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Some Formulae for Products of Geometric Polynomials with Applications

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Abstract

In this paper, we evaluate sums and integrals of products of two geometric polynomials and obtain new explicit formulas for geometric polynomials and numbers. As a consequence of these results, we give new explicit formulas for p-Bernoulli numbers, Apostol-Bernoulli functions, and integrals of products of two Apostol-Bernoulli functions.

1 Introduction

Let $\left\{ {n \atop k} \right\}$ be the Stirling numbers of the second kind [16]. Geometric polynomials are defined by [33]

$$w_n(y) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} k! y^k. \tag{1}$$

They have the exponential generating function

$$\frac{1}{1 - y(e^t - 1)} = \sum_{n=0}^{\infty} w_n(y) \frac{t^n}{n!}, \tag{2}$$

and are related to the geometric series by [5]

$$\left(y\frac{d}{dy}\right)^m \frac{1}{1-y} = \sum_{k=0}^{\infty} k^m y^k = \frac{1}{1-y} w_m\left(\frac{y}{1-y}\right), \quad |y| < 1.$$

In addition, the following recurrence relation holds for the geometric polynomials [13]:

$$w_{n+1}(y) = y\frac{d}{dy}\big(w_n(y) + yw_n(y)\big). \tag{3}$$

The nth geometric number (ordered Bell number, or Fubini number, or preferred arrangement number) [12, 17, 33], w_n, is defined by

$$w_n(1) = w_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} k!, \tag{4}$$

and counts all the possible set partitions of an n element set such that the order of the blocks matters. Besides with this combinatorial property, these numbers are seen in the evaluation of the following series

$$\sum_{k=0}^{\infty} \frac{k^n}{2^k} = 2w_n. \tag{5}$$

In the literature, numerous identities concerned with these polynomials and numbers were obtained [5, 6, 7, 8, 14, 28] and some generalizations were given [5, 15, 20, 29].

The sums of products of various polynomials and numbers with or without binomial coefficients have been studied [2, 10, 21, 25, 26, 32, 34]. One of the famous results is [19, Eq. 50.11.2]

$$\sum_{k=0}^{n} \binom{n}{k} B_k(x) B_{n-k}(y) = (1-n) B_n(x+y-1) + (x+y-1) nB_{n-1}(x+y-1), \tag{6}$$

where $B_n(x)$ is the nth Bernoulli polynomial. In addition, the integrals of products of various polynomials and functions have been studied [3, 6, 11, 24, 27]. For example, the following interesting integral for a product of two Bernoulli polynomials appears in the book by Nörlund [30, p. 31]: For all $k+m \geq 2$,

$$\int_0^1 B_k(x) B_m(x)\, dx = \frac{k!m!}{(k+m)!} B_{k+m}. \tag{7}$$

Here, B_n is the nth Bernoulli number defined by $B_n = B_n(0)$.

The main purpose of this paper is to obtain analogues of (6) and (7) for geometric polynomials, which generalize the binomial formulas [13]

$$\sum_{k=0}^{n}\binom{n}{k}w_k = 2w_n, \quad n>0, \tag{8}$$

$$2\sum_{k=0}^{n}\binom{n}{k}(-1)^k w_k = (-1)^n w_n + 1, \quad n\geq 0, \tag{9}$$

and the integral [22]

$$\int_{-1}^{0} w_n(y)\,dy = B_n, \quad n>0. \tag{10}$$

The paper is outlined as follows. We derive sums of products of two geometric polynomials and explicit formulas for these polynomials and numbers in Section 2. In Section 3, we give integrals of products of two geometric polynomials. As an application of this result, an explicit formula for p-Bernoulli numbers is obtained. Finally, in Section 4, we give a new explicit formula and integrals of products of Apostol-Bernoulli functions.

2 Sums of products of geometric polynomials

In this section, we define two variable geometric polynomials and obtain some basic properties which give us new formulas for $w_n(y)$. Moreover, we consider the sums of products of two geometric polynomials.

