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Words and Linear Recurrences

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Abstract

In previous papers, for an arithmetical function f_0, we defined the functions f_m and c_m, and designated the number of restricted words over a finite alphabet counted by these functions. In this paper, we examine the reverse problem. For each of the five specific types of restricted words, we find the initial function f_0 such that f_m and c_m enumerate these words. We derive explicit formulas for f_m and c_m.

The Fibonacci, Mersenne, Pell, Jacosthal, Tribonacci, and Padovan numbers all appear as values of f_m. We give new combinatorial interpretations and explicit formulas for f_m.

1 Introduction

This paper is a continuation of the investigations of the problem of restricted words enumeration from the author's previous papers [2, 3, 4], where two functions f_m and c_m were defined as follows. For an initial arithmetic function f_0, the function $f_m, (m \geq 1)$ is the m^{th} invert transform of f_0. The function $c_m(n,k)$ was defined as

$$c_m(n,k) = \sum_{i_1+i_2+\cdots+i_k=n} f_{m-1}(i_1) \cdot f_{m-1}(i_2) \cdots f_{m-1}(i_k), \tag{1}$$

where the sum is over positive $i_1, i_2, \ldots, i_k$.

The functions f_m and c_m depend only on the initial function f_0, and are related to the enumeration of weighted compositions. Namely, if $f_{m-1}(i), (i = 1, 2, \ldots)$ is the weight of i, then $f_m(n)$ is the number of all weighted compositions of n, and $c_m(n,k)$ is the number of weighted compositions of n into k parts.

In Janjić [2, 3, 4], weighted compositions were related to restricted words over a finite alphabet. For a given initial function f_0, we investigated restricted words counted by f_m and c_m. In this paper, we reverse the problem. Namely, for each of five types of restricted words, we first find the initial function f_0 that counts such words. We then derive formulas for f_m and c_m and give its combinatorial meanings in terms of restricted words.

We firstly restate results from our recent papers [2, 3, 4], which will be used in this work.

(A) [2, Theorem 6] Let f_0 be an arithmetic function, and let k be a positive integer. If there exist constants $a_0(1), a_0(2), \ldots, a_0(k)$ such that

$$f_0(n+k;k) = \sum_{i=1}^{k} a_0(i) f_0(n+k-i;k), (n \geq 1),$$

where $f_0(1;k), f_0(2;k), \ldots, f_0(k;k)$ are arbitrary numbers. Then we have

$$f_1(i;k) = \sum_{j=1}^{i} f_0(j;k) f_1(i-j;k), (i = 1, 2, \ldots, k), \text{ and}$$

$$f_1(n+k;k) = \sum_{i=1}^{k} a_1(i) f_1(n+k-i;k), (n \geq 1),$$

where

$$a_1(1) = a_0(1) + f_0(1;k),$$

$$a_1(i) = a_0(i) + f_0(i;k) - \sum_{j=1}^{i-1} a_0(j) f_0(i-j;k), (2 \leq i \leq k).$$

(B) [2, Corollary 9]. If $f_0(1), f_0(2), a_0(1), a_0(2)$ are arbitrary, and

$$f_0(n+2) = a_0(1) f_0(n+1) + a_0(2) f_0(n),$$

then

$$f_m(1) = f_0(1), \; f_m(2) = m f_0(1)^2 + f_0(2),$$

$$f_m(n+2) = a_m(1) f_m(n+1) + a_m(2) f_m(n),$$

where

$$a_m(1) = a_0(1) + m f_0(1), a_m(2) = a_0(2) - m a_0(1) f_0(1) + m f_0(2).$$

(C) [2, Proposition 23]. Assume that $f_0(1)=0$, and $f_0(i)=1,(i>1)$. Then we have

$$f_m(1)=0, f_m(2)=1,$$

and

$$f_m(n+2)=f_m(n+1)+mf_m(n).$$

(D) [3, Corollary 2]. The following formula holds:

$$f_m(n)=\sum_{k=1}^{n} c_m(n,k).$$

(E) [4, Proposition 6]. The following formula holds:

$$c_m(n,k)=\sum_{i=k}^{n}(m-1)^{i-k}\binom{i-1}{k-1}c_1(n,i),\ (1\leq k\leq n).$$

(F) [4, Propositions 12]. Assume that $f_0(1)=1$ and $m>1$. Assume next that, for $n\geq 1$, we have $f_{m-1}(n)$ words of length $n-1$ over a finite alphabet α. Let x be a letter which is not in α. Then, $c_m(n,k)$ is the number of words of length $n-1$ over the alphabet $\alpha\cup\{x\}$ in which x appears exactly $k-1$ times.

