal number $n(3n-1)/2$.

The fourth row $(1, 15, 60, 154, 315, 561, \ldots)$ appears under A005945 and $b_4^{[n]}$ is the number of n-step mappings with 4 inputs.

The subsequent rows are not included explicitly in the *Encyclopedia*, so our approach is useful in order to merge the above sequences into a general framework.

6 Laguerre-type derivatives

Dattoli and Ricci [9] introduced the Laguerre-type derivatives in connection with a differential isomorphism denoted by the symbol $\mathcal{T} := \mathcal{T}_x$, acting onto the space $\mathcal{A} := \mathcal{A}_x$ of analytic functions of the x variable by means of the correspondence

$$D := \frac{d}{dx} \;\rightarrow\; \hat{D}_L := DxD; \qquad x\cdot \;\rightarrow\; \hat{D}_x^{-1}, \tag{19}$$

where

$$\hat{D}_x^{-1} f(x) := \int_0^x f(\xi) d\xi,$$

$$\hat{D}_x^{-n} f(x) := \frac{1}{(n-1)!} \int_0^x (x-\xi)^{n-1} f(\xi) d\xi,$$

and $\hat{D}_x^{-n}$ denotes the n-fold R-L integration, so that

$$\mathcal{T}_x(x^n) = \hat{D}_x^{-n}(1) := \frac{1}{(n-1)!} \int_0^x (x-\xi)^{n-1} d\xi = \frac{x^n}{n!}. \tag{20}$$

According to the isomorphism $\mathcal{T}_x$, the exponential operator e^x is transformed into the first Laguerre-type exponential $e_1(x) := \sum_{k=0}^{\infty} x^k/(k!)^2$ which is an eigenfunction of the *Laguerre derivative operator* $D_L := DxD$. We have, in fact,

$$\mathcal{T}_x(e^x) = \sum_{k=0}^{\infty} \frac{\mathcal{T}_x(x^k)}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{(k!)^2} = e_1(x),$$

$$\hat{D}_L \; e_1(ax) = a e_1(ax), \qquad \forall a \in \mathbf{C}.$$

This result can be generalized by considering the rth Laguerre-type exponential $e_r(x) := \sum_{k=0}^{\infty} x^k/(k!)^{r+1}$, the rth Laguerre-type derivative operator $D_{rL} := DxDxD \cdots DxD$ (containing $r+1$ ordinary derivatives), and the iterated isomorphism $\mathcal{T}^r$, since

$$\mathcal{T}_x^r(e^x) = \sum_{k=0}^{\infty} \frac{\mathcal{T}_x(x^k)}{(k!)^r} = \sum_{k=0}^{\infty} \frac{x^k}{(k!)^{r+1}} = e_r(x),$$

$$\hat{D}_{rL} \; e_r(ax) = a e_r(ax), \qquad \forall a \in \mathbf{C}.$$

Remark 8. The above results show that, for every positive integer r, we can define a Laguerre-type exponential function $e_r(x)$, satisfying an eigenfunction property, which is an analog of the elementary property of the exponential. The Laguerre-type exponential function reduces to the classical exponential function when $r = 0$, so that we can put by definition

$$e_0(x) := e^x, \qquad \hat{D}_{0L} := D.$$

Obviously, $\hat{D}_{1L} := \hat{D}_L$.

For this reason we will refer to such functions as *L-exponential functions*, or shortly *L-exponentials*.

Bernardini et al. [5] found several applications to the theory of L-exponential functions.

7 Laguerre-type Bell polynomials and numbers

The problem of constructing Bell polynomials can be extended in the natural way to the case of the Laguerre-type derivatives.

By using the notation (1)–(2), we introduce the following definition

Definition 9. The nth Laguerre-type Bell polynomial of order r, denoted by ${}_{rL}Y_n\left(x;[f,g]_n\right)$, represents the nth Laguerre-type derivative of order r of the composite function $f(g(t))$.

