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The Dual of Spivey's Bell Number Identity from Zeon Algebra

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Abstract

In this paper, we give a new short proof of the dual of Spivey's Bell number identity due to Mező. Our approach follows from basic manipulations involving a fundamental identity representing factorials in the Zeon algebra. This work, along with a previous one due to the author and dos Anjos, shows that Spivey's and Mező's identities have at their root a common underlying algebraic origin.

1 Introduction

In this paper, we will give a new, simple, and short proof of the dual of Spivey's Bell number identity obtained by Mező [5, Chap. 3], [6]. We mention that the original proofs of the identities due to Spivey and Mező are combinatorial in nature [3, 6, 11]. Indeed, the proof given by Mező is constructed by considering the enumeration of the permutations of $m+n$ elements in terms of the number of k-permutations of n and the number of permutations of m with j disjoint cycles. This work, along with a previous work concerning the proof of Spivey's identity from Zeons [7], adds another point of view in the origin of Spivey's and Mező's results which is algebraic in nature, as a direct consequence of the use of the Zeon

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algebra [5, Chap. 5], [7], [10, Chap. 2]. Throughout this work we let $\mathbb{R}$ denote the real numbers and $\mathbb{N}$ the positive integers.

More precisely, if m, $n \in \mathbb{N}$, we will show that

$$(m+n)! = \sum_{k=0}^{n} \sum_{j=0}^{m} m^{\overline{n-k}} \begin{bmatrix} m \\ j \end{bmatrix} \binom{n}{k} k! \tag{1}$$

using the Zeon algebra. In Eq. (1) $a^{\overline{n}} = a(a+1)\cdots(a+n-1)$ is the Pochhammer symbol with $a \in \mathbb{R}$, $\binom{n}{k} = n!/(k!(n-k)!)$ is the binomial coefficient, and $\left[\begin{smallmatrix} m \\ j \end{smallmatrix}\right]$ are the unsigned Stirling numbers of the first kind [4, Chap. 6], [12, Eq. (3.5.3)].

For completeness, in the next section, we introduce the tools needed to give the proof of Eq. (1). More precisely, we give the definition of the Zeon algebra and the Grassmann-Berezin integral in the Zeon algebra.

2 Brief review of the Zeon algebra and Grassmann-Berezin integral in the Zeon algebra

Definition 1. The *Zeon algebra* $\mathcal{Z}_n \supset \mathbb{R}$ is defined as the associative and commutative algebra generated by the collection $\{\varepsilon_i\}_{i=1}^n$ $(n < \infty)$ of nilpotent elements and the scalar $1 \in \mathbb{R}$, which is the identity of the algebra.

For $\{i, j, \ldots, k\} \subset \{1, 2, \ldots, n\}$ and $\varepsilon_{ij\cdots k} \equiv \varepsilon_i \varepsilon_j \cdots \varepsilon_k$ the most general element with n generators ε_i can be written with the convention of sum over repeated indices implicit and taking $\varepsilon_\emptyset = 1$ as

$$\vartheta_n = a + a_i \varepsilon_i + a_{ij} \varepsilon_{ij} + \cdots + a_{12\cdots n} \varepsilon_{12\cdots n} = \sum_{\mathbf{i} \in 2^{[n]}} a_{\mathbf{i}} \varepsilon_{\mathbf{i}},$$

with a, a_i, a_{ij}, $\ldots$, $a_{12\cdots n} \in \mathbb{R}$, $2^{[n]}$ being the power set of $[n] := \{1, 2, \ldots, n\}$, and $1 \leq i < j < \cdots \leq n$. We refer to a as the body of ϑ_n and write $b(\vartheta_n) := a$ and to $\vartheta_n - a$ as the soul such that $s(\vartheta_n) := \vartheta_n - a$. Note that $s^{n+1}(\vartheta_n) = 0$.

As described in previous work [7] a real analytic function f can be extended to the realm of the Zeon algebra taking

$$f(\vartheta_n) := \sum_{i=0}^{n} \frac{f^{(i)}\big(b(\vartheta_n)\big)}{i!} s^i\big(\vartheta_n\big) = \sum_{i=0}^{n} \frac{f^{(i)}\big(a\big)}{i!} s^i\big(\vartheta_n\big), \tag{2}$$

where $f^{(i)}(a) = d^i f(x)/dx^i|_{x=a}$ is the i-th ordinary derivative of $f(x)$ at a. Note that $f\big(a + s\big(\vartheta_n\big)\big)|_{s(\vartheta_n)=0} = f(a)$ and $f\big(a + s\big(\vartheta_n\big)\big) \in \mathcal{Z}_n$, because $s^{n+1}(\vartheta_n) = 0$.