We consider the following five types of restricted words over a finite alphabet:

1. Words over $\{0,1,\ldots,a-1,\ldots\}$, such that no two adjacent letters from $\{0,1,\ldots,a-1\}$ are the same.

2. Words over $\{0,1,\ldots,a-1,\ldots\}$ such that letters $0,1,\ldots,a-1$ avoid a run of odd length.

3. Words over $\{0,1,\ldots,a,\ldots\}$ avoiding subwords of the form $0i,(i=1,\ldots,b)$, for $b<a$.

4. Words over $\{0,1,\ldots\}$ such that 0 and 1 appear only as subwords of the form $1i$, where i is a run of zeros.

5. Words over $\{0,1,\ldots\}$ in which 0 appears only in a run of even length, and 1 appears only in a run the length of which is divisible by 3.

We note that the initial function f_0 will be defined by a linear homogeneous recurrence in all cases.

2 Case 1

To solve the problem posed in Case 1, we consider the following linear recurrence:

$$f_0(1) = 1, f_0(2) = a,$$
$$f_0(n+2) = (a-1)f_0(n+1), (n \geq 1),$$

where $a > 0$. It is easy to see that

$$f_0(n) = a(a-1)^{n-2}, (n \geq 2).$$

Remark 1. This formula appears in Birmajer at al. [1, Example 17]. Also, the case $a = 1$ is considered in [4, Example 18].

The function f_0 has the following combinatorial interpretation:

Proposition 2. *The number $f_0(n)$ is the number of words of length $n-1$ over $\{0, 1, \ldots, a-1\}$ such that no two adjacent letters are the same.*

Proof. We have $f_0(1) = 1$, since only the empty word has length 0. Also, $f_0(2) = a$, since a word of length 1 may consist of an arbitrary letter. To obtain a word of length $n + 2$, for $n > 0$, we need to insert $a - 1$ letters in front of each word of length $n + 1$. □

As an immediate consequence of (B), we obtain

Corollary 3. *For $m \geq 0$, the following recurrence holds:*

$$f_m(1) = 1, f_m(2) = m + a,$$
$$f_m(n+2) = (m+a-1)f_m(n+1) + mf_m(n), (n \geq 1).$$

We now describe words counting by f_m.

Proposition 4. *The number $f_m(n)$ is the number of words of length $n-1$ over the alphabet $\{0, 1, \ldots, a-1, a, \ldots, m+a-1\}$, such that no two adjacent letters from $\{0, 1, \ldots, a-1\}$ are the same.*

Proof. We have $f_m(1) = 1$, since only the empty word has length 0. Also, $f_m(2) = m + a$ since a word of length 1 may consist of any letter of the alphabet. Assume that $n > 2$. Consider a word of length $n + 1$. In front of such a word, we insert a letter different from the first letter of the word. In this way, we obtain all words of length $n + 2$ beginning with two different letters. The remaining words must begin with two same letters. Since there are $mf_m(n)$ such words, the statement is true. □

Remark 5. For $a = 2$, the continued fraction $[f_0(1); f_0(2), f_0(3), \ldots]$ equals $\sqrt{2}$. The sequence $f_1(1), f_1(2), \ldots, f_1(n)$ is the numerator of the nth convergent of $\sqrt{2}$. Also, $f_1(n)$ is the number of ternary words of length $n - 1$ avoiding 00 and 11.

Since $f_m(1) = 1$, we may apply (F) to obtain

Proposition 6. *The number $c_m(n,k)$ is the number of words of length $n-1$ over $\{0,1,\ldots,a-1,\ldots,m+a-1\}$ in which $k-1$ letters equal $m+a-1$, and no two adjacent letter from $\{0,1,\ldots,a-1\}$ are identical.*

We next derive an explicit formula for $c_1(n,k)$.

Proposition 7. *We have*

$$c_1(n,n) = 1,$$
$$c_1(n,k) = \sum_{i=0}^{k-1} \binom{k}{i} \binom{n-k-1}{k-i-1} a^{k-i}(a-1)^{n-2k+i}, (k < n).$$

Proof. From (1), we firstly obtain $c_1(n,n) = 1$. Assume that $k < n$. Since at most $k-1$ of $i_t, (t = 1,2,\ldots,k)$ may equal 1, then

$$\begin{aligned} c_1(n,k) &= \sum_{i=0}^{k-1} \binom{k}{i} \sum_{j_1+j_2+\cdots+j_{k-i}=n-i} f_0(j_1)f_0(j_2)\cdots f_0(j_{k-i}) \\ &= \sum_{i=0}^{k-1} \binom{k}{i} a^{k-i}(a-1)^{n-2k+i} \sum_{j_1+j_2+\cdots+j_{k-i}=n-i} 1, \end{aligned}$$

where the last sum is taken over $j_t \geq 2$. It follows that

$$c_1(n,k) = \sum_{i=0}^{k-1} a^{k-i}(a-1)^{n-2k+i} \binom{k}{i} \binom{n-k-1}{k-i-1}.$$

$\square$

Remark 8. Note that, in the preceding formula, terms in which $i < 2k-n$ would equal zero.