The authors [19] showed that ${}_{rL}Y_n$ can be expressed as a polynomial in the independent variable x, depending on $f_1, g_1; f_2, g_2; \ldots; f_n, g_n$ in terms of the classical Bell polynomials.

According to a general result due to Viskov [29], the Laguerre derivative satisfy

$$(D_L)^n = (DxD)^n = D^n x^n D^n, \tag{21}$$

and furthermore, for any order r, it turns out that

$$(D_{rL})^n = (DxDx\cdots DxD)^n = D^n x^n D^n x^n \cdots D^n x^n D^n. \tag{22}$$

Therefore, the authors [19] proved the following representation formula for the Laguerre-type Bell polynomials of the first order, denoted by ${}_LY_n$,

Theorem 10. *The ${}_LY_n$ polynomials are expressed in terms of the ordinary Bell polynomials according to the equation*

$$ {}_LY_n\left(x;[f,g]_n\right) = \sum_{k=0}^{n} \frac{n!}{k!}\binom{n}{k} x^k \, Y_{n+k}\left([f,g]_{n+k}\right). \tag{23}$$

Definition 11. The $1st$ order Laguerre-type Bell numbers are defined by

$$ {}_Lb_n := {}_L Y_n(1;1,1,\ldots,1) = \sum_{k=0}^{n} \frac{n!}{k!}\binom{n}{k} Y_{n+k}(1,1,\ldots,1).$$

The above results can be easily generalized, since

$$\begin{aligned}(D_{2L})^n &= (DxDxD)^n = D^n x^n \left(D^n x^n D^n\right) = \\ &= \sum_{k_1=0}^{n}\sum_{k_2=0}^{n} \frac{n!}{k_1!}\frac{(n+k_1)!}{(k_1+k_2)!}\binom{n}{k_1}\binom{n}{k_2} x^{k_1+k_2} D^{n+k_1+k_2},\end{aligned} \tag{24}$$

so that the last definition becomes

Definition 12. The $2nd$ order Laguerre-type Bell numbers are defined by

$$ {}_{2L}b_n := {}_{2L} Y_n(1;1,1,\ldots,1) = \sum_{k_1=0}^{n}\sum_{k_2=0}^{n} \frac{n!}{k_1!}\frac{(n+k_1)!}{(k_1+k_2)!}\binom{n}{k_1}\binom{n}{k_2} Y_{n+k_1+k_2}(1,1,\ldots,1)$$

and in general, for every integer r,

Definition 13. The rth order Laguerre-type Bell numbers are defined by

$$\begin{aligned}
&{}_{rL}b_n :={}_{rL} Y_n\left(1; 1, 1, \ldots, 1\right) = \\
&= \sum_{k_1=0}^{n} \sum_{k_2=0}^{n} \cdots \sum_{k_r=0}^{n} \frac{n!}{k_1!} \frac{(n+k_1)!}{(k_1+k_2)!} \cdots \frac{(n+k_1+k_2+\cdots+k_{r-1})!}{(k_1+k_2+\cdots+k_r)!} \\
&\times \binom{n}{k_1}\binom{n}{k_2}\cdots\binom{n}{k_r} Y_{n+k_1+k_2+\cdots+k_r}\left(1, 1, \ldots, 1\right).
\end{aligned} \tag{25}$$

In particular, for $r = 1$, $r = 2$ and $r = 3$, we obtain respectively

$${}_Lb_n = \sum_{k=0}^{n} \frac{(n!)^2}{(k!)^2(n-k)!} b_{n+k},$$

$${}_{2L}b_n = \sum_{k_1=0}^{n} \sum_{k_2=0}^{n} \frac{(n!)^3(n+k_1)!}{(k_1!)^2(k_2)!(k_1+k_2)!(n-k_1)!(n-k_2)!} b_{n+k_1+k_2},$$

and

$$\begin{aligned}
{}_{3L}b_n &= \sum_{k_1=0}^{n} \sum_{k_2=0}^{n} \sum_{k_3=0}^{n} \\
&\frac{(n!)^4(n+k_1)!(n+k_1+k_2)!}{(k_1!)^2(k_2)!(k_1+k_2!)(k_3)!(k_1+k_2+k_3)!(n-k_1)!(n-k_2)!(n-k_3)!} b_{n+k_1+k_2+k_3}.
\end{aligned}$$

