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Number of Dissections of the Regular n-gon by Diagonals

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Abstract

This paper presents a formula for the distinct dissections by diagonals of a regular n-gon modulo the action of the dihedral group. This counting includes dissection with intersecting or non-intersecting diagonals. We utilize a corollary of the Cauchy-Frobenius theorem, which involves counting of cycles. We also give an explicit formula for the prime number case. We give as a remark the number of distinct dissections, modulo the action of the cyclic group of finite order.

1 Introduction

The theory of polygon dissection has proven to be a rich area of mathematical thoughts. Cayley derived the number of ways to dissect an $n-$gon using a specified number of diagonals. Other mathematicians gave proofs of older formulas involving polygon dissections using new techniques, such as generating functions, Legendre polynomials, and Lagrange inversion [2]. Przytycki and Sikora showed relationships between polygon dissections and special types of numbers, such as the Catalan numbers [4]. Explicit formulas for dissections of a regular polygon using non-intersecting diagonals were derived in a paper of Bowman and Regev [1]. More recently, Siegel counted the number of dissections of a regular n-gon using non-intersecting diagonals in his thesis [5].

The main aim of this paper is to count the number of distinct dissections of an unlabeled regular n-gon by diagonals modulo the dihedral group. We consider both intersecting and non-intersecting diagonals in our counting. To do this, we first label the vertices of the polygon and determine which dissections of this labeled n-gon are the same up to the canonical action of the dihedral group of degree n. We present the following definition:

Definition 1. Let $n \geq 3$. A regular polygon with n vertices is called an n-gon. A *diagonal* of an n-gon is a segment extending from a vertex to a non-adjacent vertex. A *dissection* of the n-gon is any set of crossing or non-crossing diagonals of the n-gon. A dissection without any diagonal is an *empty dissection*.

The main result of this paper is anchored on a consequence of the Cauchy-Frobenius theorem [3, Corollary 1.7A, p. 26]. We give it below as Lemma 2.

Lemma 2. *Let G be a finite group acting on a finite set Δ. Suppose Γ is a non-empty finite set and $\mathrm{Fun}(\Delta, \Gamma)$ is the set of all functions from Δ to Γ, then G acts on $\mathrm{Fun}(\Delta, \Gamma)$ by*

$$f^x(\delta) = f(\delta^{x^{-1}}) \ \ (\forall f \in \mathrm{Fun}(\Delta, \Gamma), x \in G, \delta \in \Delta.)$$

In addition, the number of orbits of this action is equal to

$$\frac{1}{|G|}\left(\sum_{g \in G} |\Gamma|^{c(g)}\right)$$

where $c(g)$ counts the number of cycles of g as it acts on Δ, including the trivial cycles, if they exist.

2 Preliminaries

Let $[n] = \{1, 2, \ldots, n\}$ be the set of vertices of a regular n-gon. It is well-known that the *dihedral group* of degree n, with presentation $D_n = \langle r, s : r^n = 1 = s^2, srs = r^{-1}\rangle$, acts on $[n]$ in a natural way. This is obvious when we express the elements of D_n as permutations of $[n]$ corresponding to the symmetries of an n-gon, i.e., $D_n \leq \mathrm{Sym}([n])$. Here, r is the $\frac{2\pi}{n}$-rotation and s is the reflection along the axis through center and vertex 1.

Definition 3. Let $i, j \in [n]$ be vertices of the n-gon. If $i < j$, then we define the *cycle length* of i and j as follows:

$$d(\{i, j\}) = \min\{j - i, n - (j - i) \bmod n\}.$$

Form $\Delta_n = \{\{i, j\} : d(\{i, j\}) \geq 2\}$. This is simply the set of all diagonals of the n-gon and it can be shown that $|\Delta_n| = \frac{n^2 - 3n}{2}$. Moreover, the group D_n acts on Δ_n in a natural way. Observe that $\{i, j\} \in \Delta_n$ if and only if i and j are non-adjacent. Since each element of D_n only *rotates* or *reflects* the n-gon, then for $x \in D_n$

$$d(\{i^x, j^x\}) = d(\{i, j\}).$$

It can then be proven that the map $\Delta_n \times D_n \to \Delta_n$ defined by

$$\{i, j\}^g = \{i^g, j^g\}$$

is an action. Let us denote the corresponding permutation representation of this action by $\rho : D_n \to \mathrm{Sym}(\Delta_n)$. That is, $\rho(r)$ and $\rho(s)$ are permutations of the set Δ_n satisfying the following:

i. $\rho(r)(\{i,j\}) = \{i+1 \bmod n, j+1 \bmod n\}$;

ii. $\rho(s)(\{i,j\}) = \{2-i \bmod n, 2-j \bmod n\}$.

