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Some Sufficient Conditions for the Log-Balancedness of Combinatorial Sequences

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Abstract

In this paper, we give some new sufficient conditions for log-balancedness of combinatorial sequences. In particular, we show that the product of two log-convex sequences is log-balanced under a mild condition. Then, we apply this result to a series of special combinatorial sequences. In addition, we show some results by using the definition of log-balancedness directly.

1 Introduction

For convenience, we first recall some concepts that will be used later on. The following definition is well known in combinatorics.

Definition 1. (i) For a sequence of real numbers $\{z_n\}_{n\geq 0}$, we say that $\{z_n\}_{n\geq 0}$ is *concave* (resp., *convex*) if $2z_n \geq z_{n-1} + z_{n+1}$ (resp., $2z_n \leq z_{n-1} + z_{n+1}$) for all $n \geq 1$.
(ii) For a sequence of positive numbers $\{z_n\}_{n\geq 0}$, we say that $\{z_n\}_{n\geq 0}$ is *log-concave* (resp., *log-convex*) if $z_n^2 \geq z_{n-1}z_{n+1}$ (resp., $z_n^2 \leq z_{n-1}z_{n+1}$) for all $n \geq 1$.

Došlić [2] gave the following definition.

Definition 2. Let $\{z_n\}_{n\geq 0}$ be a log-convex sequence. We say that $\{z_n\}_{n\geq 0}$ is *log-balanced* if $\{\frac{z_n}{n!}\}_{n\geq 0}$ is log-concave.

Log-concavity and log-convexity play important roles in many subjects. For example, in combinatorics, they are not only instrumental in obtaining the growth rate of a combinatorial sequence, but also fertile sources of inequalities. See, e.g., [1, 6] for more applications of log-concavity and log-convexity.

For a sequence of positive numbers, it is easy to see from the arithmetic-geometric mean inequality that its concavity implies its log-concavity and its log-convexity implies its convexity. Obviously, a sequence $\{z_n\}_{n\geq 0}$ is log-convex (resp., log-concave) if and only if its quotient sequence $\{\frac{z_{n+1}}{z_n}\}_{n\geq 0}$ is nondecreasing (resp., nonincreasing). A log-balanced sequence is naturally log-convex, but its quotient sequence does not grow too fast. Moreover, a sequence $\{z_n\}_{n\geq 0}$ is log-balanced if and only if $z_n^2 \leq z_{n-1}z_{n+1}$ and $(n+1)z_n^2 \geq nz_{n-1}z_{n+1}$ for every $n \geq 1$. Došlić [2] showed that many combinatorial sequences, including the Motzkin numbers, the Fine numbers, the Franel numbers of order 3 and 4, the Apéry numbers, the large Schröder numbers, and the central Delannoy numbers, are log-balanced. Zhao [7, 8] proved that the sequences of the exponential numbers and the Catalan-Larcombe-French numbers are respectively log-balanced.

The main purpose of this paper is to discuss log-balancedness of some combinatorial sequences. In the next section, we present some new sufficient conditions for log-balancedness of combinatorial sequences. In particular, we provide a sufficient condition for log-balancedness of the product of two log-convex sequences. Then, based on this result, we obtain some similar results for a series of special combinatorial sequences.

2 Main results

Zhao [7] gave a sufficient condition for log-balancedness of the product of a log-balanced sequence and a log-concave sequence. Here, we consider log-balancedness of the product of two log-convex sequences.

Theorem 3. *Suppose that the sequences $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ are both log-convex. Let $s_n = \frac{x_{n+1}y_{n+1}}{(n+1)x_ny_n}$ for $n \geq 0$. If $\{s_n\}_{n\geq 0}$ is decreasing, then $\{x_ny_n\}_{n\geq 0}$ is log-balanced.*

Proof. By the log-convexity of the sequences $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$, we know that $\{x_ny_n\}_{n\geq 0}$ is log-convex. Note that $\{s_n\}_{n\geq 0}$ is the quotient sequence of $\{\frac{x_ny_n}{n!}\}_{n\geq 0}$. Since $\{s_n\}_{n\geq 0}$ is decreasing, $\{\frac{x_ny_n}{n!}\}_{n\geq 0}$ is log-concave. Hence, the sequence $\{x_ny_n\}_{n\geq 0}$ is log-balanced. $\square$

Next, we apply Theorem 3 to deduce log-balancedness of some combinatorial sequences.

