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Arithmetic Progressions on $y^2 = x^3 + k$

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Abstract

Many authors have studied the problem of finding sequences of rational points on elliptic curves such that either the abscissae or the ordinates of these points are in arithmetic progression. In this paper we obtain upper bounds for the lengths of sequences of rational points on curves of the type $y^2 = x^3 + k$, $k \in \mathbb{Q} \setminus \{0\}$, such that the ordinates of the points are in arithmetic progression, and also when both the abscissae and the ordinates of the points are separately the terms of two arithmetic progressions.

1 Introduction

Let there exist n rational points $P_i = (x_i, y_i),\ i = 1, 2, \ldots, n$ on an elliptic curve E over $\mathbb{Q}$ such that either the n numbers $x_i,\ i = 1, 2, \ldots, n$, or the n numbers $y_i,\ i = 1, 2, \ldots, n$, are the terms of an arithmetic progression. We say that there exists an x-arithmetic progression, or a y-arithmetic progression, of length n on the elliptic curve E depending on whether the abscissae or the ordinates of the points P_i are in arithmetic progression. Further, we say there exists a simultaneous arithmetic progression of length n on the curve E if the n numbers $x_i,\ i = 1, 2, \ldots, n$ are in arithmetic progression and simultaneously, the n numbers $y_i,\ i = 1, 2, \ldots, n$ are also, in some order, the terms of an arithmetic progression.

The problem of finding x-arithmetic progressions on various models of elliptic curves has been studied by several mathematicians [3, 4, 5, 6, 10, 13, 13, 15]. The existence of simultaneous arithmetic progressions on the cubic model of elliptic curves has also been studied [7].

This paper is concerned with the existence of y-arithmetic progressions as well as simultaneous arithmetic progressions on Mordell curves which are elliptic curves defined by equations of the type,

$$y^2 = x^3 + k, \tag{1}$$

where k is a rational number. Mohanty [11] found an y-arithmetic progression of four consecutive integers on the curve (1) when $k = 1025$ and conjectured [12] that the length of an x-arithmetic progression on a Mordell curve cannot exceed 4. Later, Lee and Vélez [9] found infinitely many families of Mordell curves with x-arithmetic progressions of length 4 and y-arithmetic progressions of length 6.

In this paper we show that the maximum length of a y-arithmetic progression on a Mordell curve is 6, and we explicitly find infinitely many Mordell curves on which there exist y-arithmetic progressions of length 6. Further, we show that the maximum length of a simultaneous arithmetic progression on a Mordell curve is 3, and we explicitly find infinitely many Mordell curves on which there exist simultaneous arithmetic progressions of length 3.

2 Notation and preliminaries

Now we will introduce some basic notation. We denote by $S_x(E)$ (respectively, $S_y(E)$) as the maximal number length of x-arithmetic progression (respectively, y-arithmetic progression) on the elliptic curve E. Also, let $S_{x,y}(E)$ denote the maximal length of simultaneous arithmetic progression on the elliptic curve E

To prove one of our results we need the following theorem of Bremner [2].

Theorem 1. *The only general rational solution of the surface*

$$x^3 + y^3 + cz^3 = c \ (c \neq 1) \tag{2}$$

are given, up to symmetry, by

$$(i)\ (\lambda, -\lambda, 1)\ \textit{and}\ \ (ii)\left(\frac{9}{c}\lambda^4 - 3\lambda, -\frac{9}{c}\lambda^4, \frac{9}{c}\lambda^3 - 1\right)$$

and the additional case when $c = 2$

$$\begin{aligned} (iii)\ (x,y,z) =& (-4\lambda^2 + 6\lambda - 1, -4\lambda^2 + 2\lambda + 1, 4\lambda^2 - 4\lambda + 1) \\ (iv)\ (x,y,z) =& (\frac{2}{27}(4\lambda^4 - 4\lambda^3 - 6\lambda^2 + 17\lambda - 2), \\ & \frac{4}{27}(2\lambda^4 - 8\lambda^3 + 6\lambda^2 + 4\lambda - 13), \\ & \frac{-1}{27}(8\lambda^4 - 20\lambda^3 + 24\lambda^2 + 16\lambda - 37)) \end{aligned} \tag{3}$$

with $\lambda \in \mathbb{Q}$.

3 Main results

We obtained an upper bound for $S_y(E)$ and $S_{x,y}(E)$ where $E : y^2 = x^3 + k$ with $k \in \mathbb{Q}^*$. Indeed, we prove the following results.

Theorem 2. *Let $E : y^2 = x^3 + k$ be an elliptic curve for some $k \in \mathbb{Q}$. Then $S_y(E) \leq 6$. Moreover, there exist infinitely many such elliptic curves $E : y^2 = x^3 + k$ with $S_y(E) = 6$.*

Proof. At first we will construct an infinite family of elliptic curves of the form $E : y^2 = x^3 + k$ with six rational points such that the y-coordinates of these six rational points are in arithmetic progression. After that, we will show that there exist at most six rational points whose y-coordinates are in arithmetic progression i.e., $S_y(E) \leq 6$.