We consider specific examples of Eq. (2) important for this work. If $a \neq 0$, then ϑ_n is invertible and the inverse is given by

$$\vartheta_n^{-1} := \frac{1}{a} \left(1 - \frac{s(\vartheta_n)}{a} + \frac{s^2(\vartheta_n)}{a^2} + \cdots + (-1)^n \frac{s^n(\vartheta_n)}{a^n} \right). \tag{3}$$

Next, we will need

$$e^{\vartheta_n} := e^a \sum_{m=0}^{n} \frac{s^m(\vartheta_n)}{m!}. \tag{4}$$

As a final example, for $\vartheta_n = 1 + s(\vartheta_n)$, we consider

$$\ln(1 + s(\vartheta_n)) := \sum_{m=1}^{n} (-1)^{m+1} \frac{s^m(\vartheta_n)}{m}, \tag{5}$$

where the right-hand side of Eq. (5) is the solution of the equation

$$e^{\ln(1+s(\vartheta_n))} \equiv 1 + s(\vartheta_n).$$

We give some examples of the functions considered in Eqs. (3), (4), and (5) for $n = 3$. We have

$$\frac{1}{1 + \frac{1}{2}\varepsilon_2 + \sqrt{3}\varepsilon_{13} - 5\varepsilon_{123}} = 1 - \frac{1}{2}\varepsilon_2 - \sqrt{3}\varepsilon_{13} + \left(5 + \sqrt{3}\right)\varepsilon_{123},$$

$$e^{1+\frac{1}{2}\varepsilon_2+\sqrt{3}\varepsilon_{13}-5\varepsilon_{123}} = e\left(1 + \frac{1}{2}\varepsilon_2 + \sqrt{3}\varepsilon_{13} + \left(\frac{\sqrt{3}}{2} - 5\right)\varepsilon_{123}\right),$$

and

$$\ln\left(1 + \frac{1}{2}\varepsilon_2 + \sqrt{3}\varepsilon_{13} - 5\varepsilon_{123}\right) = \frac{1}{2}\varepsilon_2 + \sqrt{3}\varepsilon_{13} - \left(5 + \frac{\sqrt{3}}{2}\right)\varepsilon_{123}.$$

Note that $e^{\frac{1}{2}\varepsilon_2+\sqrt{3}\varepsilon_{13}-\left(5+\frac{\sqrt{3}}{2}\right)\varepsilon_{123}} = 1 + \frac{1}{2}\varepsilon_2 + \sqrt{3}\varepsilon_{13} - 5\varepsilon_{123}$.

Definition 2. The *Grassmann-Berezin integral* on $\mathcal{Z}_n$, denoted by $\int$, is the linear functional $\int : \mathcal{Z}_n \to \mathbb{R}$ such that

$$d\varepsilon_i d\varepsilon_j = d\varepsilon_j d\varepsilon_i, \ \int \vartheta_n\left(\hat{\varepsilon}_i\right) d\varepsilon_i = 0, \text{ and} \int \vartheta_n\left(\hat{\varepsilon}_i\right)\varepsilon_i d\varepsilon_i = \vartheta_n\left(\hat{\varepsilon}_i\right),$$

where $\vartheta_n\left(\hat{\varepsilon}_i\right)$ means any element of $\mathcal{Z}_n$ with no dependence on ε_i. We use throughout this work the compact notation $d\nu_n := d\varepsilon_n \cdots d\varepsilon_1$. Multiple integrals are iterated integrals, i.e.,

$$\int f(\vartheta_n) d\nu_n = \int \cdots \left(\int \left(\int f(\vartheta_n) d\varepsilon_n\right) d\varepsilon_{n-1}\right) \cdots d\varepsilon_1.$$

The standard literature on Grassmann algebra comprises, e.g., the work of Berezin [1, Chap. 1], DeWitt [2, Chap. 1], and Rogers [9, Chap. 3].

