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Improved Bounds on the Anti-Waring Number

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Abstract

The Anti-Waring number, $N(k, r)$, is defined to be the least integer such that it and every larger integer can be written as the sum of the k^{th} powers of r or more distinct positive integers. Several authors have examined this variation of the classical Waring problem. We provide improved bounds for $N(k, r)$ in general and when $k = 2$. We then connect this problem to the theory of partitions. We use traditional counting arguments, as well as a generating function methodology that has not yet been applied to finding the Anti-Waring number.

1 Introduction

In 1770, Waring conjectured that for each positive integer k there exists a $g(k)$ such that every positive integer is a sum of $g(k)$ or fewer k^{th} powers of positive integers. In 1909, Hilbert offered a valid proof of this theorem. The challenge then that became known as the Waring problem was the question that asks, "For each k, what is the smallest $g(k)$ such that this statement holds?"

The Anti-Waring number, $N(k, r)$, is defined to be the least integer such that it and every larger integer can be written as the sum of the k^{th} powers of r or more distinct positive integers. In 2010, Johnson and Laughlin [6] introduced the Anti-Waring number and provided some initial results and lower bounds. In particular, they noticed that $N(1, r) = \frac{r(r+1)}{2}$. In 2012, Looper and Saritzky [7] proved that $N(k, r)$ exists for all positive integers k and r. In 2014 and 2015, Prier et al. [5] and Fuller et al. [4] found a method of using certain conditions to verify values of $N(k,r)$. With the aid of computers, they calculated $N(k,r)$ for many values including $2 \le k \le 5$ and $1 \le r \le 36$ as well as $N(6,1)$. Currently, only computing limitations impede calculating $N(k,r)$ for a specific (k,r) pair.

In this paper, we find improved bounds for $N(k,r)$ in the general case and in the specific case when $k = 2$. We also reexamine the question of finding $N(k,r)$ under the lens of generating functions.

2 Definitions

For the remainder of the paper, let r and k be positive integers.

We say an integer n is (k,r)-*good* if one can write it as the sum of k^{th} powers of r or more distinct positive integers, and it is (k,r)-*bad* if it is not (k,r)-good. For example, 36 is $(3,2)$-good because $36 = 1^3 + 2^3 + 3^3$. However, 37 is $(3,2)$-bad because one cannot write 37 as the sum of two or more distinct cubes.

When considering which positive integers are (k,r)-good, one should notice that the smallest (k,r)-good number is the sum of k^{th} powers of the first r distinct positive integers. In order to simplify notation, we define $P_k(n)$ to be this sum. In other words, $P_k(n) = \sum_{i=1}^{n} i^k$. Here, we allow n to be any nonnegative integer including 0.

In 1994, Bateman et al. studied the sum of distinct squares [2]. We use results from that work to further the study of $N(k,r)$ in this article. However, they examined a slightly different question than that of the Anti-Waring question. Whereas $N(2,r)$ concerns integers that can be written as the sum of r *or more* distinct squares, Bateman et al. considered those integers one can write as the sum of *exactly* r distinct squares. This distinction motivates the following analogous definitions.

Define $N_0(k,r)$ to be the first integer such that it and every larger integer is the sum of k^{th} powers of exactly r distinct positive integers. An integer n is $(k,r)_0$-*good* if one can write it as the sum of k^{th} powers of exactly r distinct positive integers, and it is $(k,r)_0$-*bad* if it is not $(k,r)_0$-good. For example, 28 is $(3,2)_0$-good because $28 = 1^3 + 3^3$. However, 36 is $(3,2)_0$-bad because one cannot write 36 as the sum of exactly two distinct cubes.

Notice here that if n is $(k,r)_0$-good, then it is by definition (k,r)-good. However, the reverse implication is not true. A positive integer could be (k,r)-good but not $(k,r)_0$-good. As shown above, 36 is $(3,2)$-good, but it is not $(3,2)_0$-good.

3 Improved bounds for $N(k,r)$

The following two results represent the previously known bounds on $N(k,r)$.

Lemma 1. *[6]*

i $N(1,r) = P_1(r) = \frac{r(r+1)}{2}$.

ii If $k > 1$, *then* $P_k(r-1) + (r+1)^k \le N(k,r)$.

