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The Constant of Recognizability is Computable for Primitive Morphisms

Fabien Durand
Laboratoire Amiénois de Mathématiques Fondamentales et Appliquées
CNRS-UMR 7352
Université de Picardie Jules Verne
33 rue Saint Leu
80000 Amiens
France
fabien.durand@u-picardie.fr

Julien Leroy[1]
Institut de mathématique
Université de Liège
Allée de la découverte 12 (B37)
4000 Liège
Belgium
J.Leroy@ulg.ac.be

Abstract

Mossé proved that primitive morphisms are recognizable. In this paper we give a computable upper bound for the constant of recognizability of such a morphism. This bound can be expressed using only the cardinality of the alphabet and the length of the longest image of a letter under the morphism.

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1 Introduction

Infinite words, i.e., infinite sequences of symbols from a finite set, usually called the alphabet, form a classical object of study. They have an important power of representation: they provide a natural way to code elements of an infinite set using finitely many symbols, e.g., the coding of an orbit in a discrete dynamical system or the characteristic sequence of a set of integers. A rich family of infinite words, with a simple algorithmic description, consists of the words obtained by iterating a morphism $\sigma : A^* \to A^*$ [3], where A^* is the free monoid generated by the finite alphabet A.

If σ is *prolongable* on some letter $a \in A$, that is, if $\sigma(a) = au$ for some non-empty word u and $\lim_{n\to+\infty} |\sigma^n(a)| = +\infty$, then $\sigma^n(a)$ converges to an infinite word $x = \sigma^\omega(a) \in A^{\mathbb{N}}$ that is a fixed point of σ. Two-sided fixed points are similarly defined as infinite words of the form $\sigma^\omega(a \cdot b) \in A^{\mathbb{Z}}$, where $\sigma(a) = ua$ and $\sigma(b) = bv$ with $u, v \in A^+$ and $\lim_{n\to+\infty} |\sigma^n(a)| = \lim_{n\to+\infty} |\sigma^n(b)| = +\infty$. Such a fixed point is said to be *admissible* if ab occurs in $\sigma^n(c)$ for some $n \in \mathbb{N}$ and some $c \in A$. When the morphism is *primitive*, i.e., there exists $k \in \mathbb{N}$ such that b occurs in $\sigma^k(c)$ for all $b, c \in A$, then x is *uniformly recurrent*: each finite word u that occurs in x occurs infinitely many times in x and the gaps between two consecutive occurrences of u in x are bounded [16]. The converse almost holds: if $x = \sigma^\omega(a)$ is uniformly recurrent, then there exist a primitive morphism $\varphi : B^* \to B^*$, a letter $b \in B$ and a morphism $\psi : B^* \to A^*$ such that $x = \psi(\varphi^\omega(b))$ [4]. We let $\mathcal{L}(x)$ denote the set of *factors* of x, i.e., $\mathcal{L}(x) = \{u \in A^* \mid \exists p \in A^*, w \in A^{\mathbb{N}} : x = puw\}$ (with an analogous definition for two-sided fixed points). We also let $p_x : \mathbb{N} \to \mathbb{N}$ denote the *complexity function* of x defined by $p_x(n) = \operatorname{Card} \mathcal{L}_n(x)$ where $\mathcal{L}_n(x) = \mathcal{L}(x) \cap A^n$. A primitive morphism σ is *aperiodic* if its fixed points are not periodic, i.e., are not of the form $u^\omega = uuu\cdots$.

Recognizability is a central notion when dealing with fixed points of morphisms. It roughly means that any sufficiently long finite word that occurs in $\sigma^\omega(a)$ has a unique pre-image under σ, except for a prefix and a suffix of bounded length. Recognizability of a morphism is linked to the existence of long powers u^k in $\mathcal{L}(x)$ [13]. An infinite word $x \in A^{\mathbb{Z}}$ is said to be *k-power-free* if there is no non-empty word u such that u^k belongs to $\mathcal{L}(x)$. We refer, for example, to [1, 5, 7, 8]. A fundamental result concerning recognizability is due to Mossé, who proved that aperiodic primitive morphisms are recognizable [14, 15]. In this paper, we present a detailed proof of this result. This allows us to give a bound on the constant of recognizability.

2 Recognizability

Recognizability of a morphism deals with uniqueness of pre-images of words. More precisely, given a morphism σ, an admissible fixed point x of σ and a sufficiently long word $w \in \mathcal{L}(x)$, we would like to find some word $u \in \mathcal{L}(x)$ such that w appears in $\sigma(u)$ and such that any other word $u' \in \mathcal{L}(x)$ satisfying the same property has a large part in common with u. This large common part is the pre-image that the notion of recognizability is concerned with.

To find such a pre-image, it suffices to consider the pre-images of the letters. As a letter a can appear in several images $\sigma(b)$ and at different positions in $\sigma(b)$, we need to consider it in some *context* of length ℓ, i.e., to consider a word $uav \in \mathcal{L}(x)$ with $|u| = |v| = \ell$. We would like that the length of the context ensures the uniqueness of b and of the position $i \in \{0, 1, \ldots, |\sigma(b)| - 1\}$ such that $(\sigma(b))_i = a$. This length ℓ will be defined as the constant of recognizability in Definition 1.

