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Infinite Products Arising in Paperfolding

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Abstract

J.-P. Allouche recently examined two infinite products where the term is a rational function of the index n raised to the term of the paperfolding sequence ϵ_n. A closed form is given only for one of them. We discuss an attempt to produce the missing closed form. We give a detailed analysis of convergence and a closed form for the analogous question, where the paperfolding sequence is replaced by a periodic one.

1 Introduction

The *paperfolding sequence* ϵ_n is defined by the rules

$$\begin{aligned} \epsilon_{2n} &= (-1)^n \\ \epsilon_{2n+1} &= \epsilon_n. \end{aligned} \tag{1}$$

The first few values are $\{1,\, 1,\, -1,\, 1,\, 1\}$. For fixed $a \in \mathbb{N}$, the rules (1) determine all subsequences of the form

$$\{\epsilon_{2^a n+b} :\; a \in \mathbb{N},\, 0 \leq b < 2^a\} \tag{2}$$

in terms of constants, $\{\epsilon_n\}$ and $\{(-1)^n\}$. For example, when $a = 2$,

$$\epsilon_{4n} = 1,\; \epsilon_{4n+1} = \epsilon_{2n} = (-1)^n,\; \epsilon_{4n+2} = (-1)^{2n+1} = -1,\; \epsilon_{4n+3} = \epsilon_{2n+1} = \epsilon_n. \tag{3}$$

The work presented here is motivated by results given by Allouche [1]. In particular, the evaluation

$$B = \prod_{n=1}^{\infty} \left(\frac{2n}{2n+1} \right)^{\epsilon_n} = \frac{1}{8\sqrt{2\pi}} \Gamma \left(\frac{1}{4} \right)^2 \tag{4}$$

is obtained using the auxiliary product

$$A = \prod_{n=0}^{\infty} \left(\frac{2n+1}{2n+2} \right)^{\epsilon_n}. \tag{5}$$

Indeed, the identity

$$AB = \frac{1}{2} \prod_{n=1}^{\infty} \left(\frac{n}{n+1} \right)^{\epsilon_n} \tag{6}$$

is split according to the parity of n and (1) yields

$$AB = \frac{1}{2} A \prod_{n=1}^{\infty} \left(\frac{2n}{2n+1} \right)^{(-1)^n}. \tag{7}$$

The non-vanishing of A gives

$$B = \frac{1}{2} \prod_{n=0}^{\infty} \frac{(4n+4)(4n+3)}{(4n+5)(4n+2)}. \tag{8}$$

A classical result expressing such products in terms of the gamma function gives the value of B. Observe that the value of A does not come from this formulation. A search for a closed form for A was the motivation for the results presented here.

An early evaluation of an infinite product was produced by Wallis in his representation

$$\prod_{n=1}^{\infty} \frac{(2n)(2n)}{(2n-1)(2n+1)} = \frac{\pi}{2}. \tag{9}$$

The history of this discovery appears in Osler [7]. The literature contains a variety of infinite product evaluations. For instance,

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{F_{2^n+1}}\right)=\frac{3}{\varphi} \text{ and } \prod_{n=1}^{\infty}\left(1+\frac{1}{L_{2^n+1}}\right)=3-\varphi, \tag{10}$$

is given by Sondow [9]. Here F_n, L_n are the Fibonacci (Lucas) numbers and $\varphi=\frac{1}{2}(\sqrt{5}+1)$ is the golden ratio.

The value of infinite products usually involves classical constants of analysis. For instance, Borwein [3] evaluates the function

$$D(x)=\lim_{n\to\infty}\prod_{k=1}^{2n+1}\left(1+\frac{x}{k}\right)^{(-1)^{k+1}k} \tag{11}$$

as a generalization of the values

$$\prod_{n=1}^{\infty}\left(1+\frac{2}{n}\right)^{(-1)^{n+1}n}=\frac{\pi}{2e} \quad \text{and} \quad \prod_{n=1}^{\infty}\left(1+\frac{2}{n}\right)^{(-1)^{n}n}=\frac{6}{\pi e} \tag{12}$$

established by Melzak [6]. Some exact evaluations are given in terms of the constant

$$A_1=\exp\left(\frac{1}{4}-\int_0^{\infty}\frac{e^{-s}}{s^3}\left(1-\frac{s}{2}+\frac{s^2}{12}-\frac{s}{e^s-1}\right)ds\right) \tag{13}$$

and the Catalan constant

$$G=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}. \tag{14}$$

Examples include

$$D(1)=\frac{A_1^6}{2^{1/6}\sqrt{\pi}} \text{ and } D\left(\tfrac{1}{4}\right)=\frac{2^{1/6}\sqrt{\pi}A_1^3}{\Gamma\left(\frac{1}{4}\right)}e^{G/\pi}. \tag{15}$$

Other types of products involving gamma factors have recently been analyzed by Chamberland and Straub [4].

The question considered here deals with the evaluation of products of the form

$$\mathfrak{P}(R,s)=\prod_{n=1}^{\infty}R(n)^{s_n}. \tag{16}$$

Here R is a rational function and s is an *automatic sequence* (as studied by Allouche [1]). Examples include periodic sequences taking values in the alphabet $\{+1,-1\}$ or k-*automatic sequences*: a sequence $\{s_n : n\geq 0\}$ is k-automatic if the set of subsequences $\{s_{k^j n+\ell} : n\geq 0\}$ with $j\geq 0$, $\ell\in[0,\,k^j-1]$ is finite. More information about such sequences appears in [2].