Consider the family $\mathrm{Fun}(\Delta_n,\Gamma)$ where $\Gamma = \{0,1\}$. We can view each function $f \in \mathrm{Fun}(\Delta_n,\Gamma)$ as a way of dissecting the n-gon. Here, $f(\{i,j\}) = 1$ means that there exists a diagonal from vertex i to j. Otherwise, i and j are not connected by any diagonal. The action of an element $x \in D_n$ on $\mathrm{Fun}(\Delta_n,\Gamma)$ can be viewed as either rotating or reflecting the dissection f to f^x preserving the form of the dissection. Consequently, every orbit of this action represents a certain way of dissecting an n-gon. This only means that counting the distinct orbits is equivalent to counting the number of distinct dissections of the n-gon modulo the dihedral group.

Proposition 4. *The number $\gamma(n)$ of distinct dissections of an n-gon modulo the dihedral action is*

$$\gamma(n) = \frac{1}{2n}\left(\sum_{g\in D_n} 2^{c(g)}\right)$$

where $c(g)$ counts the number of cycles of g as it acts on Δ_n, including the trivial cycles whenever they exist.

3 Result

The following observation will be used to prove the succeeding claims:

Proposition 5. *Let $n > 4$ be a natural number. Then $\rho[D_n] \cong D_n$.*

Proof. Let r, s be the generators of D_n. When we express $\rho(r)$ as a product of disjoint cycles, we see that $(\{1,3\}\ \{2,4\}\ \{3,5\}\ \ldots\ \{n-1,1\}\ \{n,2\})$ is one of these cycles. Since this cycle is of length n and $|\rho(r)| \leq n$, then the length of each cycle is at most n and so $|\rho(r)| = n$.

We now show that $|\rho(s)| = 2$. Since $|s| = 2$, then $|\rho(s)|$ divides 2 and so the length of each cycle is at most two. If n is odd then $\rho(s)$ sends $\left\{1, \frac{n+1}{2}\right\}$ to $\left\{1, \frac{n+3}{2}\right\}$ and this creates a cycle of length two. If n is even, $\rho(s)$ sends $\left\{1, \frac{n}{2}\right\}$ to $\left\{1, \frac{n+4}{2}\right\}$ and again, this makes a cycle of length two. Hence, $|\rho(s)| = 2$.

Finally, we obtain

$$\rho(s)\rho(r)\rho(s) = \rho(srs) = \rho(r^{-1}) = \rho(r)^{-1}.$$

□

For $x \in D_n$, we now count the number of cycles in the decomposition of $\rho(x)$. We make use of the well-known properties of permutations stated as Lemma 6.

Lemma 6. *Let $\alpha \in \mathrm{Sym}([n])$ such that $\alpha = c_1 c_2 \cdots c_l$, where c_i's are disjoint cycles, then*

$$|\alpha| = \mathrm{lcm}(\mathrm{length}(c_i) : i \in \{1, 2, \ldots, l\}).$$

If $\alpha = (a_1\ a_2\ \ldots\ a_k)$, then the number of disjoint cycles of α^t, where $1 \le t \le k$, is $\gcd(k, t)$.

Lemma 7. *Let $n \geq 4$. For $i \in \{1, 2, \ldots, n\}$,*

$$c(r^i) = \begin{cases} \left(\frac{n-4}{2}\right)\gcd(n,i) + \gcd(\frac{n}{2}, i), & \textit{if } n \textit{ is even;} \\ \left(\frac{n-3}{2}\right)\gcd(n,i), & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof. We start with $n = 4$. Then $\Delta_4 = \{\{1,3\}, \{2,4\}\}$, $i \in \{1,2,3,4\}$ and we obtain the following computations:

$i = 1$. $\rho(r) = (\{1,3\}\ \{2,4\})$ and so $c(r) = 1 = \left(\frac{4-4}{2}\right)\gcd(4,1) + \gcd(\frac{4}{2}, 1)$;

$i = 2$. $\rho(r^2) = (\{1,3\})(\{2,4\}) = 1_{\Delta_4}$ and so $c(r^2) = 2 = \left(\frac{4-4}{2}\right)\gcd(4,2) + \gcd(\frac{4}{2}, 2)$;

$i = 3$. $\rho(r^3) = (\{1,3\}\ \{2,4\})$ and so $c(r^3) = 1 = \left(\frac{4-4}{2}\right)\gcd(4,3) + \gcd(\frac{4}{2}, 3)$;

$i = 4$. $\rho(r^4) = \rho(1_{[4]}) = 1_{\Delta_4} = (\{1,3\})(\{2,4\})$ and so $c(r^4) = 2 = \left(\frac{4-4}{2}\right)\gcd(4,4) + \gcd(\frac{4}{2}, 4)$.