Corollary 4. *For the sequence $\{C_n\}_{n\geq 1}$ of the Catalan numbers, we have that $\{C_n^2\}_{n\geq 3}$ is log-balanced.*

Proof. Since $\{C_n\}_{n\geq 1}$ is log-convex, $\{C_n^2\}_{n\geq 1}$ is log-convex. For $n \geq 1$, let $s_n = \frac{C_{n+1}^2}{(n+1)C_n^2}$. It is well known that

$$C_n = \frac{1}{n}\binom{2n-2}{n-1}, \quad n \geq 1.$$

Then we have

$$s_n = \frac{4(2n-1)^2}{(n+1)^3}.$$

It is not difficult to verify that $\{s_n\}_{n\geq 3}$ is decreasing. By Theorem 3, the sequence $\{C_n^2\}_{n\geq 3}$ is log-balanced. $\square$

Corollary 5. *For the sequence $\{M_n\}_{n\geq 0}$ of the Motzkin numbers, we have that $\{M_n^2\}_{n\geq 1}$ is log-balanced.*

Proof. The Motzkin numbers satisfy the recurrence

$$(n+3)M_{n+1} = (2n+3)M_n + 3nM_{n-1}, \quad M_0 = M_1 = 1. \tag{1}$$

For $n \geq 0$, let $t_n = \frac{M_{n+1}}{M_n}$ and $s_n = \frac{t_n^2}{n+1}$. It follows from (1) that

$$t_n = \frac{2n+3}{n+3} + \frac{3n}{(n+3)t_{n-1}}. \tag{2}$$

Then we have

$$s_n - s_{n+1} = \frac{(n+2)t_n^2 - (n+1)t_{n+1}^2}{(n+1)(n+2)}.$$

It follows from (2) that

$$\begin{aligned}&(n+2)t_n^2 - (n+1)t_{n+1}^2 \\ = &\frac{(n+2)(n+4)^2t_n^4 - (n+1)(2n+5)^2t_n^2 - 6(n+1)^2(2n+5)t_n - 9(n+1)^3}{(n+4)^2t_n^2}.\end{aligned}$$

For any real number x, we let

$$f(x) = (n+2)(n+4)^2x^4 - (n+1)(2n+5)^2x^2 - 6(n+1)^2(2n+5)x - 9(n+1)^3.$$

Then we have

$$f'(x) = 4(n+2)(n+4)^2x^3 - 2(n+1)(2n+5)^2x - 6(n+1)^2(2n+5)$$

and

$$f''(x) = 12(n+2)(n+4)^2x^2 - 2(n+1)(2n+5)^2.$$

Since $f''(x) > 0$ when $x \geq 1$, we know that f' is increasing over $[1, \infty)$. Došlić and Veljan [3] showed that

$$t_n \geq q_n,$$

where $q_n = \frac{6(n+1)}{2n+5}$. Since

$$f'(q_n) = \frac{18(n+1)^2[48(n+1)(n+2)(n+4)^2 - (2n+5)^4]}{(2n+5)^3} > 0,$$

the function f is increasing over $[q_n, \infty)$. Note that

$$f(q_n) = \frac{81(n+1)^3(16n^3+72n^2+24n-113)}{(2n+5)^4} > 0.$$

By the definition of f, we have

$$(n+2)t_n^2 - (n+1)t_{n+1}^2 = \frac{f(t_n)}{(n+4)^2t_n^2} > 0$$

for each n. This means that $\{s_n\}_{n\geq 0}$ is decreasing. On the other hand, $\{M_n\}_{n\geq 1}$ is log-balanced. It follows from Theorem 3 that the sequence $\{M_n^2\}_{n\geq 1}$ is log-balanced. $\square$

Denote by A_n the number of directed animals of size n (see [5, Exercise 6.46]), which satisfies the recurrence

$$(n+1)A_{n+1} = 2(n+1)A_n + 3(n-1)A_{n-1} \tag{3}$$

with $A_0 = 1$, $A_1 = 1$, and $A_2 = 2$.