The main example discussed here is the *paperfolding sequence* ϵ_n defined in (1). Splitting the evaluation of a product into even and odd indices leads, in the special case of a rational function of degree 1, to the identity

$$\prod_{n=0}^{\infty}\left(\frac{\alpha n+\beta}{\gamma n+\delta}\right)^{\epsilon_n}=\prod_{n=0}^{\infty}\left(\frac{2\alpha n+\beta}{2\gamma n+\delta}\right)^{(-1)^n}\times\prod_{n=0}^{\infty}\left(\frac{2\alpha n+\alpha+\beta}{2\gamma n+\gamma+\delta}\right)^{\epsilon_n}. \tag{17}$$

The exponent $(-1)^n$ appearing in the first product on the right is a periodic sequence of period length 2. This motivates the evaluation of products with terms of the form $R(n)^{M_n}$ where M_n is a periodic sequence. This is the topic of Sections 2–4.

Section 2 discusses the convergence of the product

$$\mathfrak{P}(R,1)=\prod_{n=0}^{\infty}R(n), \tag{18}$$

where

$$R(z)=\frac{(z+a_1)\cdots(z+a_d)}{(z+b_1)\cdots(z+b_d)}. \tag{19}$$

This section reviews the elementary arguments showing that convergence in (18) is equivalent to $R(n)\to 1$ as $n\to\infty$ and $\mathfrak{S}(R)=0$. Here

$$\mathfrak{S}(R)=\sum_{b\in R^{-1}(\infty)}b-\sum_{a\in R^{-1}(0)}a. \tag{20}$$

The value of $\mathfrak{P}(R,1)$ is then given by

$$\prod_{n=0}^{\infty}\frac{(n+a_1)\cdots(n+a_d)}{(n+b_1)\cdots(n+b_d)}=\prod_{k=1}^{d}\frac{\Gamma(b_k)}{\Gamma(a_k)}. \tag{21}$$

Section 3 discusses the convergence of products $\mathfrak{P}(R,M)$, where R is a rational function and M is a periodic sequence of period length 2. Section 4 extends the results to any periodic sequence, with special emphasis on period lengths 3 and 4. Section 5 considers some infinite products related to the paperfolding sequence, and Section 6 considers a generalization to certain k-automatic sequences. An alternative proof of the evaluation of Allouche's product B is presented and a new form of the product A is given. The question of existence of a closed form for A remains open.

2 Convergence of infinite products

This section considers the simplest type of product (16): R is a given rational function and $s_n\equiv 1$. The data for the rational function is a sequence of complex numbers $\{a_k\}$ and $\{b_k\}$ where a_k, b_k are not 0 nor a negative integer. The convergence of the partial finite products

$$\mathfrak{P}_r(R,1)=\prod_{n=1}^{r}\frac{(n+a_1)\cdots(n+a_d)}{(n+b_1)\cdots(n+b_r)} \tag{22}$$

is examined first.

Theorem 1. *The infinite product*

$$\mathfrak{P}(R,1) = \prod_{n=1}^{\infty} \frac{(n+a_1)\cdots(n+a_d)}{(n+b_1)\cdots(n+b_r)} \tag{23}$$

converges if and only if $d=r$ *and* $a_1+\cdots+a_d = b_1+\cdots+b_r$*; that is,* $R(n)\to 1$ *and* $\mathfrak{S}(R)=0$.

Proof. The convergence of a product $\prod(1+u_k)$ is equivalent to the convergence of the series $\sum u_k$. Therefore $u_k \to 0$ is a necessary condition for convergence. This implies $d=r$. On the other hand

$$\frac{(n+a_1)\cdots(n+a_d)}{(n+b_1)\cdots(n+b_r)} = 1 + (a_1+\cdots+a_d-b_1-\cdots-b_r)\,\frac{1}{n} + O(1/n^2), \tag{24}$$

and the second condition on the parameters a_k, b_k is now clear. □

The next question is the evaluation of the limiting product. The motivation for the final result is this: consider the problem of producing a function $h(z)$ with zeros at a prescribed sequence $\{z_n\}$. This is elementary if the sequence is finite: the solution is simply given as

$$P(z) = \prod_{n=1}^{N}\left(1-\frac{z}{z_j}\right) \tag{25}$$

when $z_j \neq 0$. On the other hand, if the sequence is infinite, convergence issues might appear. For instance, if one would like to have a function that vanishes precisely at the negative integers, then the natural first attempt

$$P_1(z) = \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right) \tag{26}$$

fails to converge. To fix this, introduce an exponential correction and form the partial products

$$\begin{aligned} P_{2,N}(z) &= e^{z\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{N}\right)} \prod_{n=1}^{N}\left(1+\frac{z}{n}\right)e^{-z/n} \\ &= e^{z(E_1(N)+\ln N)} \prod_{n=1}^{N}\left(1+\frac{z}{n}\right)e^{-z/n}, \end{aligned} \tag{27}$$

with

$$E_1(N) = 1+\frac{1}{2}+\cdots+\frac{1}{N} - \ln N. \tag{28}$$

The limit

$$\gamma = \lim_{N \to \infty} E_1(N) \tag{29}$$

is the famous *Euler constant*. Therefore, the modified product

$$\frac{P_{2,N}(z)}{N^z} := e^{zE_1(N)} \prod_{n=1}^{N} \left(1 + \frac{z}{n}\right) e^{-z/n} \tag{30}$$

has a limit as $N \to \infty$. The infinite product has zeros at the negative integers. It turns out to be convenient to write an infinite product with poles at the negative integers and also to include 0 as a pole. This yields the classical *gamma function* $\Gamma(z)$. The functional equation $\Gamma(z+1) = z\Gamma(z)$ is used to simplify the result.