We let $n > 4$ and consider two cases. Firstly, assume n is even. The elements of Δ_n can be partitioned according to different cycle lengths and we get the following cycle decomposition:

$$\rho(r) = \underbrace{(\{1,3\}\ \{2,4\}\ \ldots\ \{n,2\})}_{n-\text{cycle}}\underbrace{(\{1,4\}\ \{2,5\}\ \ldots\ \{n,3\})}_{n-\text{cycle}}\ldots$$
$$\underbrace{(\{1,n/2\}\ \{2,(n+2)/2\}\ \ldots\ \{n,(n-2)/2\})}_{n-\text{cycle}}\underbrace{(\{1,(n+2)/2\}\ \{2,(n+4)/2\}\ \ldots\ \{n/2,n\})}_{n/2-\text{cycle}}$$

in which there are $\frac{n-4}{2}$ n-cycles and only one $\frac{n}{2}$-cycle. For $i \in \{1, 2, \ldots, n\}$:

$$\rho(r^i) = (\{1,3\}\ \{2,4\}\ \ldots\ \{n,2\})^i(\{1,4\}\ \{2,5\}\ \ldots\ \{n,3\})^i\ldots$$
$$(\{1,n/2\}\ \{2,(n+2)/2\}\ \ldots\ \{n,(n-2)/2\})^i(\{1,(n+2)/2\}\ \{2,(n+4)/2\}\ \ldots\ \{n/2,n\})^i.$$

By Lemma 6, we obtain

$$c(r^i) = \left(\frac{n-4}{2}\right)\gcd(n,i) + \gcd(n/2, i).$$

Secondly, take n to be odd. Similar to the first case, the elements of Δ_n can be partitioned according to different cycle lengths. We obtain the following:

$$\rho(r) = \underbrace{(\{1,3\}\ \ \{2,4\}\ \ \ldots\ \ \{n,2\})}_{n-\text{cycle}}\underbrace{(\{1,4\}\ \ \{2,5\}\ \ \ldots\ \ \{n,3\})}_{n-\text{cycle}}\cdots$$

$$\underbrace{(\{1,(n+1)/2\}\ \ \{2,(n+3)/2\}\ \ \ldots\ \ \{n,(n-1)/2\})}_{n-\text{cycle}}$$

in which there are $\frac{n-3}{2}$ n-cycles. As with the above, we can compute the following:

$$c(r^i) = \left(\frac{n-3}{2}\right)\gcd(n,i).$$

□

Lemma 8. *Let $n \geq 4$ and $s_v \in D_n \backslash \langle r \rangle$ be a reflection with axis passing through the center and a vertex. Then*

$$c(s_v) = \begin{cases} \frac{n^2-2n}{4}, & \textit{if } n \textit{ is even;} \\ \frac{n^2-2n-3}{4}, & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof. Note that the case $n = 4$ is an easy computation. We consider two cases for $n > 4$. Firstly, take n to be even. The axis of s_v is the diagonal $\left\{i, i + \frac{n}{2} \bmod n\right\}$. Form

$$\Delta_o = \left\{\{i - k \bmod n, i + k \bmod n\} : k \in \left\{1, 2, \ldots, \frac{n-2}{2}\right\}\right\}.$$