Corollary 6. *Both $\{A_n^2\}_{n\geq 2}$ and $\{\frac{A_n}{n}\}_{n\geq 2}$ are log-balanced.*

Proof. It is clear that the sequence $\{\frac{1}{n}\}_{n\geq 1}$ is log-convex. Liu and Wang [4] proved that the sequence $\{A_n\}_{n\geq 0}$ is log-convex. For $n \geq 0$, let $t_n = \frac{A_{n+1}}{A_n}$ and $s_n = \frac{t_n^2}{n+1}$. By (3), we have

$$t_n = 2 + \frac{3(n-1)}{(n+1)t_{n-1}}, \quad n \geq 1. \tag{4}$$

It follows from (4) that

$$\begin{aligned}&(n+2)t_n^2 - (n+1)t_{n+1}^2\\ &= \frac{(n+2)^3t_n^4 - 4(n+1)(n+2)^2t_n^2 - 12n(n+1)(n+2)t_n - 9n^2(n+1)}{(n+2)^2t_n^2}\end{aligned}$$

and

$$\begin{aligned}&n(n+2)^2t_n-(n+1)^3t_{n+1}\\&=\frac{n(n+2)^3t_n^2-2(n+2)(n+1)^3t_n-3n(n+1)^3}{(n+2)t_n}.\end{aligned}$$

For any real number x, let

$$\begin{aligned}f(x)&=(n+2)^3x^4-4(n+1)(n+2)^2x^2-12n(n+1)(n+2)x-9n^2(n+1),\\g(x)&=n(n+2)^3x^2-2(n+2)(n+1)^3x-3n(n+1)^3.\end{aligned}$$

Then we have

$$\begin{aligned}f'(x)&=4(n+2)^3x^3-8(n+1)(n+2)^2x-12n(n+1)(n+2),\\f''(x)&=12(n+2)^3x^2-8(n+1)(n+2)^2,\\g'(x)&=2n(n+2)^3x-2(n+2)(n+1)^3.\end{aligned}$$

It is obvious that $f''(x)>0$ when $x\geq 1$ and hence f' is increasing over $[1,+\infty)$. Noting that $f'(2)>0$, we have $f'(x)>0$ when $x\geq 2$.

Liu and Wang [4] showed that

$$t_n\geq\mu_n,$$

where $\mu_n=\frac{6n}{2n+1}$. Since $f'(\mu_n)>0$, f is increasing over $[\mu_n,\infty)$. It is evident that $g'(x)>0$ for $x\geq 1$ and hence the function g is also increasing over $[1,\infty)$.

Note that

$$\begin{aligned}f(\mu_n)&=\frac{9n^2}{(2n+1)^4}\Big[144n^2(n+2)^3-16(n+1)(n+2)^2(2n+1)^2\\&\quad-8(n+1)(n+2)(2n+1)^3-(n+1)92n+1)^4\Big]\\&=\frac{9n^2}{(2n+1)^4}\Big(144n^4+370n^2-72n^2-513n-81\Big)\end{aligned}$$

and

$$\begin{aligned}g(\mu_n)&=\frac{3n}{(2n+1)^2}\Big[12n^3(n+2)^3-4(n+2)(2n+1)(n+1)^3-(2n+1)^2(n+1)^3\Big]\\&=\frac{3n}{(2n+1)^2}\Big(12n^4+27n^3-15n^2-51n-9\Big).\end{aligned}$$