Two variable geometric polynomials are defined by means of the following generating function:

$$\sum_{n=0}^{\infty} w_n(x;y)\frac{t^n}{n!} = \frac{e^{xt}}{1-y(e^t-1)}. \tag{11}$$

As some special cases of (11), we have

$$w_n(0;y) = w_n(y) \text{ and } w_n(0;1) = w_n. \tag{12}$$

We can rewrite (11) as

$$\begin{aligned}\sum_{n=0}^{\infty} w_n(x;y)\frac{t^n}{n!} &= \frac{1}{1-y(e^t-1)}e^{xt}\\ &= \sum_{n=0}^{\infty} w_n(y)\frac{t^n}{n!}\sum_{n=0}^{\infty} x^n\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\binom{n}{k}w_k(y)\,x^{n-k}\Big)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ yields

$$w_n(x;y)=\sum_{k=0}^{n}\binom{n}{k}w_k(y)\,x^{n-k}. \tag{13}$$

Besides, from (11) we have

$$\begin{aligned}\sum_{n=0}^{\infty}\bigl(w_n(x+1;y)-w_n(x;y)\bigr)\frac{t^n}{n!}&=\frac{e^{xt}\left(e^t-1\right)}{1-y\left(e^t-1\right)}\\&=\frac{1}{y}\Bigl(\frac{e^{xt}}{1-y\left(e^t-1\right)}-e^{xt}\Bigr)\\&=\frac{1}{y}\sum_{n=0}^{\infty}\bigl(w_n(x;y)-x^n\bigr)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ gives

$$yw_n(x+1;y)=(1+y)\,w_n(x;y)-x^n. \tag{14}$$

Thus, setting $x=0$ and $x=-1$ in (14), we find

$$yw_n(1;y)=(1+y)\,w_n(y)\,,\quad n>0, \tag{15}$$

$$(1+y)\,w_n(-1;y)=yw_n(y)+(-1)^n\,,\quad n\geq 0, \tag{16}$$

respectively. Combining these relations with (13) gives equations (8) and (9) which were obtained by using Euler-Seidel matrix method in [13].

Now, we want to give the generalization of the binomial formula (8). Derivative of (11) can be written as

$$\frac{\partial}{\partial t}\left(\frac{e^{xt}}{1-y\left(e^t-1\right)}\right)=\frac{xe^{xt}}{1-y\left(e^t-1\right)}+\frac{ye^t}{1-y\left(e^t-1\right)}\frac{e^{xt}}{1-y\left(e^t-1\right)}.$$

Taking $x=x_1+x_2-1$ leads to

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{e^{xt}}{1-y\left(e^t-1\right)}\right)&=\sum_{n=0}^{\infty}w_{n+1}(x_1+x_2-1;y)\,\frac{t^n}{n!},\\\frac{xe^{xt}}{1-y\left(e^t-1\right)}&=(x_1+x_2-1)\sum_{n=0}^{\infty}w_n(x_1+x_2-1;y)\,\frac{t^n}{n!},\end{aligned}$$

and

$$\begin{aligned}\frac{ye^t}{1-y\left(e^t-1\right)}\frac{e^{xt}}{1-y\left(e^t-1\right)}&=y\left(\sum_{n=0}^{\infty}w_n(x_1;y)\,\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}w_n(x_2;y)\,\frac{t^n}{n!}\right)\\&=y\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}w_k(x_1;y)\,w_{n-k}(x_2;y)\,\frac{t^n}{n!}.\end{aligned}$$

By equating the coefficients of $\frac{t^n}{n!}$, we get

$$y\sum_{k=0}^{n}\binom{n}{k}w_k\left(x_1;y\right)w_{n-k}\left(x_2;y\right)=w_{n+1}\left(x_1+x_2-1;y\right)-\left(x_1+x_2-1\right)w_n\left(x_1+x_2-1;y\right).$$

For $x_1=x_2=0$ in the above equation, by using (16) gives the sums of products of the geometric polynomials.