To obtain an explicit formula for $c_m(n,k)$, we use (E). Extracting the term for $i = n$, we obtain

$$c_m(n,k) = (m-1)^{n-k} \binom{n-1}{k-1} + \sum_{i=k}^{n-1} (m-1)^{i-k} \binom{i-1}{k-1} c_1(n,i).$$

Next, we have

$$c_m(n,k) = (m-1)^{n-k} \binom{n-1}{k-1} +$$
$$\sum_{i=k}^{n-1} \sum_{j=0}^{i-1} (m-1)^{i-k} a^{i-j}(a-1)^{n-2i+j} \binom{i}{j} \binom{i-1}{k-1} \binom{n-i-1}{i-j-1}.$$

Explicit formula for $f_m(n)$ may be obtained from (C).

The following arrays in Sloane [5] are related with this case: A154929, A113413, A054458, A116412.

3 Case 2

Let a be a positive integer. Define f_0 as follows:

$$f_0(1) = 1, f_0(2) = 0,$$
$$f_0(n+2) = af_0(n), (n \geq 1).$$

Proposition 9. *For $a > 0$, the number $f_0(n)$ is the number of words of length $n-1$ over the alphabet $\{0, 1, \dots, a-1\}$ in which there are no runs of odd length.*

Proof. Let $d(n)$ denote the number of words of length n, which we wish to count. Firstly, $d(0) = 1$ since only the empty word has length 0. Next, $d(1) = 0$ as there are no runs of length 1. Assume that $n > 2$. A word of length n must begin with two identical letters. Hence, there are $ad(n-2)$ such words. We conclude that the following recurrence holds:

$$d(0) = 1, d(1) = 0, d(n) = ad(n-2), (n \geq 2),$$

which yields $d(n-1) = f_0(n), (n \geq 1)$. □

From (3), we easily obtain the following explicit formula for f_0:

$$f_0(n) = \begin{cases} 0, & \text{if } n = 2t; \\ a^t, & \text{if } n = 2t+1. \end{cases}$$

Using (B) yields

Corollary 10. *For $m \geq 0$, the following recurrence holds:*

$$f_m(1) = 1, f_m(2) = m,$$
$$f_m(n+2) = mf_m(n+1) + af_m(n), (n \geq 1).$$

Proposition 11. *The number $f_m(n)$ is the number of words of length $n-1$ over $\{0, 1, \dots, a-1, \dots, a+m-1\}$, such that letters $0, 1, \dots, a-1$ avoid runs of odd length.*

Proof. We let $d(n)$ denote the number of desired words of length $n-1$. It is clear that $d(0) = 1$ and $d(1) = m$. A word of length $n+1$ may begin with a letter from $\{a, a+1, \dots, a+m-1\}$. There are $md(n)$ such word. If a word begins with a letter from $\{0, 1, \dots, a-1\}$, it must be followed by the same letter. Hence, there are $ad(n-1)$ such words. We conclude that $d(n) = f_m(n+1)$. □

Some well-known classes of numbers satisfy the recurrence from Corollary 10. We give the appropriate combinatorial meaning for some of them.

1. The case $a = 1, m = 1$ concerns the Fibonacci numbers. The number of binary words of length $n-1$ in which 0 avoids a run of odd length is F_n.

2. The case $a = 1, m = 2$ concerns the Pell numbers P_n (A000129). The number of ternary words of length $n-1$ in which 0 avoids runs of odd length is P_n.

3. The case $a = 2, m = 1$ concerns the Jacobsthal numbers J_n (A001045). The number of ternary words of length $n-1$ in which 0 and 1 avoid runs of odd length is J_n.

From the combinatorial interpretation, we easily derive an explicit formula for $f_m(n)$.