Clearly, $f(\mu_n)>0$ and $g(\mu_n)>0$ for $n\geq 2$. This implies that

$$(n+2)t_n^2-(n+1)t_{n+1}^2>0$$

and

$$n(n+2)^2t_n-(n+1)^3t_{n+1}>0$$

for $n\geq 2$. Then $\{s_n\}_{n\geq 2}$ and $\{\frac{nt_n}{(n+1)^2}\}_{n\geq 2}$ are both decreasing. It follows from Theorem 3 that the sequences $\{A_n^2\}_{n\geq 2}$ and $\{\frac{A_n}{n}\}_{n\geq 2}$ are both log-balanced. $\square$

Corollary 7. *For the sequence $\{B_n\}_{n\geq 0}$ of the Fine numbers, we have that $\{\frac{B_n}{n}\}_{n\geq 2}$ is log-balanced.*

Proof. The Fine numbers satisfy the recurrence

$$B_{n+1}=\frac{7n+2}{2(n+2)}B_n+\frac{2n+1}{n+2}B_{n-1},\quad B_0=1,\quad B_1=0. \tag{5}$$

For $n\geq 2$, let $t_n=\frac{B_{n+1}}{B_n}$. Došlić [2] showed that the sequence $\{B_n\}_{n\geq 2}$ is log-balanced. We next prove that $\{\frac{nt_n}{(n+1)^2}\}_{n\geq 2}$ is decreasing.

By (5), we have

$$t_n=\frac{7n+2}{2(n+2)}+\frac{2n+1}{(n+2)t_{n-1}}.$$

Then we have

$$\begin{aligned}&n(n+2)^2t_n-(n+1)^3t_{n+1}\\ =&\frac{2n(n+3)(n+2)^2t_n^2-(7n+9)(n+1)^3t_n-2(n+1)^3(2n+3)}{2(n+3)t_n}.\end{aligned}$$

For any real number x, let

$$f(x)=2n(n+3)(n+2)^2x^2-(7n+9)(n+1)^3x-2(n+1)^3(2n+3).$$

Then we obtain

$$f'(x)=4n(n+3)(n+2)^2x-(7n+9)(n+1)^3.$$

It is obvious that $f'(x)>0$ for $x\geq 3$. Then f is increasing over $[3,\infty)$.

Liu and Wang [4] proved that

$$t_n\geq\lambda_n,$$

where $\lambda_n=\frac{2(2n+5)}{n+4}$. Since

$$\begin{aligned}f(\lambda_n)&=\frac{1}{(n+4)^2}[8n(n+3)(n+2)^2(2n+5)^2\\&\quad-(7n+9)(2n+5)(n+4)(n+1)^3-2(2n+3)(n+4)^2(n+1)^3]\\&=\frac{14n^6+185n^5+968n^4+458n^3+4314n^2+1023n-596}{(n+4)^2}\\&>0,\end{aligned}$$

we have

$$2n(n+3)(n+2)^2t_n^2-(7n+9)(n+1)^3t_n-2(n+1)^3(2n+3)>0.$$

Then $n(n+2)^2t_n-(n+1)^3t_{n+1}>0$, and $\{\frac{nt_n}{(n+1)^2}\}_{n\geq 2}$ is decreasing. It follows from Theorem 3 that the sequence $\{\frac{B_n}{n}\}_{n\geq 2}$ is log-balanced. □

Theorem 8. *For a given sequence $\{z_n\}_{n\geq 0}$, if it is log-balanced, then $\{\sqrt{z_n}\}_{n\geq 0}$ is also log-balanced.*

Proof. Suppose that $\{z_n\}_{n\geq 0}$ is log-balanced, that is,

$$z_n^2\leq z_{n-1}z_{n+1},\quad (n+1)z_n^2\geq nz_{n-1}z_{n+1},\quad n\geq 1.$$

For $n\geq 1$, we immediately derive

$$z_n\leq\sqrt{z_{n-1}z_{n+1}}$$

and

$$z_n\geq\sqrt{\frac{n}{n+1}z_{n-1}z_{n+1}}>\frac{n}{n+1}\sqrt{z_{n-1}z_{n+1}}.$$