Theorem 1. *For $n\geq 0$,*

$$(y+1)\sum_{k=0}^{n}\binom{n}{k}w_k\left(y\right)w_{n-k}\left(y\right)=w_{n+1}\left(y\right)+w_n\left(y\right). \tag{17}$$

When $y=1$ this becomes

$$2\sum_{k=0}^{n}\binom{n}{k}w_kw_{n-k}=w_{n+1}+w_n. \tag{18}$$

We note that Boyadzhiev and Dil [9, Proposition 9] proved a more general form of Theorem 1 by different way.

Now, we investigate the sums of products of the geometric polynomials for different y values in the following theorem.

Theorem 2. *For $n\geq 0$ and $y_1\neq y_2$,*

$$\sum_{k=0}^{n}\binom{n}{k}w_k\left(y_1\right)w_{n-k}\left(y_2\right)=\frac{y_2w_n\left(y_2\right)-y_1w_n\left(y_1\right)}{y_2-y_1}. \tag{19}$$

Proof. We write

$$\begin{aligned}&\frac{e^{x_1t}}{\left(1-y_1\left(e^t-1\right)\right)}\frac{e^{x_2t}}{\left(1-y_2\left(e^t-1\right)\right)}\\&\qquad=\frac{y_2}{y_2-y_1}\frac{e^{(x_1+x_2)t}}{1-y_2\left(e^t-1\right)}-\frac{y_1}{y_2-y_1}\frac{e^{(x_1+x_2)t}}{1-y_1\left(e^t-1\right)}.\end{aligned} \tag{20}$$

Using the same method as in the proof of Theorem 1, we have

$$\sum_{k=0}^{n}\binom{n}{k}w_k\left(x_1;y_1\right)w_{n-k}\left(x_2;y_2\right)=\frac{y_2w_n\left(x_1+x_2;y_2\right)-y_1w_n\left(x_1+x_2;y_1\right)}{y_2-y_1}.$$

Setting $x_1=x_2=0$ in the above equation gives the desired equation. □

As we know, for $y=1$, geometric polynomials reduce to geometric numbers. We now point out (see (24)) that geometric numbers also arise for other values of y. If we take $y-1$ in place of y in (11), we have

$$w_n(x;y-1)=(-1)^n w_n(1-x;-y). \tag{21}$$

Setting $x=0$ in the above equation and using the relation (15), we have a reflection formula

$$w_n(y)=(-1)^n \frac{y}{y+1} w_n(-y-1), \quad n>0. \tag{22}$$

Therefore, employing (1) gives a new explicit formula for geometric polynomials as in the following theorem.

Theorem 3. *For $n>0$, we obtain*

$$w_n(y)=y\sum_{k=1}^{n}\left\{\begin{matrix} n \\ k \end{matrix}\right\}(-1)^{n+k} k!\,(y+1)^{k-1}. \tag{23}$$

Note that, when $y=1$ (23) reduces to [18, Theorem 4.2]. Moreover, from (22), we get two conclusions

$$w_{2k}\left(\frac{-1}{2}\right)=0 \text{ and } w_n(-2)=(-1)^n 2w_n. \tag{24}$$

The first part of (24) was given in [9, Corollary 17]. If we take $y_1=-2$ and $y_2=1$ in (19) and use the second part of (24), we obtain the alternating sums of products of geometric numbers.

Corollary 4. *For $n>0$, we have*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k w_k w_{n-k}=\begin{cases} 0, & \text{if } n \text{ is odd;} \\ \frac{4}{3}w_n, & \text{if } n \text{ is even.} \end{cases}$$

Finally, we obtain a new explicit formula for geometric polynomials and numbers in the following theorem.