Let us consider an example with the morphism τ defined by $\tau(0) = \tau(1) = 021$ and $\tau(2) = 0$. The letter 0 appears at the first position in the image of 0, 1 and 2. The contexts of length 4 of 0 are the words of length 9 that occur in $x = \tau^\omega(1 \cdot 0)$ and that have 0 for central letter, i.e.,

$$\begin{aligned} c_1 &= 0021\,0\,2100; \\ c_2 &= 0021\,0\,2102; \\ c_3 &= 0210\,0\,2102; \\ c_4 &= 1021\,0\,0210; \\ c_5 &= 1021\,0\,2100. \end{aligned}$$

In the contexts c_1, c_2 and c_5, the central occurrence of 0 is the first letter in the image $\tau(0)$. Indeed, the first occurrence of 0 in the word 00 can only be the image of 2 and the words 12, 22 and 20 do not occur in x. Therefore we necessarily have the following factorization for c_1:

$$c_1 = \underbrace{0}_{\tau(2)}\underbrace{021}_{\tau(1)}\underbrace{021}_{\tau(0)}\underbrace{0}_{\tau(2)}0.$$

Similar arguments show that in c_3 (resp., in c_4) the central occurrence of 0 is the first letter in the image $\tau(1)$ (resp., $\tau(2)$). We have

$$c_3 = \underbrace{021}_{\tau(0)}\underbrace{0}_{\tau(2)}\underbrace{021}_{\tau(1)}02 \qquad \text{and} \qquad c_4 = 1\underbrace{021}_{\tau(0)}\underbrace{0}_{\tau(2)}\underbrace{021}_{\tau(1)}0.$$

Let us now formalize the precise definition of recognizability. Let $x = (x_n)_{n\in\mathbb{Z}} \in A^{\mathbb{Z}}$ be an infinite word. Given two integers i, j with $i \leq j$, we let $x_{[i,j]}$ (resp., $x_{[i,j)}$) denote the factor $x_i x_{i+1} \cdots x_j$ (resp., $x_i x_{i+1} \cdots x_{j-1}$) $x_{[i,i)} = \varepsilon$, where ε is the empty word, i.e., the identity element of A^*.

Assume that the morphism $\sigma : A^* \to A^*$ is non-erasing and has an admissible fixed point $x = (x_n)_{n\in\mathbb{Z}} \in A^{\mathbb{Z}}$. For all $p \in \mathbb{N}$, we let $f_x^{(p)}$ denote the function

$$f_x^{(p)} : \mathbb{Z} \to \mathbb{Z}, i \mapsto f_x^{(p)}(i) = \begin{cases} |\sigma^p(x_{[0,i)})|, & \text{if } i > 0; \\ 0, & \text{if } i = 0; \\ -|\sigma^p(x_{[i,0)})|, & \text{if } i < 0. \end{cases}$$

We set $E(x, \sigma^p) = f_x^{(p)}(\mathbb{Z})$. When it is clear from the context, we write $f^{(p)}$ instead of $f_x^{(p)}$. Observe that σ being non-erasing, all functions $f_x^{(p)}$ are increasing.

Definition 1. We say that σ is *recognizable on x* if there exists some constant $L > 0$ such that for all $i, m \in \mathbb{Z}$,

$$(x_{[m-L,m+L]} = x_{[f^{(1)}(i)-L,f^{(1)}(i)+L]}) \Rightarrow (\exists j \in \mathbb{Z})((m = f^{(1)}(j)) \wedge (x_i = x_j)). \tag{1}$$

We say that the factor $x_{[f^{(1)}(i)-L,f^{(1)}(i)+L]}$ *uniquely determines the letter x_i*. By extension, we say that the factor $x_{[f^{(1)}(i)-L,f^{(1)}(j)+L]}$ uniquely determines the factor $x_{[i,j]}$.

The smallest L satisfying Equation (1) is called the *constant of recognizability* of σ for x. When σ is recognizable on all its admissible fixed points, we say that it is recognizable and its constant of recognizability is the greatest one.

In the example above, the arguments given for the contexts of the letter 0 can be similarly applied for the contexts of length 4 of the letters 1 and 2. Thus those contexts also have unique factorizations as images of τ. Therefore the constant of recognizability of τ on the admissible fixed point $\tau^\omega(1 \cdot 0)$ is at most 4. A careful inspection of the contexts of length 4 actually shows that the constant is at most 3 (no context of length 3 appears as a subword of two contexts of length 4 having different factorization). We can finally show that it cannot be equal to 2. Indeed, the word $21\,0\,21$ is a context of length 2 of 0 but appears as a subword of c_1 and c_3 that have different decompositions as images of τ. The constant of recognizability of τ is thus equal to 3.

The following result shows that any primitive aperiodic morphism is recognizable.

Theorem 2. *Let $\sigma : A^* \to A^*$ be an aperiodic primitive morphism and let $x \in A^{\mathbb{Z}}$ be an admissible fixed point of σ.*

1. *[14] There exists $M > 0$ such that, for all $i, m \in \mathbb{Z}$,*

$$x_{[f^{(1)}(i)-M,f^{(1)}(i)+M]} = x_{[m-M,m+M]} \Rightarrow m \in E(x, \sigma).$$

2. *[15] There exists $L > 0$ such that, for all $i, j \in \mathbb{Z}$,*

$$x_{[f^{(1)}(i)-L,f^{(1)}(i)+L]} = x_{[f^{(1)}(j)-L,f^{(1)}(j)+L]} \Rightarrow x_i = x_j.$$