This means that the sequence $\{\sqrt{z_n}\}_{n\geq 0}$ is log-convex and the sequence $\{\frac{\sqrt{z_n}}{n!}\}_{n\geq 0}$ is log-concave. As a result, the sequence $\{\sqrt{z_n}\}_{n\geq 0}$ is log-balanced. □

Theorem 9. *Suppose that the sequences $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ are both log-convex. If both $\{\frac{x_n}{n!}\}_{n\geq 0}$ and $\{\frac{y_n}{n!}\}_{n\geq 0}$ are concave, then $\{x_n+y_n\}_{n\geq 0}$ is log-balanced.*

Proof. Since $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ are both log-convex, the sequence $\{x_n+y_n\}_{n\geq 0}$ is log-convex. We next prove that $\{\frac{x_n+y_n}{n!}\}_{n\geq 0}$ is log-concave.

It is well known that $\{x_n\}_{n\geq 0}$ is concave if and only if its difference sequence $\{x_{n+1}-x_n\}_{n\geq 0}$ is decreasing. Therefore, by the concavity of $\{\frac{x_n}{n!}\}_{n\geq 0}$ and $\{\frac{y_n}{n!}\}_{n\geq 0}$, the sequence $\{\frac{x_{n+1}+y_{n+1}}{n!}-\frac{x_n+y_n}{n!}\}_{n\geq 0}$ is decreasing. Then the sequence $\{\frac{x_n+y_n}{n!}\}_{n\geq 0}$ is concave and it is also log-concave. Hence, the sequence $\{x_n+y_n\}_{n\geq 0}$ is log-balanced. □

It follows from Theorem 9 that the sequence $\{n!+(n+1)!\}_{n\geq 0}$ is log-balanced.

In the rest of this section, we devote to discuss the log-balancedness of some sequences by means of Definition 2 directly. Our first example is to consider some sequences related to harmonic numbers. Let H_n denote the n^{th} harmonic number. Then we have the following result.

Proposition 10. *Both $\{\frac{H_n}{n}\}_{n\geq 1}$ and $\{\frac{H_n}{n^2}\}_{n\geq 1}$ are log-balanced.*

Proof. In order to prove the log-balancedness of $\{\frac{H_n}{n}\}_{n\geq 1}$, it is sufficient to show that $\{\frac{H_n}{n}\}_{n\geq 1}$ is log-convex and the sequence $\{\frac{H_n}{nn!}\}_{n\geq 1}$ is log-concave. In fact, for $n \geq 2$, we have

$$\begin{aligned}\frac{H_n^2}{n^2} - \frac{H_{n-1}H_{n+1}}{n^2-1} &= \frac{1}{n^2(n^2-1)}\left[(n^2-1)H_n^2 - n^2\left(H_n - \frac{1}{n}\right)\left(H_n + \frac{1}{n+1}\right)\right] \\ &= -\frac{n(H_n^2 - H_n - 1) + H_n^2}{n^2(n+1)^2(n-1)}.\end{aligned}$$

Note that

$$2(H_2^2 - H_2 - 1) + H_2^2 > 0, \quad H_n > H_2 > 2 \quad (n \geq 3).$$

Now we prove that $n(H_n^2 - H_n - 1) + H_n^2 > 0$ for $n \geq 3$. For any real number x, let

$$f(x) = x^2 - x - 1.$$

It is clear that $f'(x) = 2x - 1 > 0$ for $x \geq 2$. Then f is increasing over $[2, \infty)$ and $f(H_n) > f(H_3) = \frac{19}{36} > 0$ for $n \geq 3$. Hence, the sequence $\{\frac{H_n}{n}\}_{n\geq 1}$ is log-convex. On the other hand, for $n \geq 2$, we have

$$\begin{aligned}&\frac{H_n^2}{nn!} - \frac{H_{n-1}H_{n+1}}{(n^2-1)(n-1)!(n+1)(n+1)!} \\ &= \frac{1}{n^2(n^2-1)n!(n+1)!}\left[(n^2-n-1)H_n^2 + n^2\left(\frac{H_n}{n+1} + \frac{1}{n+1}\right)\right] \\ &> 0.\end{aligned}$$

Hence the sequence $\{\frac{H_n}{nn!}\}_{n\geq 1}$ is log-concave. It follows from Definition 2 that the sequence $\{\frac{H_n}{n}\}_{n\geq 1}$ is log-balanced.