Theorem 5. *For $y\neq\frac{-1}{2}$ and $n\geq 0$,*

$$w_n(y)=\sum_{k=0}^{n}\left\{\begin{matrix} n \\ k \end{matrix}\right\}k!y^k\frac{\left(2^{n+1}(y+1)y^k+(-1)^{k+1}\right)}{(2y+1)^{k+1}}. \tag{25}$$

When $y=1$ this becomes

$$w_n=\sum_{k=0}^{n}\left\{\begin{matrix} n \\ k \end{matrix}\right\}k!\frac{\left(2^{n+2}+(-1)^{k+1}\right)}{3^{k+1}}, \quad n\geq 0. \tag{26}$$

When $y=-2$ this becomes

$$w_n=\sum_{k=0}^{n}(-1)^{n-k}\left\{\begin{matrix} n \\ k \end{matrix}\right\}k!\frac{2^{k-1}\left(2^{n+k+1}+1\right)}{3^{k+1}}, \quad n>0. \tag{27}$$

Proof. If we take $\frac{1}{y^2-1}$ in place of y in (2), we arrive at

$$\frac{1}{1-\frac{1}{y^2-1}\left(e^{2t}-1\right)}=\frac{y^2-1}{2y^2}\left(\frac{1}{y-e^t}+\frac{1}{y+e^t}\right). \tag{28}$$

Each of the functions in the above equation can be written as

$$\frac{1}{1-\frac{1}{y^2-1}\left(e^{2t}-1\right)}=\sum_{n=0}^{\infty}2^n w_n\left(\frac{1}{y^2-1}\right)\frac{t^n}{n!}, \tag{29}$$

$$\frac{1}{y-e^t}=\frac{y}{y-1}\sum_{n=0}^{\infty}w_n\left(\frac{1}{y-1}\right)\frac{t^n}{n!}, \tag{30}$$

$$\frac{1}{y+e^t}=\frac{y}{y+1}\sum_{n=0}^{\infty}w_n\left(\frac{-1}{y+1}\right)\frac{t^n}{n!}. \tag{31}$$

By equating the coefficients of $\frac{t^n}{n!}$, we have

$$w_n\left(y\right)=2^{n+1}\left(1+y\right)w_n\left(\frac{y^2}{1+2y}\right)-\left(1+2y\right)w_n\left(-y\right). \tag{32}$$

Finally, using (1) in the right hand side of the above equation yields (25). □

3 Integrals of products of geometric polynomials

In this section, we deal with an integral for a product of two geometric polynomials. First we need following lemmas.

Lemma 6. *For all $k \geq 0$ and $n \geq 1$, we have*

$$\int_{-1}^{0}y^k w_n\left(y\right)dy=\frac{(-1)^k}{k!}\sum_{j=0}^{k}\begin{bmatrix}k+1\\j+1\end{bmatrix}B_{n+j}, \tag{33}$$

where $\begin{bmatrix}n\\k\end{bmatrix}$ is the Stirling number of the fist kind [16].

Proof. We prove (33) by induction on k. The case $k=0$ of (33) is known from (10). If we integrate both sides of (3) with respect to y from -1 to 0 and apply integration by parts, we have

$$\begin{aligned}\int_{-1}^{0}w_{n+1}\left(y\right)dy&=\int_{-1}^{0}y\frac{d}{dy}\big(w_n\left(y\right)+yw_n\left(y\right)\big)dy\\&=\left[y\left(w_n\left(y\right)+yw_n\left(y\right)\right)\right]_{-1}^{0}-\int_{-1}^{0}\big(w_n\left(y\right)+yw_n\left(y\right)\big)dy.\end{aligned}$$

So, using (10) yields the case $k=1$ of (33) as

$$\int_{-1}^{0} y w_n(y)\,dy = -\left(B_{n+1}+B_n\right).$$

Multiplying both sides of (3) by y and integrating it with respect to y from -1 to 0, we obtain

$$\int_{-1}^{0} y w_{n+1}(y)\,dy = \int_{-1}^{0} y^2 \frac{d}{dy}\big(w_n(y) + y w_n(y)\big)dy.$$

