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Bases in Dihedral and Boolean Groups

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Abstract

A subset B of a group G is called a *basis* of G if $G = B^2$. The smallest cardinality of a basis of G is called the *basis size* of G. We prove upper bounds for basis sizes of dihedral and Boolean groups. We find a lower bound for the basis size of a Boolean group. We also calculate basis sizes for dihedral and Boolean groups of small orders.

1 Introduction

A subset B of a group G is called a *basis* of G if each element $g \in G$ can be written as $g = ab$ for some $a, b \in B$. The smallest cardinality of a basis of G is called the *basis size* of G and is denoted by $r[G]$. The problem of estimating $r[G]$ for a cyclic group G was first proposed by Schur and various bounds were obtained by Rohrbach [13], Moser [10], Stöhr [15], Klotz [7] and others. Bases for arbitrary groups were dealt by Rohrbach [14] and lately by Cherly [5], Bertram and Herzog [4], Nathanson [11], Kozma and Lev [8].

A family $\mathcal{G}$ of finite groups is *well-based* if there exists a constant $c \in \mathbb{R}_+$ such that $r[G] \le c\sqrt{|G|}$ for each $G \in \mathcal{G}$. Bertram and Herzog [4] showed that the families of nilpotent groups, as well as the families of the alternating and symmetric groups, are well-based. Kozma and Lev [8] proved (using the classification of finite simple groups) that $r[G] \le \frac{4}{\sqrt{3}}\sqrt{|G|}$ for any finite group G. Therefore, the family of all finite groups is well-based.

The definition of a basis B for a group G implies that $|G| \le |B|^2$, and hence $r[G] \ge \sqrt{|G|}$. The fraction

$$\delta[G] := \frac{r[G]}{\sqrt{|G|}} \ge 1$$

is called the *basis characteristic* of G.

In this paper we shall evaluate basis characteristics of dihedral and Boolean groups. We recall that the *dihedral group* D_{2n} of order $2n$ is the isometry group of a regular n-gon. The dihedral group D_{2n} contains a normal cyclic subgroup of index 2. A standard model of a cyclic group of order n is the multiplicative group

$$C_n = \{z \in \mathbb{C} : z^n = 1\}$$

of n-th roots of 1. The group C_n is isomorphic to the additive group of the ring $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. A group G is called *Boolean* if $g^{-1} = g$ for every $g \in G$. It is well-known that Boolean groups are Abelian and are isomorphic to the powers of the two-element cyclic group C_2.

Kozma and Lev [8] proved that each finite group G has the basis characteristic $\delta[G] \le \frac{4}{\sqrt{3}} \approx 2.3094$. Until now, the best estimate for an upper bound of the basis characteristic was the estimate for the class of cyclic groups: each cyclic group C_n has the basis characteristic $\delta[C_n] \le 2$. We shall show that for dihedral groups this upper bound can be improved to $\delta[D_{2n}] \le \frac{24}{\sqrt{146}} \approx 1.9862$. Moreover, if $n \ge 2 \cdot 10^{15}$, then $\delta[D_{2n}] < \frac{4}{\sqrt{6}} \approx 1.633$. Also we prove that each Boolean group G has the basis size $\frac{1+\sqrt{8|G|-7}}{2} \le r[G] < \frac{3}{\sqrt{2}}\sqrt{|G|}$ and, therefore, its basis characteristic $\delta[G] < \frac{3}{\sqrt{2}} \approx 2.1213$.

For a class $\mathcal{G}$ of finite groups the number

$$\delta[\mathcal{G}] = \sup_{G \in \mathcal{G}} \delta[G]$$

is called the *basis characteristic* of the class $\mathcal{G}$.

For a real number x we put

$$\lceil x \rceil = \min\{n \in \mathbb{Z} : n \ge x\} \text{ and } \lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \le x\}.$$

2 Known results

In this section we recall some known results on bases in finite groups. Kozma and Lev [8] proved the following fundamental fact.

Theorem 1. *Each finite group G has the basis characteristic $\delta[G] \le \frac{4}{\sqrt{3}}$.*

Bertram and Herzog [4] proved the following proposition.

Proposition 2. *Let G be a finite group. Then*

1) $r[G] \le |G : H| \cdot r[H]$ for any subgroup $H \subset G$;

2) $r[G] \le r[G/H] \cdot |H|$ for any normal subgroup $H \subset G$;

3) $r[G] \le 2r[G/H] \cdot r[H]$ for any normal subgroup $H \subset G$;

4) $r[G] \le |H| + |G:H| - 1$ for any subgroup $H \subset G$.