Proposition 12. *We have*

$$f_m(n) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} m^{n-2j-1} a^j \binom{n-1-j}{j}.$$

Proof. According to Proposition 11, in a word counted by f_m, the letters from $\{0, 1, \dots, a-1\}$ may appear only in pairs. There are a such pairs. We may choose j, $(0 \le j \le \lfloor \frac{n-1}{2} \rfloor)$ pairs in a word of length $n-1$. These j pairs may be chosen in $\binom{n-j-1}{j}$ ways. When we have chosen j pairs from $\{0, 1, \dots, a-1\}$, the remaining $n-1-2j$ letters are from $\{a, a+1, \dots, a+m-1\}$, which are m in number. $\square$

As a consequence, we obtain the following similar explicit formulas for the Fibonacci, Pell and Jacobsthal numbers:

$$F_n = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-j-1}{j}, P_n = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} 2^{n-2j-1} \binom{n-j-1}{j},$$

$$J_n = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} 2^j \binom{n-j-1}{j}.$$

From (F), we obtain

Proposition 13. *The number $c_m(n,k)$ is the number of words of length $n-1$ over the alphabet $\{0, 1, \dots, a-1, \dots, a+m-1\}$ in which the letter $a+m-1$ appears $k-1$ times, and letters from $\{0, 1, \dots, a-1\}$ avoid runs of odd length.*

We now derive an explicit formula for $c_1(n,k)$.

Proposition 14. *The following equation holds:*

$$c_1(n,k) = \begin{cases} a^{\frac{n-k}{2}} \binom{\frac{n+k}{2}-1}{k-1}, & \text{if } n-k \text{ is even;} \\ 0, & \text{if } n-k \text{ is odd .} \end{cases}$$

Proof. Each term in (1) in which i_t is even equals zero. Hence, (1) becomes

$$\begin{aligned} c_1(n,k) &= \sum_{2j_1+1+2j_2+1+\cdots+2j_k+1=n} a^{j_1}\cdot a^{j_2}\cdots a^{j_k} \\ &= a^{\frac{n-k}{2}} \sum_{s_1+s_2+\cdots+s_k=\frac{n+k}{2}} 1 = a^{\frac{n-k}{2}}\binom{\frac{n+k}{2}-1}{k-1}. \end{aligned}$$

Note that the last sum is over positive $s_1, s_2, \ldots, s_k$. ◻

As a consequence of (D), we obtain the following explicit formulas for the Fibonacci and Jacobsthal numbers:

$$F_{2n}=\sum_{k=1}^{n}\binom{n+k-1}{n-k}, F_{2n-1}=\sum_{k=1}^{n}\binom{n+k-2}{n-k},$$
$$J_{2n}=\sum_{k=1}^{n}2^{n-k}\binom{n+k-1}{n-k}, J_{2n-1}=\sum_{k=1}^{n}2^{n-k}\binom{n+k-2}{n-k}.$$

Furthermore, we derive an explicit formula for $c_2(n,k)$. Using (E), for even n, we obtain

$$\begin{aligned} c_2(2n,k) &= \sum_{i=k}^{2n}\binom{i-1}{k-1}c_1(2n,i) = \sum_{j=\lceil\frac{k}{2}\rceil}^{n}\binom{2j-1}{k-1}c_1(2n,2j) \\ &= \sum_{j=\lceil\frac{k}{2}\rceil}^{n} a^{n-j}\binom{2j-1}{k-1}\binom{n+j-1}{n-j}. \end{aligned}$$

For odd n, we have

$$\begin{aligned} c_2(2n-1,k) &= \sum_{i=k}^{2n-1}\binom{i-1}{k-1}c_1(2n,i) = \sum_{j=\lceil\frac{k+1}{2}\rceil}^{n}\binom{2j-2}{k-1}c_1(2n-1,2j-1) \\ &= \sum_{j=\lceil\frac{k+1}{2}\rceil}^{n} a^{n-j}\binom{2j-2}{k-1}\binom{n+j-2}{n-j}. \end{aligned}$$

In particular, for $a=1$, we obtain the following formulas for Pell numbers:

$$P_{2n}=\sum_{k=1}^{2n}\sum_{j=\lceil\frac{k}{2}\rceil}^{n}\binom{2j-1}{k-1}\binom{n+j-1}{n-j},$$
$$P_{2n-1}=\sum_{k=1}^{2n-1}\sum_{j=\lceil\frac{k+1}{2}\rceil}^{n}\binom{2j-2}{k-1}\binom{n+j-2}{n-j}.$$

Remark 15. Using (E), we easily obtain an explicit formula for $c_m(n,k)$.

The following arrays in [5] are related with this case: A000129,A001045,A168561, A037027, A054456, A132964, A073370.