By a careful reading of the proofs of Mossé's results, we can improve the statements as follows. The proof of Theorem 3 is given in Section 3. Given a morphism $\sigma : A^* \to A^*$, we define $|\sigma|$ and $\langle\sigma\rangle$ by, respectively,

$$|\sigma| = \max_{a \in A} |\sigma(a)| \qquad \text{and} \qquad \langle\sigma\rangle = \min_{a \in A} |\sigma(a)|.$$

Theorem 3. *Let $\sigma : A^* \to A^*$ be a morphism with an admissible fixed point $x \in A^{\mathbb{Z}}$. If x is k-power-free and if there is some constant N such that for all $n \in \mathbb{N}$, $|\sigma^n| \leq N\langle\sigma^n\rangle$, then σ is recognizable on x and its constant of recognizability for x is at most $R|\sigma^{dQ}| + |\sigma^d|$, where*

- *$R = \lceil N^2(k+1) + 2N \rceil$;*

- $Q = 1 + p_x(R)\left(\sum_{\frac{R}{N}\leq i\leq RN+2} p_x(i)\right)$;
- $d \in \{1, 2, \ldots, \mathrm{Card}\, A\}$ *is such that for all words* $u, v \in \mathcal{L}(x)$, *we have*

$$\sigma^{d-1}(u) \neq \sigma^{d-1}(v) \Rightarrow \forall n, \sigma^n(u) \neq \sigma^n(v).$$

Then, we give some computable bounds for N, R, k, Q and d in the case of primitive morphisms. These bounds are not sharp but can be expressed using only the cardinality of the alphabet and the maximal length $|\sigma|$. The proof is given in Section 4.

Theorem 4. *Each aperiodic primitive morphism* $\sigma : A^* \to A^*$ *that has an admissible fixed point* $x \in A^{\mathbb{Z}}$ *is recognizable on* x *and the constant of recognizability for* x *is at most*

$$2|\sigma|^{6(\mathrm{Card}\, A)^2 + 6(\mathrm{Card}\, A)|\sigma|^{28(\mathrm{Card}\, A)^2}} + |\sigma|^{(\mathrm{Card}\, A)}.$$

The bound given in the previous theorem is far from being sharp. When the morphism σ is injective on A, we can take $d = 1$ in Theorem 3 and the computation in the proof of Theorem 4 gives the bound

$$2|\sigma|^{6(\mathrm{Card}\, A)^2 + 6|\sigma|^{28(\mathrm{Card}\, A)^2}} + |\sigma|.$$

The notion of recognizability is also known as *circularity* in the terminology of *D0L-systems* [10]. Assume that $\sigma : A^* \to A^*$ is non-erasing (i.e., $\sigma(a) \neq \varepsilon$ for all $a \in A$) and that $a \in A$ is a letter such that the language $\mathrm{Fac}(\sigma, a)$ defined as the set of factors occurring in $\sigma^n(a)$ for some n is infinite. Given a word $u = u_1 \cdots u_{|u|} \in \mathrm{Fac}(\sigma, a)$, we say that a triple (p, v, q) is an *interpretation* of u if $v \in \mathrm{Fac}(\sigma, a)$ and $\sigma(v) = puq$. Two interpretations $(p, v, q), (p', v', q')$ are said to be *synchronized at position* k if there exist i, j such that $1 \leq i \leq |v|$, $1 \leq j \leq |v'|$ and

$$\sigma(v_1 \cdots v_i) = pu_1 \cdots u_k \quad \text{and} \quad \sigma(v'_1 \cdots v'_j) = p'u_1 \cdots u_k.$$

The word u has a *synchronizing point (at position k)* if all its interpretations are synchronized (at position k). The pair (σ, a) is said to be *circular* if σ is injective on $\mathrm{Fac}(\sigma, a)$ and if there is a constant C such that all words of length at least C have a synchronizing point. The smallest such C is called the *synchronizing delay* of σ. Thus, despite some considerations about whether we deal with fixed points or languages, recognizability and circularity are roughly the same notion and the synchronizing delay C is associated with the constant of recognizability L through $L \leq C \leq 2L + |\sigma| + 1$. Using the terminology of D0L-systems, Klouda and Medková obtained the following result which greatly improves our bounds, but for restricted cases.

Theorem 5 ([11])**.** *If* $\mathrm{Card}\, A = 2$ *and if* (σ, a) *is circular with* $\sigma : A^* \to A^*$ *a* k*-uniform morphism for some* $k \geq 2$, *then the synchronizing delay* C *of* (σ, a) *is bounded as follows, where* d *is the least divisor of* k *greater than 1:*

1. $C \leq 8$, *if* $k = 2$;
2. $C \leq k^2 + 3k - 4$, *if* k *is an odd prime number;*
3. $C \leq k^2\left(\frac{k}{d} - 1\right) + 5k - 4$, *otherwise.*

3 Proof of Theorem 3

As is the case in Mossé's original proof, the proof of Theorem 3 goes in two steps.

As a first step, we express the constant M of Theorem 2 in terms of the constants N, R, k and Q of Theorem 3. This is done in Proposition 7 with a proof following the lines of the proof of [12, Proposition 4.35]. The difference is that we take care of all the needed bounds to express the constant of recognizability.

As a second step, we show that the constant L of Theorem 2 can be taken equal to $M' + |\sigma^d|$, where d is defined as in Theorem 3 and M' is such that for all $i, m \in \mathbb{Z}$,

$$x_{[f^{(d)}(i)-M', f^{(d)}(i)+M']} = x_{[m-M,m+M]} \Rightarrow m \in E(x, \sigma^d).$$

We first start with the following lemma.