In evaluating basis characteristics of dihedral groups we shall use difference characteristics of cyclic groups. A subset B of a group G is called a *difference basis* of G if each element $g \in G$ can be written as $g = xy^{-1}$ for some $x, y \in B$. The smallest cardinality of a difference basis of G is called the *difference size* of G and is denoted by $\Delta[G]$. The definition of a difference basis B for a group G implies that $|G| \le |B|^2$, and hence $\Delta[G] \ge \sqrt{|G|}$. The fraction

$$ð[G] := \frac{\Delta[G]}{\sqrt{|G|}} \ge 1$$

is called the *difference characteristic* of G.

Difference basis have applications in the study of structure of superextensions of groups, see [1, 3].

Difference sizes of finite cyclic groups were evaluated in [2] with the help of difference sizes of the order-intervals $[1,n] \cap \mathbb{Z}$ in the additive group $\mathbb{Z}$ of integer numbers. For a natural number $n \in \mathbb{N}$ by $\Delta[n]$ we shall denote the difference size of the order-interval $[1,n] \cap \mathbb{Z}$ and by $ð[n] := \frac{\Delta[n]}{\sqrt{n}}$ its difference characteristic. The asymptotics of the sequence $(ð[n])_{n=1}^{\infty}$ was studied by Rédei and Rényi [12], Leech [9] and Golay [6] who eventually proved that

$$\sqrt{2 + \tfrac{4}{3\pi}} < \sqrt{2 + \max_{0<\varphi<2\pi} \tfrac{2\sin(\varphi)}{\varphi+\pi}} \le \lim_{n\to\infty} ð[n] = \inf_{n\in\mathbb{N}} ð[n] \le ð[6166] = \frac{128}{\sqrt{6166}} < ð[6] = \sqrt{\tfrac{8}{3}}.$$

Banakh and Gavrylkiv [2] applied difference sizes of the order-intervals $[1,n] \cap \mathbb{Z}$ to give upper bounds for difference sizes of finite cyclic groups.

Proposition 3. *For every $n \in \mathbb{N}$ the cyclic group C_n has the difference size $\Delta[C_n] \le \Delta\big[\lceil \frac{n-1}{2} \rceil\big]$, which implies that*

$$\limsup_{n\to\infty} ð[C_n] \le \frac{1}{\sqrt{2}} \inf_{n\in\mathbb{N}} ð[n] \le \frac{64}{\sqrt{3083}} < \frac{2}{\sqrt{3}}.$$

Banakh and Gavrylkiv [2] proved the following upper bound for difference characteristics of cyclic groups.

Theorem 4. *For any $n \in \mathbb{N}$ the cyclic group C_n has the difference characteristic:*

1) $ð[C_n] \le ð[C_4] = \frac{3}{2}$;

2) $ð[C_n] \le ð[C_2] = ð[C_8] = \sqrt{2}$ if $n \ne 4$;

3) $ð[C_n] \le \frac{12}{\sqrt{73}} < \sqrt{2}$ if $n \ge 9$;

4) $ð[C_n] \le \frac{24}{\sqrt{293}} < \frac{12}{\sqrt{73}}$ if $n \ge 9$ and $n \ne 292$;

5) $ð[C_n] < \frac{2}{\sqrt{3}}$ if $n \ge 2 \cdot 10^{15}$.

The following Table 1 of difference sizes and characteristics of cyclic groups C_n for ≤ 100 is taken from [2].