Theorem 2. *The infinite product representation of the gamma function is given by*

$$\prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n} = e^{\gamma z}\Gamma(z+1). \tag{31}$$

It is now easy to write the value of the infinite product

$$\mathfrak{P}(R, 1) = \prod_{n=1}^{\infty} \frac{(n + a_1) \cdots (n + a_d)}{(n + b_1) \cdots (n + b_r)} \tag{32}$$

in Theorem 1. Start with

$$\mathfrak{P}(R, 1) = \prod_{n=1}^{\infty} \frac{(1 + b_1/n)^{-1} e^{b_1/n} \cdots (1 + b_r/n)^{-1} e^{b_r/n}}{(1 + a_1/n)^{-1} e^{a_1/n} \cdots (1 + a_d/n)^{-1} e^{a_d/n}} \tag{33}$$

and observe that the added exponential terms amount to 1. Passing to the limit in (33) gives

$$\mathfrak{P}(R, 1) = \prod_{k=1}^{d} \frac{\Gamma(b_k + 1)}{\Gamma(a_k + 1)}. \tag{34}$$

To simplify the form of the result, shift n to $n + 1$ in (32) to produce the following result.

Theorem 3. *Let a_k, $b_k \in \mathbb{C}$ none of which are 0 or negative integers. Assume*

$$a_1 + \cdots + a_d = b_1 + \cdots + b_d. \tag{35}$$

Then

$$\prod_{n=0}^{\infty} \frac{(n + a_1) \cdots (n + a_d)}{(n + b_1) \cdots (n + b_d)} = \prod_{k=1}^{d} \frac{\Gamma(b_k)}{\Gamma(a_k)}. \tag{36}$$

3 The first example: Sequences of period length 2

This section considers products of the form

$$\mathfrak{P}(R,M) := \prod_{n=0}^{\infty} R(n)^{M_n} \tag{37}$$

where $M_n = (-1)^n$.

Start with the representation

$$R(z) = C\frac{(z+a_1)\cdots(z+a_d)}{(z+b_1)\cdots(z+b_r)}. \tag{38}$$

The partial products of $\mathfrak{P}(R,s)$ are

$$\prod_{n=0}^{N} R(n)^{(-1)^n} = \prod_{n=0}^{\lfloor N/2\rfloor} \frac{R(2n)}{R(2n+1)} \times \begin{cases} 1, & \text{if } N \text{ is odd;} \\ R(N+1), & \text{if } N \text{ is even.} \end{cases} \tag{39}$$

The first factor on the right in (39) is connected to the product $\mathfrak{P}(R_1,1)$, where

$$R_1(z) = \frac{R(2z)}{R(2z+1)}. \tag{40}$$

Its convergence is decided by Theorem 1. It is clear that the product on the left-hand side of (39) converges if and only if both factors on the right converge separately.

In particular, if $\mathfrak{P}(R,M)$ converges, then $\lim\limits_{n\to\infty} R(n) = 1$ and it must be that $C=1$ in (38). To complete the discussion, it suffices to determine conditions under which $\mathfrak{P}(R_1,1)$ is finite. The rational function (40) factors as

$$R_1(z) = \frac{(2z+a_1)\cdots(2z+a_d)}{(2z+b_1)\cdots(2z+b_r)} \times \frac{(2z+1+b_1)\cdots(2z+1+b_r)}{(2z+1+a_1)\cdots(2z+1+a_d)}, \tag{41}$$

with $d+r$ zeros at

$$-\tfrac{1}{2}a_1,\ \ldots, -\tfrac{1}{2}a_d,\ -\tfrac{1}{2}(1+b_1),\ldots,-\tfrac{1}{2}(1+b_r) \tag{42}$$

and $d+r$ poles at

$$-\tfrac{1}{2}b_1,\ \ldots, -\tfrac{1}{2}b_r,\ -\tfrac{1}{2}(1+a_1),\ldots,-\tfrac{1}{2}(1+a_d). \tag{43}$$

Since $R_1(z)\to 1$ as $z\to\infty$, convergence in (39) requires the relation

$$\sum_{k=1}^{d} a_k + \sum_{k=1}^{r}(1+b_k) = \sum_{k=1}^{r} b_k + \sum_{k=1}^{d}(1+a_k). \tag{44}$$

This is equivalent to the condition $d=r$.

The value of $\mathfrak{P}(R,M)$ is obtained from Theorem 3 as

$$\mathfrak{P}(R,M) = \mathfrak{P}(R_1,1) = \prod_{k=1}^{d} \frac{\Gamma(\frac{b_k}{2})\Gamma(\frac{1+a_k}{2})}{\Gamma(\frac{1+b_k}{2})\Gamma(\frac{a_k}{2})}. \tag{45}$$

This is simplified using the duplication formula for the gamma function to obtain

$$\prod_{k=1}^{d} \frac{\Gamma(\frac{b_k}{2})\Gamma(\frac{1+a_k}{2})}{\Gamma(\frac{1+b_k}{2})\Gamma(\frac{a_k}{2})} = 2^{(b_1-a_1)+\cdots+(b_d-a_d)} \prod_{k=1}^{d} \frac{\Gamma^2(\frac{b_k}{2})\Gamma(a_k)}{\Gamma^2(\frac{a_k}{2})\Gamma(b_k)}. \tag{46}$$

The discussion above is summarized in the next statement.