Lemma 2. *[7] For $k > 1$ and $r \geq 1$, $N(k, r)$ exists.*

Lemma 1 gives the exact value of $N(1, r)$, and together, Lemmas 1 and 2 imply the following bound.

$$P_k(r-1) + (r+1)^k \leq N(k, r) < \infty$$

Lemma 3. *$P_k(r)$ is the smallest (k, r)-good number, and $P_k(r-1) + (r+1)^k$ is the smallest (k, r)-good number greater than $P_k(r)$.*

Proof. By definition, $P_k(r)$ is the smallest (k, r)-good number. Any (k, r)-good number larger than $P_k(r)$ must contain a^k for some $a > r$. The least such a is $(r + 1)$, and therefore the least sum of r or more distinct k^{th} powers other than $P_k(r)$ must be $P_k(r-1) + (r+1)^k$. $\square$

Note that the above lemma is true for all positive integers r including $r = 1$. Also, for all values of $k > 1$, it is true that $P_k(r-1) + (r+1)^k - P_k(r) > 1$ which implies that there are (k, r)-bad numbers in between the two smallest (k, r)-good numbers. This result then implies part *ii* of Lemma 1.

Theorem 4. *For $k > 1$ and $r > 1$, we have $P_k(r-2) + r^k + (r+1)^k \leq N(k, r)$.*

Proof. The next (k, r)-good number after $P_k(r-1) + (r+1)^k$ must contain in its sum either an $(r+1)^k$ or an a^k for some $a > (r+1)$. The least (k, r)-good number that contains $(r+1)^k$ and is not $P_k(r-1) + (r+1)^k$ is $P_k(r-2) + r^k + (r+1)^k$. The least (k, r)-good number that contains an a^k for some $a > (r+1)$ is $P_k(r-1) + (r+2)^k$. The difference, d, between these numbers is

$$d = \Big(P_k(r-1) + (r+2)^k\Big) - \Big(P_k(r-2) + r^k + (r+1)^k\Big) = \Big((r+2)^k - (r+1)^k\Big) - \Big(r^k - (r-1)^k\Big).$$

By the binomial theorem, $\big((r+2)^k - (r+1)^k\big) = \sum_{i=0}^{k-1} \binom{k}{i}(r+1)^i$, and $\big(r^k - (r-1)^k\big) = \sum_{i=0}^{k-1} \binom{k}{i}(r-1)^i$. Therefore, $d = \sum_{i=0}^{k-1} \binom{k}{i}(r+1)^i - \sum_{i=0}^{k-1} \binom{k}{i}(r-1)^i = \sum_{i=0}^{k-1} \binom{k}{i}\big((r+1)^i - (r-1)^i\big)$. If $i = 0$, then $\binom{k}{0}\big((r+1)^0 - (r-1)^0\big) = 0$, so d can be rewritten as

$$d = \sum_{i=1}^{k-1} \binom{k}{i}\Big((r+1)^i - (r-1)^i\Big).$$

For $i > 0$, it is true that $\big((r+1)^i - (r-1)^i\big) > 0$. Therefore, d is positive and thus $P_k(r-2) + r^k + (r+1)^k$ must be the third (k, r)-good number in numerical order. As long as $k > 1$, the difference between the third (k, r)-good number $\big(P_k(r-2) + r^k + (r+1)^k\big)$, and the second (k, r)-good number $\big(P_k(r-1) + (r+1)^k\big)$ is greater than one. These results imply that $\big(P_k(r-2) + r^k + (r+1)^k\big) - 1$ is (k, r)-bad, and therefore $P_k(r-2) + r^k + (r+1)^k \leq N(k, r)$. $\square$

In the previous theorem, we required that $r \geq 2$ in order for $r - 2 \geq 0$. If $r = 1$, then a better lower bound exists.

k	4^k	$N(k,1)$
2	16	129
3	64	12759
4	256	5134241
5	1024	67898772
6	4096	11146309948
7	16384	766834015735

Table 1: 4^k compared to $N(k,1)$ for $1 \leq k \leq 7$

Theorem 5. *For $k > 1$, it is true that $4^k \leq N(k,1)$.*

Proof. For $k = 2$, Sprague proved that $N(2,1) = 129$ [10]. Clearly $4^2 \leq 129$.
For $k > 2$, the following is a list of the first eight $(k,1)$-good numbers in numerical order.