4 Case 3

Let $a > b > 0$ be integers. We define f_0 by the following recurrence:

$$f_0(1) = 1, f_0(2) = a,$$
$$f_0(n+2) = af_0(n+1) - bf_0(n), (n \geq 1).$$

Proposition 16. *The number $f_0(n)$ is the number of words of length $n-1$ over the alphabet $\{0, 1, \ldots, a\}$, avoiding subwords of the form: $0i, (i = 1, \ldots, b)$.*

Proof. We let $d(n)$ denote the number of words of length $n-1$. Firstly, $d(0) = 1$, since only the empty word has length 0. Next, $d(1) = a$, since there are no restrictions on words of length 1. Assume that $n > 1$. We have $a \cdot d(n-1)$ words beginning with arbitrary letter. From this number, we must subtract words which begin with subwords $0i, (i = 1, 2, \ldots, b)$. Hence, $d(n)$ satisfies the same recurrence as $f_0(n)$. $\square$

Example 17. 1. If $a = 2, b = 1$, we have

$$f_0(1) = 1, f_0(2) = 2,$$
$$f_0(n+2) = 2f_0(n+1) - f_0(n), (n \geq 1),$$

which yields that $f_0(n) = n$. Hence, n is the number of binary words of length $n-1$ avoiding subword 01, which is obvious.

2. If $a = 3, b = 1$, we have

$$f_0(1) = 1, f_0(2) = 3,$$
$$f_0(n+2) = 3f_0(n+1) - f_0(n), (n \geq 1),$$

which is a well-known recurrence for the Fibonacci numbers F_{2n}.

We thus obtain the following combinatorial interpretation of the bisection of the Fiboncci numbers.

Corollary 18. *The number of ternary words of length $n-1$ avoiding 01 is F_{2n}.*

We consider the particular case $a = b + 1$.

Corollary 19. *If $b > 1$ and $a = b + 1$, then*

$$f_0(n) = \frac{b^n - 1}{b - 1}.$$

Proof. We denote $\frac{b^n-1}{b-1}$ by $g_0(n)$. We have $g_0(1) = 1, g_0(2) = 1 + b = a$. Further,

$$(b+1) \cdot g_0(n+1) - b \cdot g_0(n) = (b+1) \cdot \frac{b^{n+1} - 1}{b - 1} - b \cdot \frac{b^n - 1}{b - 1} = \frac{b^{n+2} - 1}{b - 1}.$$

By induction, we conclude that $g_0 = f_0$. $\square$

In particular, for $a = 3, b = 2$, we have $f_0(n) = 2^n - 1$, which yields

Corollary 20. *The Mersenne number* $2^n - 1$ *is the number of ternary words of length* $n-1$ *avoiding* 01 *and* 02.

Using (B), we obtain

$$f_m(1) = 1, f_m(2) = m + a,$$
$$f_m(n+2) = (a+m)f_m(n+1) - bf_m(n), (n \geq 1).$$

This means that f_m counts the same sort of words as f_0, with $m + a$ instead of a.

Using (F) and (D), we obtain the following combinatorial interpretations of $c_m(n,k)$ and $f_m(n)$.

Corollary 21.
1. *The number* $c_m(n,k)$ *is the number of words of length* $n-1$ *over* $\{0, 1, \ldots, b-1, b \ldots, m+a-1\}$ *having exactly* $k-1$ *letters equal* $m+a-1$ *and avoiding subwords* $0j, (j = 1, 2, \ldots, b)$.
2. *The number* $f_m(n)$ *is the number of words of length* $n-1$ *over the alphabet* $\{0, 1, \ldots, b-1, b \ldots, m+a-1\}$ *and avoiding subwords* $0j, (j = 1, 2, \ldots, b)$.

We next derive an explicit formula for $c_1(n,k)$. A generating function for the sequence $f_0(1), f_0(2), \ldots$ is $\frac{1}{bx^2-ax+1}$. According to [4, Equation (1)], we have

$$\frac{x^k}{(bx^2 - ax + 1)^k} = \sum_{n=k}^{\infty} c_1(n,k)x^k.$$

The numbers $\alpha = \frac{a+\sqrt{a^2-4b}}{2b}$ and $\beta = \frac{a-\sqrt{a^2-4b}}{2b}$ are the solutions of the equation $bx^2 - ax + 1 = 0$. We thus obtain

Proposition 22. *The following equation holds:*

$$c_1(n,k) = \frac{1}{b^k} \sum_{j=0}^{n-k} \frac{1}{\alpha^{j+k}\beta^{n-j}} \binom{n-j-1}{k-1} \binom{k+j-1}{k-1}.$$

