A Tribonacci-Like Sequence of Composite Numbers

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Abstract

We find a new Tribonacci-like sequence of positive integers \(x_0, x_1, x_2, \ldots\) given by \(x_n = x_{n-1} + x_{n-2} + x_{n-3}\), \(n \geq 3\), and \(\gcd(x_0, x_1, x_2) = 1\) that contains no prime numbers. We show that the sequence with initial values \(x_0 = 151646890045\), \(x_1 = 836564809606\), \(x_2 = 942785024683\) is the current record in terms of the number of digits.

1 Introduction

Šiurys [10] found initial values

\[
x_0 = 99202581681909167232 \\
x_1 = 67600144946390082339 \\
x_2 = 139344212815127987596,
\]
satisfying \(\gcd(x_0, x_1, x_2) = 1\), such that the Tribonacci-like sequence given by

\[
x_n = x_{n-1} + x_{n-2} + x_{n-3} \text{ for } n \geq 3
\] (1)
contains no prime numbers. Similar problems were considered for Fibonacci-like sequences
given by $x_n = x_{n-1} + x_{n-2}$ for $n \geq 2$ (Graham [2]; Knuth [5]; Wilf [13]; Nicol [7]; Vsemirnov
[12]; Ismailescu and Son [3]), sequences given by $a_n = k2^n + 1$ (Sierpiński [8]; Jaeschke [4]),
binary linear recurrent sequences (Dubickas, Novikas, and Šiurys [1]; Somer [11]) and some
linear recurrent sequences of higher orders (Šiurys [9]).

The main result of this note is as follows.

2 The main results

Theorem 1. Let $\langle x_0, x_1, x_2, \ldots \rangle$ be defined by (1) and $\gcd(x_0, x_1, x_2) = 1$ with the following
initial values:

\[
x_0 = 151646890045, \quad x_1 = 836564809606, \quad x_2 = 942785024683.
\]

Then $\langle x_0, x_1, x_2, \ldots \rangle$ contains no prime numbers.

Remark 2. If we allow non-positive values, we can find a slightly smaller (in absolute value)
initial triple, namely

\[
x_0 = 730344594529, \quad x_1 = -45426674968, \quad x_2 = 151646890045.
\]

3 Proof of Theorem 1

In this section we complete the proof of Theorem 1.

Proof of Theorem 1. First, recall Šiurys’ idea [10]. Consider the additional sequences $(s_n)_{n=0}^\infty$
and $(t_n)_{n=0}^\infty$ defined by the same relation (1) with $(s_0, s_1, s_2) = (0, 1, 0)$ and $(t_0, t_1, t_2) =
(0, 0, 1)$.

Lemma 3 ([10]). Let $p$ be a prime. Suppose that for some integer $m \geq 2$ we have $s_mt_{2m} -
s_{2m}t_m \equiv 0 \pmod{p}$. Then there exist $a, b \in \mathbb{Z}$ such that at least one of $a, b$ is not divisible
by $p$ and $s_{km}a + t_{km}b \equiv 0 \pmod{p}$ for $k = 0, 1, 2, \ldots$.

The next step is to find a set of pairs $(p_i, m_i)$ satisfying Lemma 3 such that every integer
belongs to at least one of the arithmetic progressions

\[
A_i = m_i k + r_i, \ k \in \mathbb{Z}, \ i = 1, 2, \ldots, 11.
\]
Table 1: $p_i$ and $m_i$.

| i | $p_i$ | $m_i$ | $|s_{m_i}t_{2m_i} - s_{2m_i}t_{m_i}|$ |
|---|---|---|---|
| 1 | 2 | 2 | 2 |
| 2 | 29 | 5 | 29 |
| 3 | 17 | 6 | 2 \cdot 17 |
| 4 | 7 | 8 | 2^6 \cdot 7 |
| 5 | 11 | 10 | 2 \cdot 11 \cdot 29 |
| 6 | 107 | 12 | 2^4 \cdot 17 \cdot 107 |
| 7 | 8819 | 15 | 29 \cdot 8819 |
| 8 | 19 | 20 | 2^4 \cdot 11 \cdot 19 \cdot 29 \cdot 239 |
| 9 | 239 | 20 | 2^4 \cdot 11 \cdot 19 \cdot 29 \cdot 239 |
| 10 | 1151 | 24 | 2^6 \cdot 7 \cdot 17 \cdot 107 \cdot 1151 |
| 11 | 1621 | 30 | 2 \cdot 11 \cdot 17 \cdot 29 \cdot 1621 \cdot 8819 |

\[
s_{km_i}a_i + t_{km_i}b_i \equiv 0 \pmod{p_i} \text{ for } k = 0, 1, 2, \ldots.
\]