Theorem 4. *Let $R(z)$ be a rational function and $M_n = (-1)^n$. Then $\mathfrak{P}(R,M)$ converges if and only if $R(z) \to 1$ as $z \to \infty$. If*

$$R(z) = \prod_{k=1}^{d} \frac{(z+a_k)}{(z+b_k)} \text{ and } \mathfrak{S}(R) = \sum_{k=1}^{d} b_k - \sum_{k=1}^{d} a_k, \tag{47}$$

then

$$\mathfrak{P}(R,M) = 2^{\mathfrak{S}(R)} \prod_{k=1}^{d} \frac{\Gamma^2(\frac{b_k}{2})\Gamma(a_k)}{\Gamma^2(\frac{a_k}{2})\Gamma(b_k)}. \tag{48}$$

Example 5. Let $R(z) = (20z+5)/(20z+4)$. The convergence conditions are satisfied and Theorem 4 gives

$$\prod_{n=0}^{\infty} \left(\frac{20n+5}{20n+4} \right)^{(-1)^n} = \frac{\Gamma(\frac{1}{10})\Gamma(\frac{5}{8})}{\Gamma(\frac{1}{8})\Gamma(\frac{3}{5})}. \tag{49}$$

`Mathematica` 9.0 does not evaluate the original product, but it does give the right-hand side of (49) for

$$\mathfrak{P}(R_1,1) = \prod_{n=0}^{\infty} \frac{80n^2+58n+6}{80n^2+58n+5}. \tag{50}$$

Example 6. The infinite product

$$\mathfrak{P}(R,s) = \prod_{n=0}^{\infty} \left(\frac{2\alpha n+\beta}{2\gamma n+\delta} \right)^{(-1)^n} \tag{51}$$

encountered in the paperfolding product (17) converges if and only if $\alpha = \gamma$. The product is then

$$\mathfrak{P}(R,s) = \prod_{n=0}^{\infty} \left(\frac{n+2v}{n+2u} \right)^{(-1)^n} = 2^{2(u-v)} \frac{\Gamma^2(u)\Gamma(2v)}{\Gamma^2(v)\Gamma(2u)}, \tag{52}$$

with $u = \delta/4\alpha$ and $v = \beta/4\alpha$.

4 Convergence for periodic sequences

This section discusses the issue of convergence of the product

$$\mathfrak{P}(R, M) = \prod_{n=0}^{\infty} R(n)^{M_n} \tag{53}$$

where $\{M_n\}$ is a periodic sequence of period length ℓ of elements of the alphabet $\{+1, -1\}$.

Notation. The results are expressed in terms of

$$\begin{aligned} M^+ &= \{i : M_i = +1 \text{ and } 0 \leq i \leq \ell - 1\} = \{i_1, i_2, \ldots, i_{|M^+|}\} \\ M^- &= \{j : M_j = -1 \text{ and } 0 \leq j \leq \ell - 1\} = \{j_1, j_2, \ldots, j_{|M^-|}\}, \end{aligned} \tag{54}$$

and the period length is $\ell = |M^+| + |M^-|$.

The rational function is written as

$$R(n) = C \frac{(n + a_1) \cdots (n + a_d)}{(n + b_1) \cdots (n + b_r)} \tag{55}$$

with $a_s, b_t \notin \{0, -1, 2, \ldots\}$ and

$$\mathfrak{S}(R) = \sum_{t=1}^{r} b_t - \sum_{s=1}^{d} a_s. \tag{56}$$

The partial product associated with $\mathfrak{P}(R, M)$ is

$$\begin{aligned} \prod_{n=0}^{N} R(n)^{M_n} &= \prod_{k=0}^{\lfloor N/\ell \rfloor} \prod_{i \in M^+} R(k\ell + i)^{M_i} \prod_{j \in M^-} R(k\ell + j)^{M_j} \prod_{n=\ell \lfloor N/\ell \rfloor + 1}^{N} R(n)^{M_n} \\ &= \prod_{k=0}^{\lfloor N/\ell \rfloor} \frac{\prod_{i \in M^+} R(k\ell + i)}{\prod_{j \in M^-} R(k\ell + j)} \prod_{n=\ell \lfloor N/\ell \rfloor + 1}^{N} R(n)^{M_n}, \end{aligned} \tag{57}$$

the last product being empty if N is a multiple of the period length ℓ. An elementary argument shows that the convergence of $\mathfrak{P}(R, M)$ requires the convergence of both products in (57). The first product, which would lead to an expression of the form $\mathfrak{P}(R_1, 1)$ for a new rational function R_1 is labeled the *main term*. The second product is called the *tail product*. We analyze its convergence first.

The tail product is defined by

$$P_{N,\ell}(M) = \prod_{n=\ell \lfloor N/\ell \rfloor + 1}^{N} R(n)^{M_n}. \tag{58}$$

Its convergence implies $R(n) \to 1$ as $n \to \infty$. Observe that $P_{N,\ell}(M) = 1$ if $N \equiv 0 \pmod{\ell}$. On the other hand, in the case $N \equiv 1 \pmod{\ell}$, one obtains

$$P_{N,\ell}(M) = R(N)^{M_N} = R(N)^{M_1},$$

since $M_N = M_1$ by periodicity. Therefore, the convergence of $\mathfrak{P}(R, M)$ requires $R(N) \to 1$ for $N \equiv 1 \pmod{\ell}$. Similarly, if $N \equiv 2 \pmod{\ell}$,

$$P_{N,\ell}(M) = R(N-1)^{M_{N-1}} R(N)^{M_N} = R(N-1)^{M_1} R(N)^{M_2}.$$

The convergence of $\mathfrak{P}(R, M)$ already implies $R(N-1) \to 1$ since $N - 1 \equiv 1 \pmod{\ell}$. This time it is required that $R(N) \to 1$. Iterating this argument it follows that $R(N) \to 1$ for $N \equiv j \pmod{\ell}$ for any residue class j. This gives the next result.

Proposition 7. *Assume $\mathfrak{P}(R, M)$ converges. Then* $\lim\limits_{n\to\infty} R(n) = 1$.