n	$\Delta[C_n]$	$\eth[C_n]$	n	$\Delta[C_n]$	$\eth[C_n]$	n	$\Delta[C_n]$	$\eth[C_n]$	n	$\Delta[C_n]$	$\eth[C_n]$
1	1	1	26	6	1.1766...	51	8	1.1202...	76	10	1.1470...
2	2	1.4142...	27	6	1.1547...	52	9	1.2480...	77	10	1.1396...
3	2	1.1547...	28	6	1.1338...	53	9	1.2362...	78	10	1.1322...
4	3	1.5	29	7	1.2998...	54	9	1.2247...	79	10	1.1250...
5	3	1.3416...	30	7	1.2780...	55	9	1.2135...	80	11	1.2298...
6	3	1.2247...	31	6	1.0776...	56	9	1.2026...	81	11	1.2222...
7	3	1.1338...	32	7	1.2374...	57	8	1.0596...	82	11	1.2147...
8	4	1.4142...	33	7	1.2185...	58	9	1.1817...	83	11	1.2074...
9	4	1.3333...	34	7	1.2004...	59	9	1.1717...	84	11	1.2001...
10	4	1.2649...	35	7	1.1832...	60	9	1.1618...	85	11	1.1931...
11	4	1.2060...	36	7	1.1666...	61	9	1.1523...	86	11	1.1861...
12	4	1.1547...	37	7	1.1507...	62	9	1.1430...	87	11	1.1793...
13	4	1.1094...	38	8	1.2977...	63	9	1.1338...	88	11	1.1726...
14	5	1.3363...	39	7	1.1208...	64	9	1.125	89	11	1.1659...
15	5	1.2909...	40	8	1.2649...	65	9	1.1163...	90	11	1.1595...
16	5	1.25	41	8	1.2493...	66	10	1.2309...	91	10	1.0482...
17	5	1.2126...	42	8	1.2344...	67	10	1.2216...	92	11	1.1468...
18	5	1.1785...	43	8	1.2199...	68	10	1.2126...	93	12	1.2443...
19	5	1.1470...	44	8	1.2060...	69	10	1.2038...	94	12	1.2377...
20	6	1.3416...	45	8	1.1925...	70	10	1.1952...	95	12	1.2311...
21	5	1.0910...	46	8	1.1795...	71	10	1.1867...	96	12	1.2247...
22	6	1.2792...	47	8	1.1669...	72	10	1.1785...	97	12	1.2184...
23	6	1.2510...	48	8	1.1547...	73	9	1.0533...	98	12	1.2121...
24	6	1.2247...	49	8	1.1428...	74	10	1.1624...	99	12	1.2060...
25	6	1.2	50	8	1.1313...	75	10	1.1547...	100	12	1.2

Table 1: Difference sizes and characteristics of cyclic groups C_n for $n \le 100$.

3 Basis sizes and characteristics of dihedral groups

In this section we shall evaluate basis sizes and characteristics of dihedral groups. The following proposition yields some upper bounds for the basis size of a dihedral group.

Proposition 5. *For any numbers $n, m \in \mathbb{N}$ the dihedral group D_{2nm} has the basis size*

1) $r[D_{2nm}] \le 2n \cdot r[C_m]$;

2) $r[D_{2nm}] \le r[D_{2n}] \cdot m$;

3) $r[D_{2nm}] \le 2r[D_{2n}] \cdot r[C_m]$.

Proof. It is well-known that the dihedral group D_{2nm} contains a normal cyclic subgroup of order nm, which can be identified with the cyclic group C_{nm}. The subgroup $C_m \subset C_{nm}$ is normal in D_{2mn} and the quotient group D_{2mn}/C_m is isomorphic to D_{2n}. Applying Proposition 2(1-3), we obtain the desired upper bounds. □

Theorem 6. *For any $n \in \mathbb{N}$ the dihedral group D_{2n} has the basis size $r[D_{2n}] \le 2\Delta[C_n]$.*

Proof. It is well-known that the group D_{2n} contains a cyclic subgroup C_n and an element $s \in D_{2n} \setminus C_n$ such that $s = s^{-1}$ and $sxs = x^{-1}$ for all $x \in C_n$. Fix a difference basis $B \subset C_n$ of cardinality $|B| = \Delta[C_n]$. Therefore, $BB^{-1} = C_n$ and $sBs = B^{-1}$. We claim that $B^{-1} \cup sB$ is a basis. Indeed,

$$(B^{-1} \cup sB)(B^{-1} \cup sB) = B^{-1}B^{-1} \cup B^{-1}sB \cup sBB^{-1} \cup sBsB \supset s(BB^{-1}) \cup (sBs)B$$

$$= sC_n \cup B^{-1}B = (D_{2n} \setminus C_n) \cup BB^{-1} = (D_{2n} \setminus C_n) \cup C_n = D_{2n}.$$

Therefore, $r[D_{2n}] \le |B^{-1} \cup sB| = 2|B| = 2\Delta[C_n]$. □

In Table 2 we present the results of computer calculations of basis sizes and characteristics of dihedral groups of order ≤ 80.