Lemma 6. *Let* $\sigma : A^* \to A^*$ *be a non-erasing morphism,* $u \in A^*$ *be a word and* n *be a positive integer. If* $v = v_0 \cdots v_{t+1} \in A^*$ *is a word of length* $t + 2$ *such that* $\sigma^n(v[1,t])$ *is a factor of* $\sigma^n(u)$, *and* $\sigma^n(u)$ *is a factor of* $\sigma^n(v)$, *then*

$$\frac{\langle \sigma^n \rangle}{|\sigma^n|}|u| - 2 \leq t \leq \frac{|\sigma^n|}{\langle \sigma^n \rangle}|u|.$$

Proof. Indeed, since $\sigma^n(v[1,t])$ is a factor of $\sigma^n(u)$ we have $t\langle \sigma^n \rangle \leq |\sigma^n(v_{[1,t]})| \leq |\sigma^n(u)| \leq |u||\sigma^n|$. Hence $t \leq |u||\sigma^n|/\langle \sigma^n \rangle$. Similarly, since $\sigma^n(u)$ is a factor of $\sigma^n(v)$, we have $|u| \leq (t+2)|\sigma^n|/\langle \sigma^n \rangle$. We thus have

$$|u|\frac{\langle \sigma^n \rangle}{|\sigma^n|} - 2 \leq t \leq |u|\frac{|\sigma^n|}{\langle \sigma^n \rangle}.$$

□

Proposition 7. *Let* $\sigma : A^* \to A^*$ *be a morphism with an admissible fixed point* $x \in A^{\mathbb{Z}}$. *Assuming that* x *is* k*-power-free and that there is some constant* N *such that for all* $n \in \mathbb{N}$, $|\sigma^n| \leq N\langle \sigma^n \rangle$, *we consider the constants*

- $R = \lceil N^2(k+1) + 2N \rceil$;
- $Q = 1 + p_x(R)\left(\sum_{\frac{R}{N} \leq i \leq RN+2} p_x(i)\right)$.

The constant $M = R|\sigma^Q|$ is such that for all $i, m \in \mathbb{Z}$,

$$x_{[f^{(1)}(i)-M, f^{(1)}(i)+M]} = x_{[m-M, m+M]} \Rightarrow m \in E(x, \sigma). \tag{2}$$

Proof. We follow the lines of the proof of [12, Theorem 2]. Obviously, if Equation (2) is true when we replace M by some $l \le M$, then it is true with M. Let us show such an l exists. We proceed by contradiction, assuming that for all $l \le M$, there exist i, j such that $x_{[i-l,i+l]} = x_{[j-l,j+l]}$ with $i \in E(x, \sigma)$ and $j \notin E(x, \sigma)$. For each integer p such that $0 < p \le Q$, we consider the integer $l_p = R|\sigma^p| \le M$. Let i_p and j_p be some integers such that

$$x_{[i_p-l_p, i_p+l_p]} = x_{[j_p-l_p, j_p+l_p]}, \qquad \text{with } i_p \in E(x, \sigma) \text{ and } j_p \notin E(x, \sigma).$$

We let r_p and s_p denote the smallest integers such that

$$\begin{aligned} \mathrm{Card}\left([i_p - r_p, i_p) \cap E(x, \sigma^p)\right) &= \left\lceil \frac{R}{2} \right\rceil; \\ \mathrm{Card}\left([i_p, i_p + s_p] \cap E(x, \sigma^p)\right) &= \left\lfloor \frac{R}{2} \right\rfloor + 1. \end{aligned}$$

There is an integer i'_p such that

$$f^{(p)}(i'_p) = i_p - r_p \qquad \text{and} \qquad f^{(p)}(i'_p + R) = i_p + s_p.$$

We set

$$u_p = x_{[i'_p, i'_p+R)}.$$

We have $\sigma^p(u_p) = x_{[i_p-r_p, i_p+s_p)}$.

Notice that any interval of length l_p contains at least $R-1$ elements of $E(x, \sigma^p)$. We thus have $i_p - l_p \le i_p - r_p \le i_p + s_p \le i_p + l_p$. Consequently we also have

$$x_{[j_p-r_p, j_p+s_p)} = \sigma^p(u_p). \tag{3}$$

However, $j_p - r_p$ does not need to belong to $E(x, \sigma^p)$. Let j'_p and t_p denote the unique integers such that

$$\begin{aligned} f^{(p)}(j'_p) < {} & j_p - r_p \le f^{(p)}(j'_p + 1); \\ f^{(p)}(j'_p + t_p + 1) \le {} & j_p + s_p < f^{(p)}(j'_p + t_p + 2). \end{aligned} \tag{4}$$

Consequently, the word $\sigma^p(x_{[j'_p+1, j'_p+t_p]})$ is a factor of $\sigma^p(u_p)$ and the word $\sigma^p(u_p)$ is a factor of $\sigma^p(x_{[j'_p, j'_p+t_p+1]})$. By Lemma 6, we have

$$R\frac{\langle \sigma^p \rangle}{|\sigma^p|} - 2 \le t_p \le R\frac{|\sigma^p|}{\langle \sigma^p \rangle}. \tag{5}$$