The limiting value of the main term is $\mathfrak{P}(R_1, 1)$, where

$$R_1(n) = \frac{R(\ell n + i_1) \cdots R(\ell n + i_{|M^+|})}{R(\ell n + j_1) \cdots R(\ell n + j_{|M^-|})}. \tag{59}$$

The ingredients entering into the convergence of $\mathfrak{P}(R_1, 1)$ are discussed in the next result. We assume the condition $R(n) \to 1$.

Proposition 8. *Let $M_* = |M^+| - |M^-|$ and assume $\mathfrak{P}(R_1, 1)$ converges. Then* $\lim\limits_{n\to\infty} R_1(n) = 1$ *and*

$$\ell \mathfrak{S}(R_1) = M_* \mathfrak{S}(R). \tag{60}$$

Proof. The behavior of $R_1(n)$ as $n \to \infty$ comes directly from that of R. The identity (60) is a direct computation. □

Combining these propositions gives the following.

Theorem 9. *Let R be a rational function satisfying* $\lim\limits_{n\to\infty} R(n) = 1$ *with zeros and poles of R are outside $\{0, -1, -2, \dots\}$. There are two cases.*

1. *Assume $M_* \neq 0$. Then $\mathfrak{P}(R, M)$ converges if and only if $\mathfrak{S}(R) = 0$.*

2. *Assume $M_* = 0$. Then $\mathfrak{P}(R, M)$ always converges.*

For a general periodic sequence, the value of the product $\mathfrak{P}(R, M)$ is given by the following.

Theorem 10. *Let $R(n)$ be a rational function written in the form*

$$R(n) = \frac{(n+a_1)\cdots(n+a_d)}{(n+b_1)\cdots(n+b_d)} \tag{61}$$

with $a_i, b_j \notin \{0, -1, -2, \dots\}$. Let $\{M_n\}$ be a periodic sequence of ± 1 with period length ℓ. Assume the product

$$\mathfrak{P}(R, M) = \prod_{n=0}^{\infty} R(n)^{M_n} \tag{62}$$

converges. Then

$$\mathfrak{P}(R, M) = \ell^{\mathfrak{S}(R)} \prod_{1 \le s \le d} \frac{\Gamma(a_s)}{\Gamma(b_s)} \prod_{i \in M^+} \frac{\Gamma^2\left(\frac{b_s+i}{\ell}\right)}{\Gamma^2\left(\frac{a_s+i}{\ell}\right)}. \tag{63}$$

Proof. Splitting the product according to its residues modulo ℓ gives

$$\begin{aligned}
\prod_{n=1}^{\infty} R(n)^{M_n} &= \prod_{n=0}^{\infty} \prod_{\substack{i \in M^+ \\ j \in M^-}} \frac{R(\ell n+i)}{R(\ell n+j)} \\
&= \prod_{\substack{i \in M^+ \\ j \in M^-}} \prod_{n=0}^{\infty} \frac{\left(n+\frac{a_1+i}{\ell}\right)\cdots\left(n+\frac{a_d+i}{\ell}\right)\left(n+\frac{b_1+j}{\ell}\right)\cdots\left(n+\frac{b_d+j}{\ell}\right)}{\left(n+\frac{b_1+i}{\ell}\right)\cdots\left(n+\frac{b_d+i}{\ell}\right)\left(n+\frac{a_1+j}{\ell}\right)\cdots\left(n+\frac{a_d+j}{\ell}\right)}.
\end{aligned}$$

The products may be expressed in terms of the gamma function to obtain

$$\prod_{n=1}^{\infty} R(n)^{M_n} = \prod_{\substack{i \in M^+ \\ j \in M^-}} \frac{\Gamma\left(\frac{b_1+i}{\ell}\right)\cdots\Gamma\left(\frac{b_d+i}{\ell}\right)\Gamma\left(\frac{a_1+j}{\ell}\right)\cdots\Gamma\left(\frac{a_d+j}{\ell}\right)}{\Gamma\left(\frac{a_1+i}{\ell}\right)\cdots\Gamma\left(\frac{a_d+i}{\ell}\right)\Gamma\left(\frac{b_1+j}{\ell}\right)\cdots\Gamma\left(\frac{b_d+j}{\ell}\right)} \tag{64}$$

and the result is simplified using Gauss' multiplication formula

$$(2\pi)^{\frac{\ell-1}{2}} \ell^{\frac{1}{2}-\ell z} \Gamma(\ell z) = \prod_{j=0}^{\ell-1} \Gamma\left(z + \frac{j}{\ell}\right). \tag{65}$$

Take $z = a_s/\ell$ to produce

$$\begin{aligned}
(2\pi)^{\frac{\ell-1}{2}} \ell^{1/2-a_s} \Gamma(a_s) &= \Gamma\left(\frac{a_s}{\ell}\right)\Gamma\left(\frac{a_s+1}{\ell}\right)\cdots\Gamma\left(\frac{a_s+\ell-1}{\ell}\right) \\
&= \Gamma\left(\frac{a_s+i_1}{\ell}\right)\cdots\Gamma\left(\frac{a_s+i_{|M^+|}}{\ell}\right)\Gamma\left(\frac{a_s+j_1}{\ell}\right)\cdots\Gamma\left(\frac{a_s+j_{|M^+|}}{\ell}\right)
\end{aligned}$$

since every residue modulo ℓ appears exactly once in the sets M^+ and M^-. It follows that

$$\prod_{j \in M^-} \Gamma\left(\frac{a_s+j}{\ell}\right) = \frac{(2\pi)^{(\ell-1)/2} \ell^{1/2-a_s} \Gamma(a_s)}{\prod_{i \in M^+} \Gamma\left(\frac{a_s+i}{\ell}\right)}, \tag{66}$$

for $1 \le s \le d$. A similar result holds for b_s. Replacing in (64) concludes the proof. □