According to the above definitions, we compute the following integer sequences relevant to the Laguerre-type Bell numbers of increasing order:

${}_Lb_1 = b_1 + b_2 = 3$
${}_Lb_2 = 2b_2 + 4b_3 + b_4 = 39$
${}_Lb_3 = 6b_3 + 18b_4 + 9b_5 + b_6 = 971$
${}_Lb_4 = 24b_4 + 96b_5 + 72b_6 + 16b_7 + b_8 = 38140$
${}_Lb_5 = 120b_5 + 600b_6 + 600b_7 + 200b_8 + 25b_9 + b_{10} = 2126890$
${}_Lb_6 = 720b_6 + 4320b_7 + 5400b_8 + 2400b_9 + 450b_{10} + 36b_{11} + b_{12} = 157874467$
${}_Lb_7 = 14928602309$
${}_Lb_8 = 1741809491235$
${}_Lb_9 = 244735956424795$
${}_Lb_{10} = 40624759074089022$

Table 3: Laguerre-type Bell numbers ${}_Lb_n$, for $n = 1, 2, \ldots, 10$.

$$
\begin{aligned}
{}_{2L}b_1 &= b_1 + 3b_2 + b_3 = 12 \\
{}_{2L}b_2 &= 4b_2 + 32b_3 + 38b_4 + 12b_5 + b_6 = 1565 \\
{}_{2L}b_3 &= 36b_3 + 540b_4 + 1242b_5 + 882b_6 + 243b_7 + 27b_8 + b_9 = 597948 \\
{}_{2L}b_4 &= 576b_4 + 13824b_5 + 50688b_6 + 59904b_7 + 30024b_8 + 7200b_9 + \\
&\quad + 856b_{10} + 48b_{11} + b_{12} = 476170277 \\
{}_{2L}b_5 &= 665045751420 \\
{}_{2L}b_6 &= 1466218536786553 \\
{}_{2L}b_7 &= 4751410403456508380 \\
{}_{2L}b_8 &= 21492464638761800105545 \\
{}_{2L}b_9 &= 130485451947856798889765548 \\
{}_{2L}b_{10} &= 1031029017676447641584236719317
\end{aligned}
$$

Table 4: Laguerre-type Bell numbers ${}_{2L}b_n$, for $n = 1, 2, \ldots, 10$

$$
\begin{aligned}
{}_{3L}b_1 &= b_1 + 7b_2 + 6b_3 + b_4 = 60 \\
{}_{3L}b_2 &= 8b_2 + 208b_3 + 652b_4 + 576b_5 + 188b_6 + 24b_7 + b_8 = 104140 \\
{}_{3L}b_3 &= 216b_3 + 13608b_4 + 94284b_5 + 186876b_6 + 149580b_7 + 56808b_8 + \\
&\quad + 11025b_9 + 1017b_{10} + 54b_{11} + b_{12} = 811796953 \\
{}_{3L}b_4 &= 17520336532435 \\
{}_{3L}b_5 &= 821820979710053847 \\
{}_{3L}b_6 &= 72322447015782400673157 \\
{}_{3L}b_7 &= 10811794473114190401496880596 \\
{}_{3L}b_8 &= 2555589594863343375850080281766604 \\
{}_{3L}b_9 &= 904589617839233661347244697559337192973 \\
{}_{3L}b_{10} &= 459477819180446167103818476324632065253323967
\end{aligned}
$$

Table 5: Laguerre-type Bell numbers ${}_{3L}b_n$, for $n = 1, 2, \ldots, 10$

and so on, for the Laguerre-type Bell numbers of higher order.

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