Now we consider the sequence $\{\frac{H_n}{n^2}\}_{n\geq 1}$. Since both $\{\frac{H_n}{n}\}_{n\geq 1}$ and $\{\frac{1}{n}\}_{n\geq 1}$ are log-convex, $\{\frac{H_n}{n^2}\}_{n\geq 1}$ is log-convex. On the other hand, for $n \geq 2$, we get

$$\begin{aligned}&\left(\frac{H_n}{n^2 n!}\right)^2 - \frac{H_{n-1}H_{n+1}}{(n-1)^2(n-1)!(n+1)^2(n+1)!} \\ &= \frac{1}{n^2(n-1)^2(n+1)^2 n!(n+1)!}\left[(n^4 - 2n^3 - 2n^2 + 3n + 1)H_n^2\right. \\ &\quad \left. + n^4\left(\frac{H_n}{n+1} + \frac{1}{n+1}\right)\right].\end{aligned}$$

For $n = 2$,

$$\left(\frac{H_n}{n^2 n!}\right)^2 - \frac{H_{n-1}H_{n+1}}{(n-1)^2(n-1)!(n+1)^2(n+1)!} = \frac{25}{20736}.$$

We find that $n^4 - 2n^3 - 2n^2 + 3n + 1 > 0$ for $n \geq 3$. Thus the sequence $\{\frac{H_n}{n^2 n!}\}_{n\geq 1}$ is log-concave. It follows from Definition 2 that the sequence $\{\frac{H_n}{n^2}\}_{n\geq 1}$ is log-balanced. □

Our second example is to consider some sequences related to the Fibonacci (Lucas) sequence. The Binet form of the Fibonacci sequence $\{F_n\}_{n\geq 0}$ and the Lucas sequence $\{L_n\}_{n\geq 0}$ are

$$F_n = \frac{\alpha^n - (-1)^n \alpha^{-n}}{\sqrt{5}}, \quad L_n = \alpha^n + (-1)^n \alpha^{-n},$$

where $\alpha = \frac{1+\sqrt{5}}{2}$. It is well known that log-convexity and log-concavity of $\{F_n\}_{n\geq 0}$ and $\{L_n\}_{n\geq 0}$ depend on the parity of n. In fact, by using the definition of log-convexity, we can easily prove that both $\{F_{2n+1}\}_{n\geq 0}$ and $\{L_{2n}\}_{n\geq 2}$ are log-convex. Now we discuss the log-balancedness of some sequences related to F_n and L_n. We first give a lemma.

Lemma 11. *For $n \geq 1$, we have*

$$F_{2n+1} \geq 2n \tag{6}$$

and

$$L_{2n} \geq 3n. \tag{7}$$

Proof. It is well known that $\{F_n\}_{n\geq 0}$ and $\{L_n\}_{n\geq 0}$ satisfy the recurrence relation

$$W_{n+1} = W_n + W_{n-1}, \quad n \geq 1. \tag{8}$$

We can prove (6)–(7) by induction. We only give a proof of (6) and (7) can be shown in a similar way. In fact, it is clear that $F_{2n+1} \geq 2n$ for $1 \leq n \leq 5$. Assume that $F_{2n+1} \geq 2n$ for $n \geq 5$. By (8), we have

$$F_{2n+3} = F_{2n+1} + F_{2n+2}.$$

Then we have $F_{2n+3} \geq F_{2n+1} + 2 \geq 2n + 2$. By mathematical induction, (6) holds for each $n \geq 1$. $\square$