Applying integration by parts and using (10) yield the case $k=2$ of (33) as

$$2\int_{-1}^{0} y^2 w_n(y)\,dy = B_{n+2}+3B_{n+1}+2B_n.$$

If we multiply both sides of (3) by y^k and integrating it with respect to y from -1 to 0, we obtain

$$\int_{-1}^{0} y^k w_{n+1}(y)\,dy = \int_{-1}^{0} y^{k+1} \frac{d}{dy}\big(w_n(y) + y w_n(y)\big)dy.$$

Applying integration by parts to the right hand side of the above equation and considering

$$\int_{-1}^{0} y^k w_n(y)\,dy = \frac{(-1)^k}{k!}\sum_{j=0}^{k} \begin{bmatrix} k+1 \\ j+1 \end{bmatrix} B_{n+j},$$

we have

$$\int_{-1}^{0} y^{k+1} F_{n+1}(y)\,dy = \frac{(-1)^{k+1}}{(k+1)!}\sum_{j=0}^{k} \begin{bmatrix} k+1 \\ j+1 \end{bmatrix} B_{n+j+1} + \frac{(-1)^{k+1}}{(k+1)!}\sum_{j=0}^{k} (k+1) \begin{bmatrix} k+1 \\ j+1 \end{bmatrix} B_{n+j}.$$

Finally, the well-known relations

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = n\begin{bmatrix} n \\ k \end{bmatrix} + \begin{bmatrix} n \\ k-1 \end{bmatrix} \text{ and } \begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)!,$$

give that the statement is true for $k+1$. □

Lemma 7. *For any non-negative integer m and j,*

$$\sum_{k=j}^{m} \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \begin{bmatrix} k+1 \\ j+1 \end{bmatrix} (-1)^k = (-1)^m \binom{m}{j}. \tag{34}$$

Proof. We rewrite this equation into matrix form by using the matrices

$$(\mathcal{S}_1)_{i,j} = (-1)^{i+j} \begin{bmatrix} i+1 \\ j+1 \end{bmatrix}, \quad (\mathcal{S}_2)_{i,j} = \begin{Bmatrix} i \\ j \end{Bmatrix}, \quad (\mathcal{B})_{i,j} = \binom{i}{j}.$$

These can be considered as infinite matrices so that the statement we are going to prove takes the form

$$\mathcal{S}_2 \mathcal{S}_1 = \mathcal{B}^{-1},$$

where the element-wise inverse of the matrix $\mathcal{B}$ is $(\mathcal{B})^{-1}_{i,k} = (-1)^{i+k}\binom{i}{k}$. The above equation is equivalent to

$$\mathcal{S}_1 = \mathcal{B}^{-1}\mathcal{S}_2^{-1} = (\mathcal{S}_2\mathcal{B})^{-1}.$$

The matrix on the right hand side is easily decipherable. Element-wise, it is

$$((\mathcal{S}_2\mathcal{B})^{-1})_{i,j} = \sum_{k=0}^{i} \begin{Bmatrix} i \\ k \end{Bmatrix} \binom{k}{j}.$$

The latter sum simply equals to

$$\sum_{k=0}^{i} \begin{Bmatrix} i \\ k \end{Bmatrix} \binom{k}{j} = \begin{Bmatrix} i+1 \\ j+1 \end{Bmatrix}$$

[16, Eq. 6.15]. Hence, our original statement equals to the matrix equation

$$(\mathcal{S}_1)^{-1}_{i,j} = \begin{Bmatrix} i+1 \\ j+1 \end{Bmatrix}.$$

This is nothing but the reformulation of the fact that the second and signed first kind Stirling matrices are inverses of each other. □