$2n$	$r[D_{2n}]$	$2\Delta[C_n]$	$\delta[D_{2n}]$	$2n$	$r[D_{2n}]$	$2\Delta[C_n]$	$\delta[D_{2n}]$
2	2	2	1.4142...	42	9	10	1.3887...
4	3	4	1.5	44	9	12	1.3568...
6	3	4	1.2247...	46	9	12	1.3269...
8	4	6	1.4142...	48	9	12	1.2990...
10	4	6	1.2649...	50	9	12	1.2727...
12	5	6	1.4433...	52	10	12	1.3867...
14	5	6	1.3363...	54	10	12	1.3608...
16	5	8	1.25	56	10	12	1.3363...
18	6	8	1.4142...	58	10	14	1.3130...
20	6	8	1.3416...	60	10	14	1.2909...
22	7	8	1.4924...	62	10	12	1.2700...
24	6	8	1.2247...	64	11	14	1.375
26	7	8	1.3728...	66	10	14	1.2309...
28	7	10	1.3228...	68	11	14	1.3339...
30	7	10	1.2780...	70	10	14	1.1952...
32	7	10	1.2374...	72	11	14	1.2963...
34	8	10	1.3719...	74	11	14	1.2787...
36	8	10	1.3333...	76	12	16	1.3764...
38	8	10	1.2977...	78	11	14	1.2455...
40	8	12	1.2649...	80	12	16	1.3416...

Table 2: Basis sizes and characteristics of dihedral groups D_{2n} for $2n \le 80$.

Theorem 7. *Let D_{2n} be a dihedral group of order $2n$. Then for any number $n \in \mathbb{N}$, the basis characteristic of the dihedral group is $\delta[D_{2n}] \le \frac{24}{\sqrt{146}}$. If $n \ge 2 \cdot 10^{15}$, then $\delta[D_{2n}] < \frac{4}{\sqrt{6}}$.*

Proof. It is well-known that the dihedral group D_{2n} contains a normal cyclic subgroup of order n, which can be identified with the cyclic group C_n. Applying Theorem 6, we obtain the upper bound

$$\delta[D_{2n}] = \frac{r[D_{2n}]}{\sqrt{2n}} \le \frac{2\Delta[C_n]}{\sqrt{2n}} = \sqrt{2} \cdot \eth[C_n].$$

If $n \ge 9$, then using part 3 of Theorem 4, we get $\eth[C_n] \le \frac{12}{\sqrt{73}}$. Therefore,

$$\delta[D_{2n}] \le \sqrt{2} \cdot \frac{12}{\sqrt{73}} = \frac{24}{\sqrt{146}}.$$

Analyzing the data from Table 2, one can check that $\delta[D_{2n}] \le \frac{24}{\sqrt{146}} \approx 1.9863$ for all $n \le 8$.

If $n \ge 2 \cdot 10^{15}$, then using part 5 of Theorem 4, we get $\eth[C_n] < \frac{2}{\sqrt{3}}$, and hence

$$\delta[D_{2n}] \le \sqrt{2} \cdot \eth[C_n] < \frac{4}{\sqrt{6}}.$$

□

The results of computer calculations given in Table 2 and Theorem 7 imply the following lower and upper bounds for the basis characteristic of the class $\mathsf{Dihedral}$ of dihedral groups.

Proposition 8. *The class* $\mathsf{Dihedral}$ *of dihedral groups has the basis characteristic*

$$1.5 = \frac{3}{\sqrt{4}} = \delta[D_4] \le \delta[\mathsf{Dihedral}] \le \frac{24}{\sqrt{146}} \approx 1.9863.$$

Proof. Theorem 7 yields the upper bound

$$\delta[\mathsf{Dihedral}] = \sup_{n\in\mathbb{N}} \delta[D_{2n}] \le \frac{24}{\sqrt{146}}.$$

On the other hand, the basis size $r[D_4] = 3$ of the dihedral group D_4 witnesses that

$$1.5 = \delta[D_4] \le \delta[\mathsf{Dihedral}].$$

□

Question 9. Is $\delta[\mathsf{Dihedral}] = \delta[D_4] = 1.5$?

4 Basis sizes and characteristics of Boolean groups

In this section we evaluate basis sizes and characteristics of finite Boolean groups. A group G is called *Boolean* if $g^{-1} = g$ for every $g \in G$. It is well-known that Boolean groups are Abelian and are isomorphic to the powers of the two-element cyclic group C_2. Let $\mathsf{Boolean}$ denote the class of finite Boolean groups and let $\mathsf{Boolean}_0 := \{G \in \mathsf{Boolean} : \sqrt{|G|} \in \mathbb{N}\}$.

We start with the lower bound for $r[G]$.