Hence

$$\frac{R}{N} - 2 \le t_p \le RN.$$

Let $v_p = x_{[j'_p, j'_p+t_p+1]}$. The number of possible pairs of words (u_p, v_p) is at most

$$p_x(R)\left(\sum_{\frac{R}{N} \le i \le RN+2} p_x(i)\right) < Q.$$

Therefore, there exist p and q in $[1, Q]$ such that $p < q$ and $(u_p, v_p) = (u_q, v_q)$. In particular, we also have $t_p = t_q$. We write

$$t = t_p, \qquad u = u_p, \qquad v = v_p, \qquad \tilde{v} = x_{[j'_p+1, j'_p+t]}.$$

Using the above notation we recall that we have

$$u = \quad x_{[i'_p, i'_p+R)} \quad = x_{[i'_q, i'_q+R)}; \tag{6}$$
$$v = \quad x_{[j'_p, j'_p+t+1]} \quad = x_{[j'_q, j'_q+t+1]}. \tag{7}$$

Let A_p, B_p, A_q and B_q be the words

$$\begin{aligned} A_p &= x_{[j_p - r_p, f^{(p)}(j'_p+1))}; \\ B_p &= x_{[f^{(p)}(j'_p+t+1), j_p+s_p)}; \\ A_q &= x_{[j_q - r_q, f^{(q)}(j'_q+1))}; \\ B_q &= x_{[f^{(q)}(j'_q+t+1), j_q+s_q)}. \end{aligned}$$

We thus have

$$x_{[j_p - r_p, j_p+s_p)} = A_p \sigma^p(\tilde{v}) B_p \qquad \text{and} \qquad x_{[j_q - r_q, j_q+s_q)} = A_q \sigma^q(\tilde{v}) B_q, \tag{8}$$

with, using (4),

$$\max\{|A_p|, |B_p|\} \le |\sigma^p| \qquad \text{and} \qquad \max\{|A_q|, |B_q|\} \le |\sigma^q|. \tag{9}$$

From (3) and (8), we obtain

$$\sigma^{q-p}(A_p)\sigma^q(\tilde{v})\sigma^{q-p}(B_p) = A_q \sigma^q(\tilde{v}) B_q.$$

We claim that

$$A_q = \sigma^{q-p}(A_p) \quad \text{(and hence } B_q = \sigma^{q-p}(B_p)\text{)}. \tag{10}$$

If not, the word $\sigma^q(\tilde{v})$ has a prefix which is a power w^r with $r = \left\lfloor \frac{|\sigma^q(\tilde{v})|}{||A_q| - |\sigma^{q-p}(A_p)||} \right\rfloor$. Since, using (5) and (9),

$$|\sigma^q(\tilde{v})| \ge t\langle \sigma^q \rangle \ge \left(\frac{R}{N} - 2\right)\langle \sigma^q \rangle \qquad \text{and} \qquad ||A_q| - |\sigma^{q-p}(A_p)|| \le |\sigma^q|,$$

we deduce from the choice of R that $r \geq k+1$, which contradicts the definition of k. We thus have $A_q = \sigma^{q-p}(A_p)$ and $B_q = \sigma^{q-p}(B_p)$.

We now show that the elements of $E(x,\sigma)$ occur in the two intervals $[i_q - r_q, i_q + s_q]$ and $[j_q - r_q, j_q + s_q]$ at the same relative positions, i.e.,

$$[j_q - r_q, j_q + s_q] \cap E(x,\sigma) = ([i_q - r_q, i_q + s_q] \cap E(x,\sigma)) - (i_q - j_q). \tag{11}$$

This will contradict the fact that i_q belongs to $E(x,\sigma)$ and j_q does not.

By (7), we have

$$\sigma^p(v) = x_{[f^{(p)}(j'_p), f^{(p)}(j'_p+t+2))} = x_{[f^{(p)}(j'_q), f^{(p)}(j'_q+t+2))}.$$

Since $\sigma^p(u)$ is a factor of $\sigma^p(v)$, we deduce from (4) that there exists $m_q \in \mathbb{Z}$ such that

$$f^{(p)}(j'_q) < m_q - r_p < m_q + s_p < f^{(p)}(j'_q + t + 2) \tag{12}$$

and

$$x_{[m_q - r_p, m_q + s_p)} = \sigma^p(u) = A_p \sigma^p(\tilde{v}) B_p.$$

By applying σ^{q-p}, we obtain

$$x_{[f^{(q-p)}(m_q - r_p), f^{(q-p)}(m_q + s_p))} = A_q \sigma^q(\tilde{v}) B_q,$$

and, from (12),

$$f^{(q)}(j'_q) < f^{(q-p)}(m_q - r_p) < f^{(q-p)}(m_q + s_p) < f^{(q)}(j'_q + t + 2).$$

As we also have

$$x_{[j_q - r_q, j_q + s_q)} = A_q \sigma^q(\tilde{v}) B_q$$

with, by (4),

$$f^{(q)}(j'_q) < j_q - r_q \leq f^{(q)}(j'_q + 1) \leq f^{(q)}(j'_q + t + 1) \leq j_q + s_q < f^{(q)}(j'_q + t + 2),$$

we apply the same argument as to show (10) and get $j_q - r_q = f^{(q-p)}(m_q - r_p)$ (hence $j_q + s_q = f^{(q-p)}(m_q + s_p)$). We thus get that $j_q - r_q$ belongs to $E(x, \sigma^{q-p}) \subset E(x,\sigma)$. Since we also have

$$\begin{aligned} x_{[f^{(1)^{-1}}(j_q - r_q), f^{(1)^{-1}}(j_q + s_q))} &= \sigma^{q-p-1}(x_{[m_q - r_p, m_q + s_p)}) = \sigma^{q-p-1}(A_p \sigma^p(\tilde{v}) B_p), \\ x_{[f^{(1)^{-1}}(i_q - r_q), f^{(1)^{-1}}(i_q + s_q))} &= \sigma^{q-1}(x_{[i'_q, i'_q + R)}) = \sigma^{q-p-1}(A_p \sigma^p(\tilde{v}) B_p), \end{aligned}$$

we get

$$x_{[f^{(1)^{-1}}(j_q - r_q), f^{(1)^{-1}}(j_q + s_q))} = x_{[f^{(1)^{-1}}(i_q - r_q), f^{(1)^{-1}}(i_q + s_q))},$$

with $j_q - r_q, i_q - r_q$ belonging to $E(x,\sigma)$. By applying σ to these two words, we thus obtain (11), which ends the proof. $\square$