Some examples of the integration in Definition 2 are given below. We have

$$\int \frac{1}{1 + \frac{1}{2}\varepsilon_2 + \sqrt{3}\varepsilon_{13} - 5\varepsilon_{123}} d\varepsilon_1 = -\sqrt{3}\varepsilon_3 + \left(5 + \sqrt{3}\right)\varepsilon_{23}$$

and

$$\int e^{1+\frac{1}{2}\varepsilon_2+\sqrt{3}\varepsilon_{13}-5\varepsilon_{123}} d\varepsilon_1 d\varepsilon_3 = e\left(\sqrt{3}+\left(\frac{\sqrt{3}}{2}-5\right)\varepsilon_2\right).$$

A particularly simple result which follows from the multinomial theorem is

$$\int \varphi_n^n d\nu_n = n! \tag{6}$$

with $\vartheta_n = \varphi_n := \varepsilon_1 + \cdots + \varepsilon_n$.

3 Proof of Eq. (1)

We are now ready to prove Eq. (1). Let us take $\phi_m = \epsilon_1 + \cdots + \epsilon_m$ with $\epsilon_i := \varepsilon_{n+i}$, $i = 1, \ldots, m$, such that $\varphi_{n+m} = \phi_m + \varphi_n \in \mathcal{Z}_{m+n}$. We start with the following identity

$$(m+n)! = \int (\phi_m + \varphi_n)^{m+n} d\mu_m d\nu_n = \int \sum_{k=0}^{m+n} (\phi_m + \varphi_n)^k d\mu_m d\nu_n = \int \frac{1}{1-\phi_m-\varphi_n} d\mu_m d\nu_n, \tag{7}$$

using Eqs. (3) and (6). Next, we observe that

$$\begin{aligned}
(m+n)! &= \int \frac{1}{1-\varphi_n} \frac{1}{1-\frac{\phi_m}{1-\varphi_n}} d\mu_m d\nu_n \\
&= \sum_{k=0}^{n} \int \varphi_n^k \frac{1}{1-\frac{\phi_m}{1-\varphi_n}} d\mu_m d\nu_n \\
&= \int \frac{1}{1-\frac{\phi_m}{1-\varphi_n}} d\mu_m d\nu_n + \sum_{k=1}^{n} \sum_{1\le j_1<\cdots<j_k\le n} k! \int \varepsilon_{j_1\cdots j_k} \frac{1}{1-\frac{\phi_m}{1-\varphi_n}} d\mu_m d\nu_n \\
&= \sum_{k=0}^{n} \binom{n}{k} k! \int \frac{1}{1-\frac{\phi_m}{1-\varphi_{n-k}}} d\mu_m d\nu_{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} k! \int \frac{1-\varphi_{n-k}}{1-\phi_m-\varphi_{n-k}} d\mu_m d\nu_{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} k! \left(\int \frac{1}{1-\phi_m-\varphi_{n-k}} d\mu_m d\nu_{n-k} - \int \frac{n-k}{1-\phi_m-\varphi_{n-k-1}} d\mu_m d\nu_{n-k-1}\right) \\
&= \sum_{k=0}^{n} \binom{n}{k} k! \left((m+n-k)! - (n-k)(m+n-k-1)!\right) \\
&= \sum_{k=0}^{n} \binom{n}{k} k! m (m+n-k-1)!,
\end{aligned}$$

using Eq. (7). Finally, recalling the definition of $m^{\overline{n-k}}$ in Eq. (1), we obtain

$$(m+n)! = m! \sum_{k=0}^{n} \binom{n}{k} k! m^{\overline{n-k}}. \tag{8}$$

Now, we consider the unsigned Stirling numbers of the first kind $\begin{bmatrix} n \\ j \end{bmatrix}$ given by the generating function [12, Eq. (3.5.3)]

$$\frac{1}{j!}\left(\ln\frac{1}{1-x}\right)^j = \sum_{n=j}^{\infty}\begin{bmatrix} n \\ j \end{bmatrix}\frac{x^n}{n!}. \tag{9}$$

Using Eqs. (5) and (6) we arrive at

$$\begin{bmatrix} m \\ j \end{bmatrix} = \frac{1}{j!}\int\left(\ln\frac{1}{1-\phi_m}\right)^j d\mu_m,$$

extending Eq. (9) to the context of Zeons by following previous work [7, 8]. Note that

$$\sum_{j=0}^{m}\begin{bmatrix} m \\ j \end{bmatrix} = \int e^{\ln\left(\frac{1}{1-\phi_m}\right)} d\mu_m = \int \frac{1}{1-\phi_m} d\mu_m = m!, \tag{10}$$

using Eqs. (3), (4), (5), and (6).

Finally, from Eqs. (8) and (10), we arrive at Eq. (1).

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