Example 11. Consider the sequence $\overline{\{1, -1, -1\}}$, where the bar indicates the fundamental period; that is,

$$M_n = \begin{cases} 1, & \text{if } n \equiv 0 \pmod 3; \\ -1, & \text{if } n \equiv 1, 2 \pmod 3). \end{cases} \tag{67}$$

Therefore $M^+ = \{0\}$, $M^- = \{1, 2\}$ so that $M_* = -1$. Theorem 9 states that the convergence of $\mathfrak{P}(R_1, 1)$ is equivalent to $\mathfrak{S}(R) = 0$. Take $R(z) = \dfrac{(z+1)(z+3)}{(z+2)^2}$. The conditions for convergence of $\mathfrak{P}(R, M)$ are satisfied, and its value is

$$\prod_{n=0}^{\infty} \left(\frac{(n+1)(n+3)}{(n+2)^2} \right)^{M_n} = \frac{\Gamma(1)\Gamma^2\left(\frac{2}{3}\right)}{\Gamma(2)\Gamma^2\left(\frac{1}{3}\right)} \frac{\Gamma(3)\Gamma^2\left(\frac{2}{3}\right)}{\Gamma(2)\Gamma^2\left(\frac{3}{3}\right)} = 2 \cdot \frac{\Gamma^4\left(\frac{2}{3}\right)}{\Gamma^2\left(\frac{1}{3}\right)} = \frac{3}{2\pi^2}\Gamma^6\left(\frac{2}{3}\right). \tag{68}$$

by Theorem 10.

Example 12. Let $R(z) = \dfrac{(z+2)(z+3)}{(z+1)(z+4)}$ and $M = \overline{\{1, 1, 1, -1\}}$. Then $M^+ = \{0, 1, 2\}$ and $M^- = \{3\}$. Thus $M_* \neq 0$. The product $\mathfrak{P}(R, M)$ converges by Theorem 9, and Theorem 10 gives

$$\prod_{n=0}^{\infty} \left(\frac{(n+2)(n+3)}{(n+1)(n+4)} \right)^{M_n} = \frac{1}{24\pi}\Gamma^4\left(\frac{1}{4}\right). \tag{69}$$

5 The paperfolding sequence

The paperfolding sequence is defined by the rules

$$\epsilon_{2n} = (-1)^n \text{ and } \epsilon_{2n+1} = \epsilon_n. \tag{70}$$

Allouche [1] considered the products

$$A = \prod_{n=0}^{\infty} \left(\frac{2n+1}{2n+2} \right)^{\epsilon_n} \text{ and } B = \prod_{n=1}^{\infty} \left(\frac{2n}{2n+1} \right)^{\epsilon_n}, \tag{71}$$

and proved

$$B = \frac{\Gamma\left(\frac{1}{4}\right)^2}{8\sqrt{2\pi}}. \tag{72}$$

The *closed-form* evaluation of A remains an open problem.

The goal of this section is to present a new proof of (72) and to present an alternative product expression for A. Observe that

$$\prod_{n=0}^{\infty} \left(\frac{an+b}{cn+d} \right)^{\epsilon_n} = \prod_{n=0}^{\infty} \left(\frac{2an+b}{2cn+d} \right)^{(-1)^n} \times \prod_{n=0}^{\infty} \left(\frac{2an+(a+b)}{2cn+(c+d)} \right)^{\epsilon_n}. \tag{73}$$

The convergence of the first product requires $a = c$ and its value has been obtained in Theorem 4 as

$$\prod_{n=0}^{\infty} \left(\frac{2an+b}{2cn+d} \right)^{(-1)^n} = 2^{d/2c-b/2a} \frac{\Gamma^2\left(\frac{d}{4c}\right)\Gamma\left(\frac{b}{2a}\right)}{\Gamma^2\left(\frac{b}{4a}\right)\Gamma\left(\frac{d}{2c}\right)}. \tag{74}$$

Iterating this procedure converts the second factor in (73) into

$$\prod_{n=0}^{\infty} \left(\frac{2an+(a+b)}{2cn+(c+d)} \right)^{\epsilon_n} = \prod_{n=0}^{\infty} \left(\frac{4an+(a+b)}{4cn+(c+d)} \right)^{(-1)^n} \times \prod_{n=0}^{\infty} \left(\frac{4an+(3a+b)}{4cn+(3c+d)} \right)^{\epsilon_n}. \tag{75}$$

The first product on the right-hand side of (75) converges and Theorem 4 gives

$$\prod_{n=0}^{\infty} \left(\frac{4an+(a+b)b}{4cn+(c+d)} \right)^{(-1)^n} = 2^{d/4c-b/4a} \frac{\Gamma^2\left(\frac{c+d}{8c}\right)\Gamma\left(\frac{a+b}{4a}\right)}{\Gamma^2\left(\frac{a+b}{8a}\right)\Gamma\left(\frac{c+d}{4c}\right)}. \tag{76}$$

Now observe that

$$\frac{c+d}{8c} = \frac{1}{4} + \frac{d-c}{8c} \tag{77}$$

so (76) can be written as

$$\prod_{n=0}^{\infty} \left(\frac{4an+(a+b)b}{4cn+(c+d)} \right)^{(-1)^n} = 2^{d/4c-b/4a} \frac{\Gamma^2\left(\frac{1}{4}+\frac{d-c}{8c}\right)\Gamma\left(\frac{1}{2}+\frac{b-a}{2a}\right)}{\Gamma^2\left(\frac{1}{4}+\frac{b-a}{8a}\right)\Gamma\left(\frac{1}{2}+\frac{d-c}{4c}\right)}. \tag{78}$$