Proposition 7 gives a more precise statement than Item 1 in Theorem 2. It also makes Item 2 more precise in the case of morphisms that are injective on A by taking $L = M + |\sigma|$. For non-injective morphisms, a key argument in Mossé's original proof is to prove the existence of an integer d such that for all $a, b \in A$, if $\sigma^n(a) = \sigma^n(b)$ for some n, then $\sigma^d(a) = \sigma^d(b)$. Theorem 8 below ensures that we can take $d = (\mathrm{Card}\, A) - 1$.

Theorem 8 ([6, Theorem 3])**.** *Let $\sigma : A^* \to A^*$ be a morphism such that $\sigma(A) \neq \{\varepsilon\}$. For all words $u, v \in A^*$, we have*

$$\sigma^{(\mathrm{Card}\, A)-1}(u) \neq \sigma^{(\mathrm{Card}\, A)-1}(v) \Rightarrow \forall n, \sigma^n(u) \neq \sigma^n(v).$$

We now give the proof of Item 2 in Theorem 2.

Proposition 9. *Let $\sigma : A^* \to A^*$ be a morphism with an admissible fixed point $x \in A^{\mathbb{Z}}$. Let $d \in \{1, 2, \ldots, \mathrm{Card}\, A\}$ be such that for all words $u, v \in \mathcal{L}(x)$,*

$$\sigma^{d-1}(u) \neq \sigma^{d-1}(v) \Rightarrow \forall n, \sigma^n(u) \neq \sigma^n(v).$$

If M is a constant such that for all $i, m \in \mathbb{Z}$,

$$x_{[f^d(i)-M, f^d(i)+M]} = x_{[m-M, m+M]} \Rightarrow m \in E(x, \sigma^d),$$

then σ is recognizable on x and its constant of recognizability for x is at most $M + |\sigma^d|$.

Proof. Let $i, m \in \mathbb{Z}$ such that

$$x_{[f^{(1)}(i)-M-|\sigma^d|, f^{(1)}(i)+M+|\sigma^d|]} = x_{[m-M-|\sigma^d|, m+M+|\sigma^d|]}.$$

By the definition of M, there exists $j \in \mathbb{Z}$ such that $m = f^{(1)}(j)$. Our goal is to show that $x_i = x_j$.

There exists $k \in \mathbb{Z}$ such that

$$f^{(1)}(i) - |\sigma^d| < f^{(d)}(k) \leq f^{(1)}(i) < f^{(d)}(k+1) \leq f^{(1)}(i) + |\sigma^d|.$$

In particular, this implies that $f^{(d-1)}(k) \leq i < f^{(d-1)}(k+1)$.

Consider $c = f^{(1)}(i) - f^{(d)}(k)$ and $d = f^{(d)}(k+1) - f^{(1)}(i)$. We have

$$\begin{aligned} x_{[f^{(d)}(k)-M, f^{(d)}(k)+M]} &= x_{[f^{(1)}(j)-c-M, f^{(1)}(j)-c+M]}; \\ x_{[f^{(d)}(k+1)-M, f^{(d)}(k+1)+M]} &= x_{[f^{(1)}(j)+d-M, f^{(1)}(j)+d+M]}. \end{aligned}$$

By the definition of M, there exists $l \in \mathbb{Z}$ such that

$$f^{(d)}(l) = f^{(1)}(j) - c \quad \text{and} \quad f^{(d)}(l+1) = f^{(1)}(j) + d.$$

We thus have $f^{(d-1)}(l) \leq j < f^{(d-1)}(l+1)$, and,

$$x_{[f^{(d)}(k), f^{(d)}(k+1))} = x_{[f^{(d)}(l), f^{(d)}(l+1))}.$$

Hence $\sigma^d(x_k) = \sigma^d(x_l)$. By the definition of d, we also have $\sigma^{d-1}(x_k) = \sigma^{d-1}(x_l)$. Hence

$$x_{[f^{(d-1)}(k), f^{(d-1)}(k+1))} = x_{[f^{(d-1)}(l), f^{(d-1)}(l+1))}.$$

Since we have $f^{(1)}(i) - f^{(d)}(k) = f^{(1)}(j) - f^{(d)}(l)$, we also have $i - f^{(d-1)}(k) = j - f^{(d-1)}(l)$. Hence $x_i = x_j$. □

4 Proof of Theorem 4

In this section, we show that the constants appearing in Theorem 3 can all be bounded by some computable constants. In all what follows, we assume that $\sigma : A^* \to A^*$ is a primitive morphism. By taking a power of σ if needed, we can assume that it has an admissible fixed point $x \in A^{\mathbb{Z}}$. Furthermore, we have $\mathcal{L}(x) = \mathcal{L}(y)$ for all admissible fixed points y of σ. We let $\mathcal{L}(\sigma)$ denote this set and we write $\mathcal{L}_n(\sigma) = \mathcal{L}(\sigma) \cap A^2$. The constants appearing in Theorem 3 are thus the same whatever the admissible fixed point we consider and the morphism is recognizable.