Theorem 10. *Each Boolean group G has $r[G] \ge \frac{1+\sqrt{8|G|-7}}{2}$.*

Proof. Take a basis $B \subset G$ of cardinality $|B| = r[G]$ and consider the surjective map $\xi : B \times B \to G$, $\xi : (x, y) \mapsto xy$. Observe that for the unit e of the group G the preimage $\xi^{-1}(e)$ coincides with the diagonal $\{(x, y) \in B \times B : x = y\}$ of the square $B \times B$, and hence has cardinality $|\xi^{-1}(e)| = |B|$. Observe also that for any element $g \in G \setminus \{e\}$ and any $(x, y) \in \xi^{-1}(g)$, we get $x \ne y$ and $yx = xy = g$, which implies that $|\xi^{-1}(g)| \ge 2$. Then

$$|B|^2 = |B \times B| = |\xi^{-1}(e)| + \sum_{g \in G\setminus\{e\}} |\xi^{-1}(g)| \ge |B| + 2(|G| - 1).$$

Therefore, $|B|^2 - |B| - 2(|G| - 1) \ge 0$, and hence $r[G] = |B| \ge \frac{1+\sqrt{1+8(|G|-1)}}{2} = \frac{1+\sqrt{8|G|-7}}{2}$. □

Theorem 11. *Each finite Boolean group G has*

$$\frac{1 + \sqrt{8|G| - 7}}{2} \le r[G] < \begin{cases} 2\sqrt{|G|}, & \text{if } G \in \mathsf{Boolean}_0; \\ \frac{3}{\sqrt{2}}\sqrt{|G|}, & \text{otherwise.} \end{cases}$$

Proof. The lower bound $\frac{1+\sqrt{8|G|-7}}{2} \le r[G]$ follows from Theorem 10.

The group G, being Boolean, is isomorphic to $(C_2)^n$ for some $n \ge 0$. Let $k = \lfloor \frac{n}{2} \rfloor$ and find a subgroup $H \subset G$ of order 2^k. By Proposition 2(4), $r[G] \le |H| + |G/H| - 1$.

If $G \in \mathsf{Boolean}_0$, then $n = 2k$ is even and

$$r[G] \le |H| + |G/H| - 1 = 2^k + 2^k - 1 < 2\sqrt{2^{2k}} = 2\sqrt{|G|}.$$

If $G \notin \mathsf{Boolean}_0$, then $n = 2k + 1$ is odd and

$$r[G] \le |H| + |G/H| - 1 = 2^k + 2^{k+1} - 1 < \frac{3}{\sqrt{2}}\sqrt{2^{2k+1}} = \tfrac{3}{\sqrt{2}}\sqrt{|G|}.$$

□

In the following Table 3 we present the results of computer calculations of basis sizes and characteristics of Boolean groups $(C_2)^n$ for $n \le 6$. In this table

$$lb[G] := \left\lceil \frac{1 + \sqrt{8|G| - 7}}{2} \right\rceil \quad \text{and} \quad ub[G] := \left\lfloor \sqrt{\frac{9|G|}{2}} \right\rfloor$$

are the lower and upper bounds given in Theorems 10 and 11.

Theorem 11 implies the following corollaries.

G	C_2	$(C_2)^2$	$(C_2)^3$	$(C_2)^4$	$(C_2)^5$	$(C_2)^6$
$lb[G]$	2	3	5	6	9	12
$r[G]$	2	3	5	6	10	14
$ub[G]$	3	4	6	8	12	16
$\delta[G]$	1.4142...	1.5	1.7677...	1.5	1.7677...	1.75

Table 3: Basis sizes and characteristics of Boolean groups $(C_2)^n$ for $n \le 6$.

Corollary 12. *The class* Boolean *of finite Boolean groups has the basis characteristic*

$$1.7677... = \frac{5}{\sqrt{8}} = \delta[(C_2)^3] \le \delta[\mathsf{Boolean}] \le \frac{3}{\sqrt{2}} = 2.1213...$$

Proof. The upper bound $\delta[\mathsf{Boolean}] \le \frac{3}{\sqrt{2}}$ follows from Theorem 11 and the lower bound $\delta[\mathsf{Boolean}] \ge \delta[(C_2)^3] = \frac{5}{\sqrt{8}}$ follows from the known value $r[(C_2)^3] = 5$. □

Corollary 13. *The class* $\mathsf{Boolean}_0$ *has the basis characteristic*

$$1.75 = \frac{14}{\sqrt{64}} = \delta[(C_2)^6] \le \delta[\mathsf{Boolean}_0] \le 2$$

Proof. The upper bound $\delta[\mathsf{Boolean}_0] \le 2$ follows from Theorem 11 and the lower bound $\delta[\mathsf{Boolean}_0] \ge \delta[(C_2)^6] = 1.75$ follows from the known value $r[(C_2)^6] = 14$. □

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