$$1^k < 2^k < 2^k + 1^k < 3^k < 3^k + 1^k < 3^k + 2^k < 3^k + 2^k + 1^k < 4^k$$

The only non-obvious inequality in this list is the last one claiming that $3^k + 2^k + 1^k < 4^k$. Indeed, consider the difference $d = (4^k) - (3^k + 2^k + 1^k)$. By the binomial theorem, $4^k - 3^k = \sum_{i=0}^{k-1} \binom{k}{i} 3^k = \sum_{i=1}^{k-1} \binom{k}{i} 3^k + 1$. Therefore, $d = \sum_{i=1}^{k-1} \binom{k}{i} 3^k + 1 - 2^k - 1 = \sum_{i=1}^{k-1} \binom{k}{i} 3^k - 2^k$. Since $k > 2$, it is true that $d = \binom{k}{k-1} 3^{k-1} + \sum_{i=1}^{k-2} \binom{k}{i} 3^k - 2^k = k3^{k-1} + \sum_{i=1}^{k-2} \binom{k}{i} 3^k - 2^k$. Again, as $k > 2$, it must be that $k3^{k-1} - 2^k > 2 \cdot 3^{k-1} - 2^k > 2 \cdot 2^{k-1} - 2^k = 0$. Also, $\sum_{i=1}^{k-2} \binom{k}{i} 3^k \geq 1$ for $k > 2$. Thus $d \geq 2$. Therefore, not only is $3^k + 2^k + 1^k < 4^k$, but there must also be at least one $(k,1)$-bad number between $3^k + 2^k + 1^k$ and 4^k. This claim is true because no $(k,1)$-good number not listed above can be less than 4^k. Specifically $4^k - 1$ must be $(k,1)$-bad, and therefore $4^k \leq N(k,1)$. □

Values of $N(k,1)$ are known for $1 \leq k \leq 7$ and are one more than the tabulated values in the sequence A001661 referenced in the On-Line Encyclopedia of Integer Sequences. The value of $N(8,1)$ is known to be greater than 74^8 [9]. Upon examining the values for $N(k,1)$ in Table 1, one can see that there is significant room for improvement upon the lower bound of 4^k.

Theorems 4 and 5 offer improved lower bounds for $N(k,r)$ in general, while the following results offer improved bounds in the special case when $k = 2$.

In Section 2, we mentioned that Bateman et al. examined $N_0(2,r)$, which is the first integer such that it and every larger integer is the sum of *exactly* r distinct positive squares. In actuality, this paper examined a number denoted $N(r)$ which, using our notation, is defined to be largest $(2,r)_0$-bad number. Therefore $N(r) = N_0(2,r) - 1$. Theorems 8 and 10 stated below have been rewritten to match the notation of this paper.

As previously stated, if one can write a number as the sum of k^{th} powers of exactly r distinct positive integers, then one can certainly write that number as the sum of k^{th} powers of r or more distinct positive integers. Hence, we have the following lemma.

Lemma 6. *As long as $N_0(k,r)$ exists, $N(k,r) \leq N_0(k,r)$.*

For example, $N_0(2,5) = 246$, but $N(2,5) = 198$. Indeed, $245 = 1^2 + 2^2 + 3^2 + 5^2 + 6^2 + 7^2 + 11^2$ is not the sum of exactly 5 distinct squares but is the sum of 5 *or more* distinct squares.

The following lemma is not a new result. See, for instance, Conway and Fung [3, pp. 137–140].

Lemma 7. *The number, $N_0(2,r)$, does not exist for $r \in \{1,2,3,4\}$.*

Proof. For $r = 1$, any non-square natural number is not expressible as the sum of one square.

For $r = 2$, Fermat's two-square theorem implies that numbers with prime decomposition containing a prime of the form $4a + 3$ raised to an odd power, for some integer a, are not expressible as the sum of two squares of not necessarily distinct integers.

For $r = 3$, Legendre's three-square theorem implies that numbers of the form $4^a(8b + 7)$, for integers a and b, are not expressible as the sum of three squares of not necessarily distinct integers.

For $r = 4$, the numbers not expressible as the sum of four positive squares are $1, 3, 5, 9, 11, 17, 29, 41$ and numbers of the form $2(4^a), 6(4^a)$, or $14(4^a)$, for some integer a [3]. □

Bateman et al. [2] proved the following concerning $N_0(2,r)$.

Theorem 8. *[2] $N_0(2,r) \leq P_2(r) + 2r\sqrt{2r} + 44r^{5/4} + 108r$ for $r \geq 5$.*

Bateman et al. actually proved $N_0(2,r) \leq P_2(r) + 2r\sqrt{2r} + 44r^{5/4} + 108r$ for $r \geq 166$. However, they also calculated the exact value of $N_0(2,r)$ for $5 \leq r \leq 400$. For $5 \leq r \leq 165$, $N_0(2,r)$ satisfies this inequality. Therefore, Theorem 8 is true for $r \geq 5$.

Using the theorem above, we prove a new bound on $N(2,r)$ in general.