Repeated application of this process gives

$$\prod_{n=0}^{\infty} \left(\frac{an+b}{cn+d} \right)^{\epsilon_n} = 2^{(d/c-b/a)\sum_{k=1}^{N} 1/2^k} \times \prod_{k=2}^{N} \frac{\Gamma^2\left(\frac{1}{4}+\frac{d-c}{c2^k}\right)\Gamma\left(\frac{1}{2}+\frac{b-a}{a2^{k-1}}\right)}{\Gamma^2\left(\frac{1}{4}+\frac{b-a}{a2^k}\right)\Gamma\left(\frac{1}{2}+\frac{d-c}{c2^{k-1}}\right)} \times \prod_{n=0}^{\infty} \left(\frac{2^N an + a(2^N-1)+b}{2^N cn + c(2^N-1)+d} \right)^{\epsilon_n}. \tag{79}$$

A direct argument shows that the last product converges to 1 when $N \to \infty$. This completes the proof of the next statement.

Theorem 13. *The infinite product associated with the paperfolding sequence is given by*

$$\prod_{n=0}^{\infty} \left(\frac{an+b}{cn+d} \right)^{\epsilon_n} = 2^{(d/c-b/a)} \prod_{k=2}^{\infty} \frac{\Gamma^2\left(\frac{1}{4}+\frac{d-c}{c2^k}\right)\Gamma\left(\frac{1}{2}+\frac{b-a}{a2^{k-1}}\right)}{\Gamma^2\left(\frac{1}{4}+\frac{b-a}{a2^k}\right)\Gamma\left(\frac{1}{2}+\frac{d-c}{c2^{k-1}}\right)}. \tag{80}$$

The product appearing in Theorem 13 does not seem to admit a simple closed form for general choice of the parameters a, b, d (recall that $a = c$ is required for the convergence of the product). Such a closed form is obtained in the special situation where the factors telescope. This occurs when $2d = a + b$. The next corollary (equivalent to a theorem of Allouche [1, Theorem 1]) gives such a closed form, with $\alpha = d/a$. In that situation

$$\prod_{k=2}^{N} \frac{\Gamma^2\left(\frac{1}{4} + \frac{d-c}{c2^k}\right)}{\Gamma^2\left(\frac{1}{4} + \frac{b-a}{a2^k}\right)} \to \frac{\Gamma^2(\frac{1}{4})}{\Gamma^2(\frac{\alpha}{2} - \frac{1}{4})} \tag{81}$$

and

$$\prod_{k=2}^{N} \frac{\Gamma\left(\frac{1}{2} + \frac{b-a}{a2^{k-1}}\right)}{\Gamma\left(\frac{1}{2} + \frac{d-c}{a2^{k-1}}\right)} \to \frac{\Gamma(\alpha - \frac{1}{2})}{\Gamma(\frac{1}{2})}. \tag{82}$$

Corollary 14. *A special case of the paperfolding product is given by*

$$\prod_{n=0}^{\infty} \left(\frac{n+2\alpha-1}{n+\alpha}\right)^{\epsilon_n} = 2^{1-\alpha} \frac{\Gamma^2\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}\right)} \frac{\Gamma\left(\alpha - \frac{1}{2}\right)}{\Gamma^2\left(\frac{\alpha}{2} - \frac{1}{4}\right)}. \tag{83}$$

Example 15. Take $\alpha = 3$ to obtain

$$\prod_{n=0}^{\infty} \left(\frac{n+5}{n+3}\right)^{\epsilon_n} = 3. \tag{84}$$

Example 16. The infinite product B in (71) comes by taking the limit as $\alpha \to \frac{1}{2}$. Indeed, write (83) as

$$\prod_{n=1}^{\infty} \left(\frac{n+2\alpha-1}{n+\alpha}\right)^{\epsilon_n} = \frac{\alpha}{\alpha - \frac{1}{2}} \frac{\Gamma^2\left(\frac{1}{4}\right)}{2^{\alpha}\Gamma\left(\frac{1}{2}\right)} \frac{\Gamma\left(\alpha - \frac{1}{2}\right)}{\Gamma^2\left(\frac{\alpha}{2} - \frac{1}{4}\right)}. \tag{85}$$

The limit

$$\lim_{x \to 0} \frac{\Gamma(x)}{x\Gamma^2(x/2)} = \frac{1}{4} \tag{86}$$

gives

$$\prod_{n=1}^{\infty} \left(\frac{2n}{2n+1}\right)^{\epsilon_n} = \frac{\Gamma^2\left(\frac{1}{4}\right)}{8\sqrt{2\pi}}, \tag{87}$$

confirming (72).

Example 17. The method described above does not produce a closed form for the product A in (71). A direct use of the expression in Theorem 13 gives

$$A = \prod_{n=0}^{\infty} \left(\frac{2n+1}{2n+2}\right)^{\epsilon_n} = \sqrt{2} \prod_{k=2}^{\infty} \left(\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{4} - \frac{1}{2^{k+1}}\right)}\right)^2 \times \frac{\Gamma\left(\frac{1}{2} - \frac{1}{2^k}\right)}{\Gamma\left(\frac{1}{2}\right)}. \tag{88}$$

Iterating the duplication formula for the gamma function yields the so-called Knar formula [5, volume 1, page 6, formula 6]

$$\Gamma(1+z) = 2^{2z} \prod_{k=1}^{\infty} \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2} + \frac{z}{2^k}\right) \tag{89}$$

and $z = -\frac{1}{2}$ gives

$$\prod_{k=2}^{\infty} \frac{\Gamma\left(\frac{1}{2} - \frac{1}{2^k}\right)}{\Gamma\left(\frac{1}{2}\right)} = 2\sqrt{\pi}. \tag{90}$$

Then (88) becomes

$$A = \prod_{n=0}^{\infty} \left(\frac{2n+1}{2n+2}\right)^{\epsilon_n} = 2\sqrt{2\pi} \prod_{k=3}^{\infty} \left(\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{4} - \frac{1}{2^k}\right)}\right)^2. \tag{91}$$

The authors have been unable to reduce this any further.