With the morphism σ, one associates its *incidence matrix* M_σ defined by $(M_\sigma)_{a,b} = |\sigma(b)|_a$, where $|u|_a$ denotes the number of occurrences of the letter a in the word u. If σ is a primitive morphism, it is well known that for all $a \in A$ there exists a constant c_a such that $|\sigma^n(a)| \leq c_a \alpha^n$ for all n, where α is the dominant eigenvalue of M_σ (see [2] for a detailed study of $|\sigma^n(a)|$).

Lemma 10 ([9]). *A $d \times d$ matrix M is primitive if and only if there is an integer $k \leq d^2 - 2d + 2$ such that M^k contains only positive entries.*

Given an infinite word $x \in A^{\mathbb{Z}}$ and a word $u \in \mathcal{L}(x)$, a *return word* to u in x is a word r such that ru belongs to $\mathcal{L}(x)$, u is a prefix of ru and ru contains exactly two occurrences of u. The infinite word x is *linearly recurrent* if it is *recurrent* (all words in $\mathcal{L}(x)$ appear infinitely many times in x) and there exists some constant K such that for all $u \in \mathcal{L}(x)$, any return word to u has length at most $K|u|$.

The next two results give bounds on the constants appearing in Theorem 3.

Theorem 11 ([5]). *If $x \in A^{\mathbb{Z}}$ is a aperiodic and linearly recurrent sequence for the constant K, then x is $(K+1)$-power-free and $p_x(n) \leq Kn$ for all n.*

Proposition 12 ([4]). *Let $\sigma : A^* \to A^*$ be an aperiodic primitive morphism and x be one of its admissible fixed points. Then, for all n, we have*

$$|\sigma^n| \leq |\sigma|^{(\mathrm{Card}\, A)^2} \langle \sigma^n \rangle.$$

Furthermore, x is linearly recurrent for some constant

$$K_\sigma < |\sigma|^{4(\mathrm{Card}\, A)^2}.$$

Proof. Durand [4] showed that the constant of linear recurrence K_σ is at most equal to $TN|\sigma|$, where

- N is a constant such that $|\sigma^n| \leq N \langle \sigma^n \rangle$ for all n;
- T is the maximal length of a return word to a word in $\mathcal{L}_2(\sigma)$.

Here we only prove that $N \leq |\sigma|^{(\mathrm{Card}\, A)^2}$ and $T \leq 2|\sigma|^{2(\mathrm{Card}\, A)^2}$. The constant of linear recurrence is thus at most $2|\sigma|^{1+3(\mathrm{Card}\, A)^2} < |\sigma|^{4(\mathrm{Card}\, A)^2}$.

Let us write $d = \mathrm{Card}\, A$. By Lemma 10, the matrix $M_\sigma^{d^2}$ contains only positive entries. For all $n \geq 0$ and all $a \in A$, we have $|\sigma^{n+d^2}(a)| = \sum_{b \in A} |\sigma^{d^2}(a)|_b |\sigma^n(b)| \geq |\sigma^n|$. Since this is true for all a, we get $|\sigma^n| \leq \langle \sigma^{n+d^2} \rangle \leq |\sigma^{d^2}| \langle \sigma^n \rangle$, so $N \leq |\sigma^{d^2}|$.

Let $a \in A$ such that σ is prolongable on a. Thus for all n, any word that occurs in $\sigma^n(a)$ also occurs in $\sigma^{n+1}(a)$. Let us show that for all $n > d^2$, any word $u \in \mathcal{L}_2(\sigma)$ occurs in $\sigma^n(a)$. For all n, the words of $\mathcal{L}_2(\sigma)$ that occur in $\sigma^{n+1}(a)$ occur in images under σ of the words of $\mathcal{L}_2(\sigma)$ that occur in $\sigma^n(a)$. As any word occurring in $\sigma^n(a)$ also occurs in $\sigma^{n+1}(a)$, the words of $\mathcal{L}_2(\sigma)$ that occur in $\sigma^{n+1}(a)$ are those that occur in $\sigma^n(a)$ together with those occurring in the images under σ of these words. Thus, if there is a word of $\mathcal{L}_2(\sigma)$ that does not occur in $\sigma^n(a)$, there is a sequence $(u_1, u_2, \ldots, u_n)$ of words in $\mathcal{L}_2(\sigma)$ such that for all $i \leq n$, u_i occurs in $\sigma^i(a)$ and does not occur in $\sigma^{i-1}(a)$. Hence all words $u_1, \ldots, u_n$ are distinct. For $n > d^2$, this is a contradiction since there are at most d^2 words in $\mathcal{L}_2(\sigma)$. Thus, for any letter $b \in A$, all words $u \in \mathcal{L}_2(\sigma)$ occur in $\sigma^{2d^2}(b)$. We deduce that $T \leq 2|\sigma^{2d^2}|$. □