Observe that $(i \pm k \bmod n)^{s_v} = i \mp k \bmod n$ and preserves both i and $i + \frac{n}{2} \bmod n$. This implies that s_v fixes setwise each element of $\Delta_o \cup \left\{\left\{i, i + \frac{n}{2} \bmod n\right\}\right\}$. Let $\{\alpha, \beta\}$ be an element of $\Delta_n \backslash \left(\Delta_o \cup \left\{\left\{i, i + \frac{n}{2} \bmod n\right\}\right\}\right)$, we consider three subcases. Let $\alpha = i$. It follows that $\beta \in \left\{i \pm k \bmod n : k \in \left\{2, \ldots, \frac{n-2}{2}\right\}\right\}$. If $\beta = i + k \bmod n$ then $\{i, i + k \bmod n\}^{s_v} = \{i, i - k \bmod n\}$. If $\beta = i - k \bmod n$ then $\{i, i - k \bmod n\}^{s_v} = \{i, i + k \bmod n\}$. Similar argument when $\alpha = i + \frac{n}{2} \bmod n$. Suppose $\{\alpha, \beta\} \cap \left\{i, i + \frac{n}{2} \bmod n\right\} = \varnothing$. It implies that $\alpha, \beta \in \left\{i \pm k \bmod n : k \in \left\{1, 2, \ldots, \frac{n-2}{2}\right\}\right\}$. If $\alpha = i + k_1 \bmod n$ and $\beta = i + k_2 \bmod n$ where $k_1, k_2 \in \left\{1, 2, \ldots, \frac{n-2}{2}\right\}$, then $\{i + k_1 \bmod n, i + k_2 \bmod n\}^{s_v} = \{i - k_1 \bmod n, i - k_2 \bmod n\}$. Similar argument can be used for $\alpha = i - k_1 \bmod n$ and $\beta = i - k_2 \bmod n$. Without loss of generality, assume $\alpha = i - k_1 \bmod n$ and $\beta = i + k_2 \bmod n$. It means that $k_1 \neq k_2$ and so $\{i - k_1 \bmod n, i + k_2 \bmod n\}^{s_v} = \{i + k_1 \bmod n, i - k_2 \bmod n\}$. In all these subcases, we obtain $\{\alpha, \beta\}^{s_v} \neq \{\alpha, \beta\}$.

Proposition 5 and Lemma 6 assure that the length of every cycle in $\rho(s_v)$ is at most two. The above results tell us that each element of $\Delta_o \cup \left\{i, i + \frac{n}{2} \bmod n\right\}$ creates an 1-cycle in $\rho(s_v)$, while each element of $\Delta_n \backslash \left(\Delta_o \cup \left\{i, i + \frac{n}{2} \bmod n\right\}\right)$ creates a 2-cycle. Hence,

$$c(s_v) = \frac{n^2 - 2n}{4}.$$

For the second case, assume n is an odd integer. The axis of s_v is the segment extending from vertex i to the midpoint of the edge $\{i+(n-1)/2 \bmod n, i-(n-1)/2 \bmod n\}$. Form

$$\Delta_o = \left\{\{i+k \bmod n, i-k \bmod n\} : k \in \left\{1, 2, \dots, \frac{n-1}{2}\right\}\right\}.$$

Observe that $i^{s_v} = i$ and $(i \pm k \bmod n)^{s_v} = i \mp k \bmod n$. Thus, each element of

$$\Delta_o \backslash \left\{\left\{i + \frac{n-1}{2} \bmod n, i - \frac{n-1}{2} \bmod n\right\}\right\}$$

creates an 1-cycle in $\rho(s_v)$. Let $\{\alpha, \beta\} \in \Delta_n \backslash \Delta_o$. We consider two subcases. Without loss of generality, assume $\alpha = i$. It follows that $\beta \in \{i \pm k \bmod n : k \in \{2, \dots, (n-1)/2\}\}$ and either $\{i, i+k \bmod n\}^{s_v} = \{i, i-k \bmod n\}$ or $\{i, i-k \bmod n\}^{s_v} = \{i, i+k \bmod n\}$. Let $i \notin \{\alpha, \beta\}$. It means that $\alpha, \beta \in \{i \pm k \bmod n : k \in \{1, 2, \dots, (n-1)/2\}\}$. As with the above, we always obtain $\{\alpha, \beta\}^{s_v} \neq \{\alpha, \beta\}$ in different subcases.

Since the length of each cycle of $\rho(s_v)$ is at most two, then the two subcases above imply that every $\{\alpha, \beta\} \in \Delta_n \backslash \Delta_o$ creates a 2-cycle in $\rho(s_v)$. Hence,

$$c(s_v) = \frac{n^2 - 2n - 3}{4}.$$

□

Lemma 9. *Let $n \geq 6$ be even. Suppose $s_e \in D_n \backslash \langle r \rangle$ to be a reflection with axis passing through the origin and midpoints of opposing edges. Then*

$$c(s_e) = \frac{n^2 - 2n - 4}{4}$$

Proof. The axis of s_e is the segment extending from the midpoint of an edge $\{i, i+1 \bmod n\}$ to the midpoint of $\left\{i - \left(\frac{n}{2} - 1\right) \bmod n, i + \frac{n}{2} \bmod n\right\}$. We note that for $j \in [n]$, $j^{s_e} = (2i+1) - j \bmod n$. Let

$$\Delta_o = \{\{i+k \bmod n, i-k+1 \bmod n\} : k \in \{2, 3, \dots, (n-2)/2\}\}.$$

It should be noted that s_e fixes setwise each element of Δ_o and creates an 1-cycle in $\rho(s_e)$.