Theorem 9. *If $r > 1$, then $P_2(r-2)+r^2+(r+1)^2 \leq N(2,r) \leq P_2(r)+2r\sqrt{2r}+44r^{5/4}+108r$.*

Proof. If $1 \leq r \leq 4$, then $N(2,r) = 129$ [5], which is less than $P_2(r) + 2r\sqrt{2r} + 44r^{5/4} + 108r$. This observation along with Theorem 4, Lemma 6 and Theorem 8 directly implies the result. □

Though the main results of Bateman et al. [2] involved sums of distinct squares, they did prove the following result concerning sums of distinct k^{th} powers for integers $k \geq 2$.

Theorem 10. *[2] For sufficiently large r, $N_0(k,r) = \frac{r^{k+1}}{k+1} + O(r^k)$.*

This result implies that $N_0(k,r)$ is asymptotic to $P_k(r)$. As $P_k(r) \leq N(k,r) \leq N_0(k,r)$, we see that $N(k,r)$ tends to be "close" to $P_k(r)$ for large enough r. If one developed a more precise relationship of this type, then one could significantly reduce the complexity in computation of $N(k,r)$.

4 Generating functions

Many, including Euler and Ramanujan, have studied the theory of generating functions concerning partitions of all types [1]. For this discussion, define the q-Pochhammer symbol to be the product $(a;q)_m = \prod_{p=0}^{m-1}(1-aq^p)$ and $[x^n]f(x)$ to be the n^{th} coefficient of $f(x)$ in the associated Laurent series of $f(x)$. Combinatorially, the q-Pochhammer symbol relates to the generating function of many partition counting functions. For instance, $[q^n](q;q)_m^{-1}$ gives the number of ways one can express n as the sum of not necessarily distinct nonnegative integers of size at most m [11, Ch. 3]. $[a^rq^n](-aq;q)_\infty$ gives the number of ways one can express n as the sum of exactly r distinct natural numbers [11, Ch. 3]. Thus, n is $(1,r)_0$-good if $[a^rq^n](-aq;q)_\infty > 0$. We may now give an alternative proof to part i of Lemma 1, by first examining $N_0(1,r)$.

Theorem 11. $N_0(1,r) = \binom{r+1}{2}$.

Proof. To obtain a formula for $N_0(1,r)$, it suffices to find the smallest power of q in $[a^r](-aq;q)_\infty$ such that it and all following powers have nonzero coefficients, and thus are $(1,r)_0$-good. To find this power, we need the following result. The q-binomial theorem [1]:

$$(a;q)_\infty = \sum_{i=0}^{\infty} \frac{(-1)^i q^{\binom{i}{2}}}{(q;q)_i} a^i$$

We may thus express:

$$(-aq;q)_\infty = \sum_{i=0}^{\infty} \frac{(-1)^i q^{\binom{i}{2}}}{(q;q)_i}(-aq)^i = \sum_{i=0}^{\infty} \frac{q^{\binom{i+1}{2}}}{(q;q)_i} a^i$$

One can represent every nonnegative integer as the sum of not necessarily distinct nonnegative integers. Therefore, $[q^n](q;q)_i^{-1} \geq 1$ for $i \geq 0$. However, the first nonzero coefficient of $[a^r](-aq;q)_\infty$ is a coefficient of $q^{\binom{r+1}{2}}$. Therefore, $[a^rq^n](-aq;q)_\infty \neq 0$ if and only if $n \geq \binom{r+1}{2}$. Thus, $N_0(1,r) = \binom{r+1}{2}$. □

Corollary 12. $N(1,r) = \binom{r+1}{2}$.

Proof. $P_1(r) = \binom{r+1}{2} \leq N(1,r) \leq N_0(1,r) = \binom{r+1}{2}$. □

If we consider a generalized q-Pochhammer symbol, defined as $(a;q)_{m,k} = \prod_{p=1}^{m-1}(1-aq^{p^k})$, we can obtain information for $N_0(k,r)$. In the case of $k=1$, the original q-Pochhammer symbol is recovered. The coefficient, $[a^rq^n](-a;q)_{\infty,k}$, gives the number of ways one can express n as the sum of exactly r k^{th} powers of distinct natural numbers [11, Ch. 3]. Hence:

Lemma 13. *n is $(k,r)_0$-good if $[a^rq^n](-a;q)_{\infty,k} > 0$.*

The case of $k \geq 2$ seems significantly more challenging than the case of $k = 1$. An explicit summation formula for the product generating function can be found in terms of Bell polynomials of sums of the form $\sum_{p=1}^{\infty} x^{p^k}$. These sums reduce easily into a closed form when $k = 1$, but they become much more complex for $k \geq 2$. When $k = 2$, they develop into a closed expression in terms of the well studied Jacobi Theta functions. Although this method did not yield any results concerning a generalized formula for $N_0(k, r)$, it can be helpful in computing $N_0(k, r)$ for specific, small values of r and k.

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