Proof. We expand $\frac{x^k}{b^k(\alpha-x)^k(\beta-x)^k}$ into powers of x. Since

$$\frac{1}{(\gamma - x)^k} = \sum_{i=0}^{\infty} \binom{k+i-1}{k-1} \frac{x^i}{\gamma^{i+k}},$$

we easily obtain

$$\frac{x^k}{b^k(\alpha - x)^k(\beta - x)^k} = \sum_{i=0}^{\infty} \left[\sum_{j=0}^{i} \frac{1}{b^k\alpha^{j+k}\beta^{i-j+k}} \binom{k+j-1}{k-1} \binom{k+i-j-1}{k-1} \right] x^{i+k},$$

and the statement follows by replacing i by $n-k$. ∎

In the case $a = b + 1$, we have $\alpha = 1$ and $\beta = \frac{1}{b}$. Therefore, the following formula holds:

$$c_1(n,k) = \sum_{i=0}^{n-k} b^{n-k-i} \binom{n-i-1}{k-1} \binom{k+i-1}{k-1}. \tag{2}$$

Using (1), we obtain the following identity:

Identity 23.

$$\sum_{i_1+i_2+\cdots+i_k=n} \left[\prod_{t=1}^{k} (b^{i_t} - 1)\right] = \sum_{i=0}^{n-k} b^{n-k-i} \binom{n-i-1}{k-1} \binom{k+i-1}{k-1},$$

where $i_t > 0, (t = 1, 2, \ldots, k)$.

Remark 24. Using (D) and (E), we obtain explicit formulas for $f_m(n)$ and $c_m(n,k)$.

The following arrays in [5] are related with this case: A078812, A125662, A207823, A207824, A110441, A116414.

5 Case 4

We let $\mathcal{R}_4$ denote the condition given in this case. We first solve the problem for binary words.

Proposition 25. *Let* $f_0(n)$ *be the number of binary words satisfying* $\mathcal{R}_4$*. Then,*

$$f_0(1) = 1, f_0(2) = 0,$$
$$f_0(n+2) = f_0(n+1) + f_0(n), (n > 1).$$
$$f_0(n) = F_{n-2}, (n > 1).$$

Proof. We have $f_0(1) = 1$, since the empty word has length 0. Next, $f_0(2) = 0$, since no words of length 1 satisfy $\mathcal{R}_4$. Also, $f_0(3) = 1$, since 10 is the only word of length 2 satisfying $\mathcal{R}_4$. Next, $f_0(4) = 1$, since 100 is the only word of length 3 satisfying $\mathcal{R}_4$. Assume that $n > 1$. Then,

$$f_0(n+4) = f_0(n+2) + f_0(n+1) + \cdots,$$

since a word of length greater than 3 must begin with a subword of the form $10\ldots0$. Analogously, we obtain

$$f_0(n+5) = f_0(n+3) + f_0(n+2) + \cdots.$$

Comparing these two equations, we get

$$f_0(n+5) = f_0(n+4) + f_0(n+3).$$

The explicit formula follows from the preceding recurrence. ∎

Since $f_0(1) = 1$, and so $f_m(1) = 1$, using (D) and (F), we obtain the following combinatorial interpretations of f_m and $c_m(n,k)$.

Corollary 26. *1. The number $c_m(n,k)$ is the number of words of length $n-1$ over $\{0,1,\ldots,m+1\}$ having $k-1$ letters equal $m+1$ and satisfying $\mathcal{R}_4$.*

2. The number $f_m(n)$ is the number of words of length $n-1$ over the alphabet $\{0,1,\ldots,m+1\}$ satisfying $\mathcal{R}_4$.

We next derive an explicit formula for $c_1(n,k)$. It is known that $c_1(n,k)$ is the coefficient of x^n in the expansion of $\left(\sum_{i=1}^{\infty} F_{i-2}x^i\right)^k$ into powers of x.

We consider the following auxiliary initial function:

$$\overline{f}_0(1) = 0, \overline{f}_0(n) = 1, (n > 1).$$

From [2, Proposition 23], we obtain $\overline{f}_1(n) = F_{n-1}$. It is proved in [3, Proposition 13] that

$$\overline{c}_1(n,k) = \binom{n-k-1}{k-1}, \left(k = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right),$$

and $\overline{c}_1(n,k) = 0$ otherwise.