6 Generalization to certain k-automatic sequences

This section extends the results on the paperfolding sequence to certain k-automatic sequences. As usual, let $R(z)$ be a rational function written in the form

$$R(z) = \frac{(z+a_1)\cdots(z+a_d)}{(z+b_1)\cdots(z+b_d)} \tag{92}$$

and assume that a_i and b_j are not in $\{0, -1, -2, \dots\}$.

Consider the case in which M_n is a 3-automatic sequence defined by the rules

$$\begin{aligned} M_{3n} &= q_0(n), \\ M_{3n+1} &= q_1(n), \\ M_{3n+2} &= M_n, \end{aligned} \tag{93}$$

where q_j takes values in $\{+1, -1\}$ and $q_j(n)$ is periodic of period length ℓ_j. Now split the product according to residues modulo 3 to produce

$$\begin{aligned} \prod_{n=0}^{\infty} R(n)^{M_n} &= \prod_{n=0}^{\infty} R(3n)^{M_{3n}} \times \prod_{n=0}^{\infty} R(3n+1)^{M_{3n+1}} \times \prod_{n=0}^{\infty} R(3n+2)^{M_{3n+2}} \\ &= \prod_{n=0}^{\infty} R(3n)^{q_0(n)} \times \prod_{n=0}^{\infty} R(3n+1)^{q_1(n)} \times \prod_{n=0}^{\infty} R(3n+2)^{M_n}. \end{aligned}$$

The convergence and values of the first two products are provided by Theorem 9 and Theorem 10.

Assume the convergence of the product

$$\mathbb{P}_0 = \prod_{n=0}^{\infty} R(3n)^{q_0(n)}. \tag{94}$$

Theorem 9 shows that this happens if $|q_0| = 0$, where $|q_0|$ is the number of $+1$ minus the number of -1 in one period. In the remaining case, it is required that $\mathfrak{S}(R(3z)) = 0$, where $\mathfrak{S}(R)$ is defined in (20). The exact form of the product is obtained from Theorem 10 which yields, with $R_0(z) = R(3z)$,

$$\mathbb{P}_0 = \mathfrak{P}(R_0, q_0) = \ell_0^{\mathfrak{S}(R_0)} \prod_{1 \leq s \leq d} \frac{\Gamma(a_s/3)}{\Gamma(b_s/3)} \prod_{i \in q_0^+} \frac{\Gamma^2\left(\frac{b_s+3i}{3\ell_0}\right)}{\Gamma^2\left(\frac{a_s+3i}{3\ell_0}\right)}. \tag{95}$$

A similar process gives an analytic formula for the second product. Repeating the previous process yields a decomposition of the third product as

$$\prod_{n=0}^{\infty} R(n)^{M_n} \quad = \quad \prod_{n=0}^{\infty} R(9n+2)^{q_0(n)} \times \prod_{n=0}^{\infty} R(9n+5)^{q_1(n)} \times \prod_{n=0}^{\infty} R(9n+8)^{M_n}.$$

As before, the first two products have an explicit analytic expression and the last one has to be split again.

This process can be iterated to obtain a formula for the original product. For simplicity, the results are given for $R(z)$ a rational function of degree 1 and only in the case in which all the periodic pieces $q_i(n)$ have a period length that is a power of a fixed even integer. In this situation, the final formula can be simplified.

Theorem 18. *Let $R(z) = \dfrac{z+b}{z+d}$, with $b,\, d \in \mathbb{R}^+$ and let M_n be a k-automatic sequence satisfying the rules*

$$\begin{aligned} M_{kn} &= q_0(n) \\ M_{kn+1} &= q_1(n) \\ &\vdots \\ M_{kn+k-2} &= q_{k-2}(n) \\ M_{kn+k-1} &= M_n. \end{aligned}$$

Assume there is an even integer L such that each sequence $q_i(n)$ is a periodic sequence of period length $L_i = L^{\alpha_i}$ some power of L. In addition, assume that $|q_i^+| = |q_i^-|$ for all $0 \leq i \leq k-2$. Then

$$\mathfrak{P}(R, M) = \prod_{n=0}^{\infty} R(n)^{M_n} \tag{96}$$

converges. Moreover, if $d = \dfrac{b+k-1}{k}$ *the product in* (96) *can be evaluated as*

$$\prod_{n=0}^{\infty} R(n)^{M_n} = \prod_{i=0}^{k-2} \left(L_i^{\frac{1-b}{k}} \frac{\Gamma(\frac{b+i}{k})}{\Gamma(\frac{i+1}{k})} \prod_{j \in q_i^+} \frac{\Gamma^2\left(\frac{i+1}{L_i k} + \frac{j}{L_i}\right)}{\Gamma^2\left(\frac{b+i}{L_i k} + \frac{j}{L_i}\right)} \right). \tag{97}$$

Note that the paperfolding sequence satisfies the hypothesis of the theorem. In this case $k = 2$ and $q_0(n) = (-1)^n$, and $L = 2$. The rational function is

$$R(n) = \frac{n+b}{n + \frac{b+1}{2}}$$

and (97) reduces to the result of Allouche. The idea of the proof is the argument presented in the case of the 3-automatic sequence above. Complete details may be found in [8].

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