Proposition 12. *The sequences $\{\frac{F_{2n+1}}{n}\}_{n\geq 1}$ and $\{\frac{L_{2n}}{n}\}_{n\geq 2}$ are log-balanced.*

Proof. Because $\{F_{2n+1}\}_{n\geq 0}$, $\{L_{2n}\}_{n\geq 2}$ and $\{\frac{1}{n}\}_{n\geq 1}$ are log-convex, the sequences $\{\frac{F_{2n+1}}{n}\}_{n\geq 1}$ and $\{\frac{L_{2n}}{n}\}_{n\geq 2}$ are log-convex. Next we show that $\{\frac{F_{2n+1}}{nn!}\}_{n\geq 1}$ and $\{\frac{L_{2n}}{nn!}\}_{n\geq 2}$ are log-concave.

For $n \geq 2$, we obtain

$$\begin{aligned}
\left(\frac{F_{2n+1}}{nn!}\right)^2 - \frac{F_{2n-1}F_{2n+3}}{(n^2-1)(n-1)!(n+1)!} &= \frac{(n+1)(n^2-1)F_{2n+1}^2 - n^3 F_{2n-1}F_{2n+3}}{n^2(n^2-1)n!(n+1)!} \\
&= \frac{n^3(F_{2n+1}^2 - F_{2n-1}F_{2n+3}) + (n^2-n-1)F_{2n+1}^2}{n^2(n^2-1)n!(n+1)!}
\end{aligned}$$

and

$$\begin{aligned}\left(\frac{L_{2n}}{nn!}\right)^2-\frac{L_{2n-2}L_{2n+2}}{(n^2-1)(n-1)!(n+1)!} &= \frac{(n+1)(n^2-1)L_{2n}^2-n^3L_{2n-2}L_{2n+2}}{n^2(n^2-1)n!(n+1)!}\\ &= \frac{n^3(L_{2n}^2-L_{2n-2}L_{2n+2})+(n^2-n-1)L_{2n}^2}{n^2(n^2-1)n!(n+1)!}.\end{aligned}$$

By means of the equalities

$$F_{2n+1}^2-F_{2n-1}F_{2n+3}=-1 \quad \text{and} \quad L_{2n}^2-L_{2n-2}L_{2n+2}=-5,$$

we have

$$\begin{aligned}\left(\frac{F_{2n+1}}{nn!}\right)^2-\frac{F_{2n-1}F_{2n+3}}{(n^2-1)(n-1)!(n+1)!} &= \frac{-n^3+(n^2-n-1)F_{2n+1}^2}{n^2(n^2-1)n!(n+1)!},\\ \left(\frac{L_{2n}}{nn!}\right)^2-\frac{L_{2n-2}L_{2n+2}}{(n^2-1)(n-1)!(n+1)!} &= \frac{-5n^3+(n^2-n-1)L_{2n}^2}{n^2(n^2-1)n!(n+1)!}.\end{aligned}$$

For $n\geq 2$, put

$$R(n)=-n^3+(n^2-n-1)F_{2n+1}^2 \quad \text{and} \quad S(n)=-5n^3+(n^2-n-1)L_{2n}^2.$$

It follows from Lemma 11 that

$$R(n)\geq n^2(4n^2-5n-4), \quad S(n)\geq n^2(9n^2-14n-9).$$

Note that

$$R(n)>0\ (n\geq 2), \quad S(n)>0\ (n\geq 3).$$

This implies that $\{\frac{F_{2n+1}}{nn!}\}_{n\geq 1}$ and $\{\frac{L_{2n}}{nn!}\}_{n\geq 2}$ are both log-concave. By Definition 2, $\{\frac{F_{2n+1}}{n}\}_{n\geq 1}$ and $\{\frac{L_{2n}}{n}\}_{n\geq 2}$ are both log-balanced. This completes the proof. □

3 Conclusions

We have derived some new sufficient conditions for log-balancedness of combinatorial sequences. We have further applied these results to show log-balancedness of some special sequences. One future work is to study log-balancedness of the partial sums for log-balanced sequences.

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