Using (E) yields

$$\overline{c}_2(n,k) = \sum_{i=k}^{\lfloor \frac{n}{2} \rfloor} \binom{i-1}{k-1}\binom{n-i-1}{i-1}.$$

Hence,

$$\left(\sum_{i=1}^{\infty} F_{i-1}x^i\right)^k = \sum_{n=k}^{\infty} \overline{c}_2(n,k)x^n. \tag{3}$$

We let X denote $\sum_{i=1}^{\infty} F_{i-1}x^i$. We have to expand the expression Y^k, where $Y = \sum_{i=1}^{\infty} F_{i-2}x^i$. Since $F_{-1} = 1$, we have that $Y = x(1+X)$, which yields

$$Y^k = x^k\left(1 + \sum_{i=1}^{k}\binom{k}{i}X^i\right)^k = \sum_{n=k}^{\infty} c_1(n,k)x^n.$$

Using the binomial theorem and (3) yields

$$Y^k = x^k + \sum_{i=1}^{k}\sum_{j=i}^{\infty}\binom{k}{i}\overline{c}_2(j,i)x^{j+k}.$$

For $j+k = n$, the coefficient of x^n on the right-hand side of this equation equals $\sum_{i=1}^{k}\binom{k}{i}\overline{c}_2(n-k,i)$. We thus obtain

Proposition 27. *The following equations hold:*

$$c_1(n,n) = 1,$$
$$c_1(n,k) = \sum_{i=1}^{k} \sum_{j=i}^{\lfloor \frac{n-k}{2} \rfloor} \binom{k}{i} \binom{j-1}{i-1} \binom{n-k-j-1}{j-1}, (n > k).$$

Proposition 28. *We have*

$$c_1(n,n) = 1,$$
$$c_1(n,k) = \sum_{t=1}^{k} \sum_{j=t}^{\lfloor \frac{n-k}{2} \rfloor} \binom{k}{t} \binom{j-1}{t-1} \binom{n-k-j-1}{j-1}, (n > k).$$

Using (B), we easily obtain the following recurrence for f_m:

$$f_m(1) = 1, f_m(2) = m,$$
$$f_m(n+2) = (m+1)f_m(n+1) - (m-1)f_m(n).$$

We examine two particular cases. In the case $m = 1$, we obtain

$$f_1(1) = 1, f_1(2) = 1,$$
$$f_1(n+2) = 2f_1(n+1), (n > 1),$$

which implies

$$f_1(1) = f_1(2) = 1, f_1(n) = 2^{n-2}, (n > 2).$$

We thus obtain the following property of powers of 2.

Corollary 29. *For $n \geq 2$, the number 2^{n-2} is the number of ternary words of length $n-1$ satisfying $\mathcal{R}_4$.*

As a consequence, the following Euler-type identity holds:

Identity 30. *For $n > 2$, the number of binary words of length $n-2$ is the number of ternary words of length $n-1$, in which 0 and 1 appear only in a run of the form $1i$, where i is the run of zeros of length $i \geq 1$.*

From Propositions 28 and (D), we obtain the following identity for the Mersenne numbers:

Identity 31.

$$2^{n-2} - 1 = \sum_{k=1}^{n} \sum_{i=1}^{k} \sum_{j=i}^{\lfloor \frac{n-k}{2} \rfloor} \binom{k}{i} \binom{j-1}{i-1} \binom{n-k-j-1}{j-1}, (n > 2).$$

We now consider the case $m = 2$. We have

$$f_2(1) = 1, f_2(2) = 2,$$
$$f_2(n+2) = 3f_2(n+1) - f_2(n),$$

which is the recurrence for Fibonacci numbers F_{2n-1}. We thus have

Corollary 32. *The Fibonacci number F_{2n-1} is the number of quaternary words of length $n-1$ satisfying $\mathcal{R}_4$.*

Calculating values for $c_2(n, k)$, we obtain an odious expression for F_{2n-1}.

Identity 33.

$$F_{2n-1} = \sum_{k=1}^{n} \sum_{i=k}^{n} \sum_{t=0}^{i} \sum_{j=t}^{\lfloor \frac{n-i}{2} \rfloor} \binom{i-1}{k-1} \binom{i}{t} \binom{j-1}{t-1} \binom{n-i-j-1}{j-1}.$$

Remark 34. Using (E) and (D), we obtain the explicit formulas for $c_m(n, k)$ and $f_m(n)$.

The following arrays in [5] are related with this case: A105422, A105306, A062110, A188137.

6 Case 5

We let $\mathcal{R}_5$ denote the given condition. Again, we first consider the binary words.

Proposition 35.

$$f_0(1) = 1, f_0(2) = 0, f_0(3) = 1.$$

$$f_0(n+3) = f_0(n+1) + f_0(n), (n \geq 1). \tag{4}$$

We have $f_0(n) = p_{n+2}$, where p_n is the nth Padovan number (A000931).

Proof. It is easy to see that the initial conditions are fulfilled. A word of length $n+2$ begins with either two zeros or three ones and (4) follows.

Since (4) is the recurrence for the Padovan numbers, the second statement is true. ☐

This means that the Padovan number p_{n+2} is the number of binary words of length $n-1$ in which 0 appears in runs of even length, while 1 appears in runs, the lengths of which are divisible by 3. This is equivalent to the fact that the Padovan numbers count the compositions into parts 2 and 3, which is known fact (see comment of A000931).