Next, we construct a sequence $\left( x_n \right)_{n=0}^{\infty}$ satisfying

\[
x_n \equiv s_{m_i-r_i+n}a_i + t_{m_i-r_i+n}b_i \pmod{p_i}, \quad i = 1, 2, \ldots, 11, \quad \text{for } n = 0, 1, 2, \ldots.
\]

The initial values satisfy

\[
x_0 \equiv s_{m_i-r_i}a_i + t_{m_i-r_i}b_i \pmod{p_i}, \quad x_1 \equiv s_{m_i-r_i+1}a_i + t_{m_i-r_i+1}b_i \pmod{p_i},
\]

\[
x_2 \equiv s_{m_i-r_i+2}a_i + t_{m_i-r_i+2}b_i \pmod{p_i}, \quad \text{for } i = 1, 2, \ldots, 11.
\]

We can find initial terms $(x_0, x_1, x_2)$ by the Chinese remainder theorem.

In the method described above there is some freedom in the choice of $a_i$ and $b_i$ (up to a common factor). Šiurys [10] used all $a_i$ equal to 1.

We show how to optimize the choice of $a_i$ and $b_i$. Let $P = \prod_{i=1}^{11} p_i$.

Let us consider the system:

\[
\begin{align*}
x'_0 & \equiv Dx_0 \pmod{P} \\
x'_1 & \equiv Dx_1 \pmod{P} \\
x'_2 & \equiv Dx_2 \pmod{P}
\end{align*}
\]

subject to the constraint

\[
\gcd(D, P) = 1. \tag{3}
\]

The new triple $(x'_0, x'_1, x'_2)$ also satisfies the above properties, i.e., each term of the sequence (1) with starting values $(x'_0, x'_1, x'_2)$ is divisible by at least one of $p_1, \ldots, p_{11}$.
For a moment, let us forget about condition (3). Then the problem can be formulated as follows: find the minimum vector of the form
\[ D(x_0, x_1, x_2) + U_1(P, 0, 0) + U_2(0, P, 0) + U_3(0, 0, P), \]
i.e., the vector in the lattice generated by the vectors \((x_0, x_1, x_2), (P, 0, 0), (0, P, 0), (0, 0, P)\). The smallest vector can be found by the LLL-algorithm [6].

For any admissible covering (2) with the above \(m_i\)’s we build the initial values \((x_0, x_1, x_2)\) for the sequence (1) by using Siurys’ method. Then using the LLL-algorithm we find the smallest lattice basis. Coordinates \((x'_0, x'_1, x'_2)\) for each of the new three basis vectors will suit us, if the condition (3) is satisfied and \((x'_0, x'_1, x'_2)\) are of the same sign (if all of them are negative, then replace \((x'_0, x'_1, x'_2)\) by \((-x'_0, -x'_1, -x'_2)\)). Thus, searching through all possible coverings (the total amount is 23040) we find sets \((p_i, m_i, r_i, a_i, b_i)\). Those listed in Table 2 give rise to the smallest initial triple \(x_0 = 151646890045, x_1 = 836564809606, x_2 = 942785024683\), as stated in Theorem 1.

<table>
<thead>
<tr>
<th>i</th>
<th>1</th>
<th>2</th>
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<td>6</td>
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<td>1077</td>
<td>180</td>
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<tr>
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<td>8</td>
<td>103</td>
<td>964</td>
<td>291</td>
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</table>

Table 2: \(p_i, m_i, r_i, a_i, b_i\).

If we allow non-positive terms in the sequence, the same method gives sets \((p'_i, m'_i, r'_i, a'_i, b'_i)\), which give the sequence mentioned in the Remark 2: \(x_0 = 730344594529, x_1 = -45426674968, x_2 = 151646890045\) (see Table 3).

<table>
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Table 3: \(p'_i, m'_i, r'_i, a'_i, b'_i\).

It is worth noting that in the sequence mentioned in Remark 2 \(x_2 = 151646890045, x_3 = 836564809606, x_4 = 942785024683\), so this means that the sequence in Theorem 1 is a shift of the sequence in Remark 2.
Both sequences can be extended to the left. It can be shown that in both cases mentioned above these extended sequences also contain no primes. Since the sequences modulo $P$ are periodic with period $\text{lcm}(m_1, \ldots, m_{11}) = 120$, it is enough to check that $x_j \neq k$ modulo $P$, $-8819 \leq k \leq 8819 = \max(p_i)$, $j = 0, \ldots, 119$.

References


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