Proof of Theorem 4. We just have to carry out the computation. Using Theorem 11, Proposition 12 and the notation of Theorem 3, we can take $d = \mathrm{Card}\, A$ and we successively have

$$\begin{aligned}
k &\leq 1 + K_\sigma \leq |\sigma|^{4d^2};\\
N &\leq |\sigma|^{d^2};\\
R &= \lceil N^2(k+1) + 2N \rceil \leq |\sigma|^{2d^2}(|\sigma|^{4d^2} + 1) + 2|\sigma|^{d^2} \leq 2|\sigma|^{6d^2};\\
Q &= 1 + p_x(R) \left(\sum_{\frac{R}{N} \leq i \leq RN+2} p_x(i) \right) \leq K_\sigma 2|\sigma|^{6d^2} \left(\sum_{0 \leq i \leq 2+2|\sigma|^{7d^2}} iK_\sigma \right) \leq 6|\sigma|^{28d^2}.
\end{aligned}$$

We finally get that the constant of recognizability of σ is at most

$$2|\sigma|^{6d^2}|\sigma|^{6d|\sigma|^{28d^2}} + |\sigma|^d = 2|\sigma|^{6d^2+6d|\sigma|^{28d^2}} + |\sigma|^d.$$

□

5 Recognizability of powers of morphisms

Theorem 3 gives a general bound on the constant of recognizability of recognizable morphisms. Powers of a recognizable morphism σ are obviously recognizable. However, the bound given in Theorem 3 applied to σ^p is far from being optimal. In this section we give two results concerning the constant of recognizability of σ^p.

Proposition 13. *If $\sigma : A^* \to A^*$ is recognizable on $x \in A^{\mathbb{Z}}$ and if L is the constant of recognizability of σ for x, then for all $p > 0$, σ^p is recognizable on x and its constant of recognizability for x is at most $L \sum_{i=0}^{p-1} |\sigma^i|$.*

Proof. The result holds by induction on $p > 0$. The case $p = 1$ is trivial. Let us assume that the result holds for $p-1$ and let us prove it for p. The infinite word x is obviously an admissible fixed point of σ^p. With $L_p = L\sum_{i=0}^{p-1}|\sigma^i|$, let us show that for all $i \in \mathbb{Z}$, the word

$$x_{[f^{(p)}(i)-L_p, f^{(p)}(i)+L_p]}$$

uniquely determines the letter x_i.

Let m and M be, respectively, the smallest and the largest integer such that

$$f^{(p)}(i) - L_p \leq f^{(p-1)}(m) - L_{p-1} < f^{(p-1)}(M) + L_{p-1} \leq f^{(p)}(i) + L_p \tag{13}$$

and let us show that

$$m \leq f^{(1)}(i) - L < f^{(1)}(i) + L \leq M. \tag{14}$$

We consider the first inequality with $m > 0$; the other cases are similar. The integer m being the smallest one satisfying (13), we have

$$f^{(p-1)}(m-1) - L_{p-1} < f^{(p)}(i) - L_p.$$

Since we have $f^{(p-1)}(m) = f^{(p-1)}(m-1) + |\sigma^{p-1}(x_{m-1})|$ and $L_p - L_{p-1} = L|\sigma^{p-1}|$, we get

$$f^{(p)}(i) > f^{(p-1)}(m) + L|\sigma^{p-1}| - |\sigma^{p-1}(x_{m-1})| \geq f^{(p-1)}(m) + (L-1)|\sigma^{p-1}|. \tag{15}$$

Assume by contradiction that $f^{(1)}(i) < m + L$, hence that $f^{(1)}(i) \leq m + L - 1$. The function $f^{(p-1)}$ being increasing, we have

$$f^{(p-1)}(f^{(1)}(i)) = f^{(p)}(i) \leq f^{(p-1)}\left(m + L - 1\right) = f^{(p-1)}(m) + |\sigma^{p-1}(x_{[m,m+L-1)})|.$$

We thus get

$$f^{(p)}(i) \leq f^{(1)}(m) + (L-1)|\sigma^{p-1}|,$$

which contradicts (15).

Let us now finish the proof of the result. Using the induction hypothesis, the word $x_{[f^{(p-1)}(m)-L_{p-1}, f^{(p-1)}(M)+L_{p-1}]}$ uniquely determines $x_{[m,M]}$. As $x_{[f^{(p-1)}(m)-L_{p-1}, f^{(p-1)}(M)+L_{p-1}]}$ is a factor of $x_{[f^{(p)}(i)-L_p, f^{(p)}(i)+L_p]}$ by (13) and $x_{[f^{(1)}(i)-L, f^{(1)}(i)+L]}$ is a factor of $x_{[m,M]}$ by (14), the word $x_{[f^{(p)}(i)-L_p, f^{(p)}(i)+L_p]}$ uniquely determines $x_{[f^{(1)}(i)-L, f^{(1)}(i)+L]}$. We conclude by the definition of recognizability. □

Corollary 14. *If $\sigma : A^* \to A^*$ is aperiodic primitive and if L is its constant of recognizability on $x \in A^{\mathbb{Z}}$, then σ^p is recognizable on x and its constant of recognizability for x is at most $LC\frac{\alpha^p - 1}{\alpha - 1}$, where α is the dominant eigenvalue of M_σ and C is a constant such that $|\sigma^p| \leq C\alpha^p$ for all p. In particular, the constant C can be taken equal to*

$$\frac{\max_{1 \leq k \leq \mathrm{Card}\, A} y_k}{\min_{1 \leq k \leq \mathrm{Card}\, A} y_k},$$

where $y = (y_k)_{1 \leq k \leq \mathrm{Card}\, A}$ is a positive eigenvector of M_σ.

Proof. The proof directly follows from [9, Corollary 8.1.33]. □

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