For $\{\alpha, \beta\} \in \Delta_n \backslash \Delta_o$, there exists $k \in \left\{1, 2, \dots, \frac{n}{2}\right\}$ such that if $\alpha = i + k \bmod n$, then $\beta \in [n] \backslash \{i+k \bmod n, i-k+1 \bmod n\}$ and so

$$\{i+k \bmod n, \beta\}^{s_e} = \{i-k+1 \bmod n, \beta^{s_e}\} \neq \{\alpha, \beta\}.$$

Also, if $\alpha = i - k + 1 \bmod n$ then $\beta \in [n] \backslash \{i+k \bmod n, i-k+1 \bmod n\}$ and so

$$\{i-k+1 \bmod n, \beta\}^{s_e} = \{i+k \bmod n, \beta^{s_e}\} \neq \{\alpha, \beta\}.$$

Hence, each element of $\Delta_n \backslash \Delta_o$ creates a 2-cycle of $\rho(s_e)$. That is,

$$c(s_e) = \frac{n^2 - 2n - 4}{4}.$$

□

We now collect the properties from Lemmas 7, 8 and 9 and plug them in to the equation in Proposition 4 to obtain our main result.

Theorem 10. *Let $n \geq 3$. The number $\gamma(n)$ of distinct ways of dissecting an n-gon modulo the action of the dihedral group D_n is:*

$$\gamma(n) = \begin{cases} \frac{1}{2n}\left(\left(\sum_{i=1}^{n} 2^{\left(\frac{n-4}{2}\right)\gcd(n,i)+\gcd(\frac{n}{2},i)}\right) + \frac{n}{2}\left(2^{\frac{n^2-2n}{4}} + 2^{\frac{n^2-2n-4}{4}}\right)\right), & \textit{if } n \textit{ is even;} \\ \frac{1}{2n}\left(\left(\sum_{i=1}^{n} 2^{\left(\frac{n-3}{2}\right)\gcd(n,i)}\right) + n\left(2^{\frac{n^2-2n-3}{4}}\right)\right), & \textit{if } n \textit{ is odd.} \end{cases}$$

Corollary 11. *The number of dissections of a regular p-gon modulo the dihedral action, where p is prime with $p \geq 3$, is*

$$\frac{(p-1)\cdot 2^{\frac{p-3}{2}} + 2^{\frac{p^2-3p}{2}} + p \cdot 2^{\frac{p^2-2p-3}{4}}}{2p}.$$

4 Remark

The number $\gamma_c(n)$ of distinct ways of dissecting an n-gon modulo the action of the cyclic group $\langle(1\ 2\ \ldots\ n)\rangle$ is

$$\gamma_c(n) = \begin{cases} \frac{1}{n}\left(\sum_{i=1}^{n} 2^{\left(\frac{n-4}{2}\right)\gcd(n,i)+\gcd(\frac{n}{2},i)}\right), & \text{if } n \text{ is even;} \\ \frac{1}{n}\left(\sum_{i=1}^{n} 2^{\left(\frac{n-3}{2}\right)\gcd(n,i)}\right), & \text{if } n \text{ is odd.} \end{cases}$$

Moreover, when $n = p \geq 3$, then

$$\gamma_c(p) = \frac{(p-1)\cdot 2^{\frac{p-3}{2}} + 2^{\frac{p^2-3p}{2}}}{p}.$$

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References

[1] D. Bowman and A. Regev, Counting symmetry classes of dissections of a convex regular polygon, preprint, 2012, http://arxiv.org/abs/1209.6270.

[2] D. Callan, Polygon dissections and marked Dyck paths, preprint, 2005, https://pdfs.semanticscholar.org/d405.

[3] J. Dixon and B. Mortimer, *Permutation Groups*, Springer-Verlag, 1996.

[4] J. Przytycki and A. Sikora, Polygon dissections and Euler, Fuss, Kirkman and Cayley numbers, preprint, 1998, https://arxiv.org/abs/math/9811086.

[5] A. Siegel, Counting the number of distinct dissections of a regular n-gon, thesis, University of Akron, 2014.

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