Now, we are ready to give the integrals of products of geometric polynomials. Using (1), we have

$$\int_{-1}^{0} w_m(y)\, w_n(y)\, dy = \int_{-1}^{0} \sum_{k=0}^{m} \begin{Bmatrix} m \\ k \end{Bmatrix} k! y^k w_n(y)\, dy.$$

Then, interchanging the sum and integral in the above equation and using (33) yield

$$\int_{-1}^{0} w_m(y)\, w_n(y)\, dy = \sum_{j=0}^{m} \sum_{k=j}^{m} \begin{Bmatrix} m \\ k \end{Bmatrix} \begin{bmatrix} k+1 \\ j+1 \end{bmatrix} (-1)^k B_{n+j}.$$

Finally, using Lemma 7 gives the following theorem.

Theorem 8. *For all $m \geq 0$ and $n \geq 1$, we have*

$$\int_{-1}^{0} w_m(y)\, w_n(y)\, dy = (-1)^m \sum_{j=0}^{m} \binom{m}{j} B_{n+j}. \tag{35}$$

Using the representation (1) in (35) and integrating termwise, one obtains

$$\sum_{k=0}^{n} \sum_{j=0}^{m} \left\{ {n \atop k} \right\} \left\{ {m \atop j} \right\} \frac{(-1)^{k+j} k!j!}{k+j+1} = (-1)^m \sum_{j=0}^{m} \binom{m}{j} B_{n+j}.$$

This double sum identity extends to the explicit formula [16, p. 560]

$$B_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} (-1)^k \frac{k!}{k+1}.$$

In order to give an application of Lemma 6, we now emphasize the summation in the right hand of (33). Rahmani [31] defined p-Bernoulli numbers as

$$\sum_{n=0}^{\infty} B_{n,p} \frac{t^n}{n!} = {}_2F_1\left(1, 1; p+2; 1-e^t\right),$$

where ${}_2F_1(a, b; c; z)$ denotes the Gaussian hypergeometric function [1]. These numbers can be written in terms of Stirling numbers of the first kind as follows:

$$\sum_{j=0}^{p} (-1)^j \begin{bmatrix} p \\ j \end{bmatrix} B_{n+j} = \frac{p!}{p+1} B_{n,p}, \quad n, p \geq 0.$$

From the above equation, we have

$$\sum_{j=0}^{p} (-1)^{j+1} \begin{bmatrix} p+1 \\ j+1 \end{bmatrix} B_{n+j} = \frac{(p+1)!}{p+2} B_{n-1,p+1}, \quad n \geq 1, p \geq 0. \tag{36}$$

Moreover, when n is odd or even, we have

$$(-1)^{j+1} B_{n+j} = B_{n+j} \text{ or } (-1)^{j+1} B_{n+j} = -B_{n+j}, \quad n > 1,$$

respectively. Therefore, we obtain

$$\sum_{j=0}^{p} \begin{bmatrix} p+1 \\ j+1 \end{bmatrix} B_{n+j} = \begin{cases} \frac{(p+1)!}{p+2} B_{n-1,p+1}, & \text{if } n \text{ is odd;} \\ -\frac{(p+1)!}{p+2} B_{n-1,p+1}, & \text{if } n \text{ is even.} \end{cases}$$

Using the above equation, (33) can be written as

$$\int_{-1}^{0} y^p w_n(y)\,dy = \begin{cases} (-1)^p \frac{p+1}{p+2} B_{n-1,p+1}, & \text{if } n \text{ is odd;} \\ (-1)^{p+1} \frac{p+1}{p+2} B_{n-1,p+1}, & \text{if } n \text{ is even,} \end{cases}$$

where $n > 1$, $p \geq 0$. On the other hand, using (1) in the left part of (3), a new explicit formula for p-Bernoulli numbers is obtained.