Corollary 36. *1. The function f_m satisfies the following recurrence:*

$$f_m(1) = 1, f_m(2) = m, f_m(3) = m^2 + 1,$$
$$f_m(n+3) = mf_m(n+2) + f_m(n+1) + f_m(n), (n > 1).$$

2. *Then,* $c_m(n,k)$ *is the number of words of length* $n-1$ *over* $\{0,1,\ldots,m+1\}$ *having* $k-1$ *letters equal to* $m+1$*, and satisfying* $\mathcal{R}_5$.

3. *Then,* $f_m(n)$ *is the number of words of length* $n-1$ *over* $\{0,1,\ldots,m+1\}$ *satisfying* $\mathcal{R}_5$.

Proof. The claim 1. easily follows from (A). The claims 2. and 3. follow from (F) and (D).

We add a short combinatorial proof for 2. Equation $f_m(1)=1$ means that the empty word satisfies $\mathcal{R}_5$. Further, $f_m(2)=m$ means that a word of length 1 may consist of any letter except 0 and 1. Next, $f_m(3)=m^2+1$ means that a word of length 2 may consist of pairs from $\{2,3,\ldots,m+1\}$, which are m^2 in number, plus the word 00. Finally, a word of length $n>2$ may begin with any letter from $\{2,3,\ldots,m+1\}$, or from 00, or from 111. □

The case $m=1$ in Corollary 36 is the recurrence for Tribonacci numbers. Hence,

Corollary 37. *The sequence* $1,1,2,4,7,\ldots$ *of the Tribonacci numbers is the invert transform of the sequence* $1,0,1,1,1,2,\ldots$ *of the Padovan numbers.*

Also, Tribonacci numbers count ternary words satisfying $\mathcal{R}_5$.

Finally, we calculate $c_1(n,k)$. We define the arithmetic function $\overline{f}_0$ such that $\overline{f}_0(2)=\overline{f}_0(3)=1$, and $\overline{f}_0(n)=0$ otherwise. It is proved in [3, Propositon 5] that $\overline{c}_1(n,k)=\binom{k}{n-2k}$.

$$\overline{f}_1(1)=0, \overline{f}_1(2)=1, \overline{f}_1(3)=1,$$
$$\overline{f}_1(n+3)=\overline{f}_0(n+1)+\overline{f}_0(n).$$

This implies that $\overline{f}_1(n)=f_0(n-1),(n>1)$. The sequence $f_0(1),f_0(2),\ldots$ is thus obtained by inserting 1 at the beginning of the sequence $\overline{f}_1(1),\overline{f}_1(2),\ldots$.

Using (E), we obtain

$$\overline{c}_2(n,k)=\sum_{i=k}^{n}\binom{i-1}{k-1}\cdot\binom{i}{n-2\cdot i},$$

which yields

$$\left(\sum_{i=1}^{\infty}\overline{f}_1(i)x^i\right)^k=\sum_{n=k}^{\infty}\overline{c}_2(n,k)x^n. \tag{5}$$

To obtain an explicit formula for $c_1(n,k)$, we need to expand the expression X given by $X=\left(\sum_{i=1}^{\infty}f_0(i)x^i\right)^k$ into powers of x. We have

$$X=\left(x+\sum_{i=2}^{\infty}f_0(i)x^i\right)^k=(x+xY)^k,$$

where $Y = \sum_{i=1}^{\infty} \overline{f}_1(i)x^i$. Hence,

$$X = x^k \sum_{i=0}^{k} \binom{k}{i} Y^i.$$

Applying Equation(5) yields

$$X = \sum_{i=0}^{k} \binom{k}{i} \sum_{j=i}^{\infty} \overline{c}_2(j,i)x^{j+k}.$$

Taking $n = j + k$, we get

Proposition 38. *The following formulas hold:*

$$c_1(n,n) = 1,$$
$$c_1(n,k) = \sum_{i=0}^{k} \sum_{j=i}^{n-k} \binom{k}{i} \binom{j-1}{i-1} \binom{j}{n-k-2j}, (k < n).$$

In particular, the following identity for the Tribonacci numbers T_n holds:

Identity 39.

$$T_n = 1 + \sum_{k=1}^{n-1} \sum_{i=0}^{k} \sum_{j=i}^{n-k} \binom{k}{i} \binom{j-1}{i-1} \binom{j}{n-k-2j}.$$

Remark 40. Using (E) and (D), we obtain explicit formulas for $c_m(n,k)$ and $f_m(n)$.

The following arrays in Sloane [5] are related with this case: A104578, A104580.

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