Theorem 9. *For $n > 1$ and $p > 0$,*

$$B_{2n-1,p} = \frac{p+1}{p} \sum_{k=0}^{2n-1} \left\{ {2n-1 \atop k+1} \right\} \frac{(-1)^{k+1}(k+1)!}{k+p+1}$$

and

$$B_{2n,p} = \frac{p+1}{p} \sum_{k=0}^{2n} \left\{ {2n+1 \atop k+1} \right\} \frac{(-1)^{k}(k+1)!}{k+p+1}.$$

4 Applications

Apostol-Bernoulli functions $\mathcal{B}_n(\lambda)$ have the following explicit expression:

$$\mathcal{B}_n(\lambda) = \frac{n}{\lambda-1} \sum_{k=0}^{n-1} \left\{ {n-1 \atop k} \right\} k! \left(\frac{\lambda}{1-\lambda} \right)^k, \quad \lambda \in \mathbb{C} \backslash \{1\}. \tag{37}$$

Thus, for $\lambda \neq 1$,

$$\mathcal{B}_0(\lambda) = 0,\ \mathcal{B}_1(\lambda) = \frac{1}{\lambda-1},\ \mathcal{B}_2(\lambda) = \frac{-2\lambda}{(\lambda-1)^2},\ \ldots$$

The functions $\mathcal{B}_n(\lambda)$ are rational functions in the variable λ. Apostol [4] introduced these functions in order to evaluate the Lerch transcendent (also known as the Lerch zeta function) for negative integer values of s. Also, these functions were studied and generalized recently in a number of papers, under the name Apostol-Bernoulli numbers.

Comparing (37) and (1), Boyadzhiev [7] showed that Apostol-Bernoulli functions can be expressed by geometric polynomials as

$$\mathcal{B}_{n+1}(\lambda) = \frac{n+1}{\lambda-1} w_n\left(\frac{\lambda}{1-\lambda} \right), \qquad \lambda \in \mathbb{C} \backslash \{1\}. \tag{38}$$

We can use this relation to obtain some new properties of $\mathcal{B}_n(\lambda)$. For example, setting $y = \frac{-\lambda}{\lambda-1}$ in (23), we have [18, Theorem 4.3]

$$\frac{\mathcal{B}_{n+1}(\lambda)}{(n+1)} = (-1)^n \lambda \sum_{k=0}^{n} \left\{ {n \atop k} \right\} k! \left(\frac{1}{\lambda-1} \right)^{k+1}, \quad \lambda \neq 1,\ n \geq 0.$$

Similarly, from Theorem 1, we get the sums of products of Apostol-Bernoulli functions as given in [23, Corollary 1.3]. Moreover, using equation (25) of Theorem 5 gives a new explicit formula for Apostol-Bernoulli functions.

Corollary 10. *For $\lambda \neq \pm 1$ and $n \geq 0$,*

$$\frac{\mathcal{B}_{n+1}(\lambda)}{(n+1)} = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} k! \frac{(-\lambda)^k \left(2^{n+1}\lambda^k + (\lambda - 1)^{k+1}\right)}{(\lambda^2 - 1)^{k+1}}.$$

To give a different application of the relation (38), we first deal with Lemma 6. Replacing y with $\frac{\lambda}{1-\lambda}$ in (33), we have

$$\int_{-\infty}^{0} \frac{\lambda^k}{(\lambda - 1)^{k+1}} \mathcal{B}_{n+1}(\lambda)\, d\lambda = \frac{n+1}{k!} \sum_{j=0}^{k} \left[{k+1 \atop j+1} \right] B_{n+j},$$

where $k \geq 0$ and $n \geq 1$. Then, from Theorem 8, we obtain the integrals of products of Apostol-Bernoulli functions as given in the following corollary.

Corollary 11. *For all $m \geq 0$ and $n \geq 1$, we have*

$$\int_{-\infty}^{0} \mathcal{B}_m(\lambda)\, \mathcal{B}_n(\lambda)\, d\lambda = (-1)^m (m+1)(n+1) \sum_{j=0}^{m} \binom{m}{j} B_{n+j}.$$

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