Let r and d be two arbitrary non-zero rational numbers and $k = r^2 - d^3$. Consequently $(d, \pm r)$ lie on E. Now we want $(x_1, \pm 3r)$ and $(x_2, \pm 5r)$ (with $x_1, x_2 \in \mathbb{Q}$) to be on E and that would give,

$$8r^2 + d^3 = x_1^3, \quad 24r^2 + d^3 = x_2^3.$$

Hence,

$$16r^2 = x_2^3 - x_1^3 \tag{4}$$

and

$$2d^3 = 3x_1^3 - x_2^3. \tag{5}$$

Considering $X_1 = \frac{x_1}{d}$ and $X_2 = \frac{x_2}{d}$ in (5), we get

$$3X_1^3 - X_2^3 = 2. \tag{6}$$

Now $(X_1, X_2) = (\frac{1}{4}, \frac{-5}{4})$ is a point on (6) and correspondingly one gets $x_1 = \frac{d}{4}$ and $x_2 = \frac{-5d}{4}$. Now putting the values of x_1 and x_2 in (4) we get

$$2^{10}r^2 = -126d^3. \tag{7}$$

At this point if we choose $d = -14q^2$ for any $q \in \mathbb{Q}^*$, then $r = \pm\frac{147}{8}q^3$. Hence

$$k = r^2 - d^3 = \frac{197225}{64}q^6. \tag{8}$$

Therefore corresponding to every rational number q we can find d, r and k; i.e., we can find infinitely many elliptic curves of the form $y^2 = x^3 + k$ with at least six rational points whose y-coordinates are in arithmetic progression.

Next we will show that there do not exist more than six rational points whose y-coordinates are in arithmetic progression.

Let d be any non-zero rational number. Let us assume that there exist seven rational points on E, whose y-coordinates are in arithmetic progression. Without loss of generality, we may take the points to be

$$P_1 = (x_1, y_1), P_2 = (x_2, y_1 + d), P_3 = (x_3, y_1 - d), P_4 = (x_4, y_1 + 2d),$$

$$P_5 = (x_5, y_1 - 2d), P_6 = (x_6, y_1 + 3d), P_7 = (x_7, y_1 - 3d).$$

One has

$$y_1^2 = x_1^3 + k, \tag{9}$$

$$(y_1 + d)^2 = x_2^3 + k, \tag{10}$$

$$(y_1 - d)^2 = x_3^3 + k, \tag{11}$$

$$(y_1 + 2d)^2 = x_4^3 + k, \tag{12}$$

$$(y_1 - 2d)^2 = x_5^3 + k, \tag{13}$$

$$(y_1 + 3d)^2 = x_6^3 + k, \tag{14}$$

$$(y_1 - 3d)^2 = x_7^3 + k. \tag{15}$$

Now calculating $[(10) - (11)], [(12) - (13)],$ and $[(14) - (15)]$ one has

$$4y_1d = x_2^3 - x_3^3, \tag{16}$$

$$8y_1d = x_4^3 - x_5^3, \tag{17}$$

$$12y_1d = x_6^3 - x_7^3. \tag{18}$$

Case 1: In this case we are considering $y_1 \neq 0$. From (17) it is clear that one of x_4 and x_5 has to be non-zero as $d \neq 0$ and $y_1 \neq 0$. So, without loss of generality we can assume that x_5 is non-zero.

Now, eliminating y_1 from the equations (16) and (17), we have

$$\begin{aligned}
&2(x_2^3 - x_3^3) = x_4^3 - x_5^3 \\
\Rightarrow &8(x_2^3 - x_3^3) = 4(x_4^3 - x_5^3) \\
\Rightarrow &-8x_2^3 + 8x_3^3 + 4x_4^3 = 4x_5^3 \\
\Rightarrow &\left(\frac{-2x_2}{x_5}\right)^3 + \left(\frac{2x_3}{x_5}\right)^3 + 4\left(\frac{x_4}{x_5}\right)^3 = 4 \\
\Rightarrow &X^3 + Y^3 + 4Z^3 = 4
\end{aligned} \tag{19}$$

where $X = \frac{-2x_2}{x_5}, Y = \frac{2x_3}{x_5}$, and $Z = \frac{x_4}{x_5}$.

Again, eliminating y_1 from the equations (18) and (17), we get

$$\begin{aligned} &2(x_7^3 - x_6^3) = 3(x_5^3 - x_4^3) \\ \Rightarrow &8x_7^3 - 8x_6^3 + 12x_4^3 = 12x_5^3 \\ \Rightarrow &\left(\frac{2x_7}{x_5}\right)^3 + \left(\frac{-2x_6}{x_5}\right)^3 + 12\left(\frac{x_4}{x_5}\right)^3 = 12 \\ \Rightarrow &U^3 + V^3 + 12W^3 = 12 \end{aligned} \tag{20}$$

where $U = \frac{2x_7}{x_5}, V = \frac{-2x_6}{x_5}$, and $W = \frac{x_4}{x_5}$.

To find rational solution of (19) and (20) we use Bremner's result. Using Theorem (1) we get all possible rational solutions of (19) and (20). The rational solutions of (19) are

$$(i)\ (\lambda, -\lambda, 1) \text{ and } \ (ii)\left(\frac{9}{4}\lambda^4 - 3\lambda, -\frac{9}{4}\lambda^4, \frac{9}{4}\lambda^3 - 1\right)$$

with $\lambda \in \mathbb{Q}$.

The rational solutions of (20) are

$$(i)\ (\mu, -\mu, 1) \text{ and } \ (ii)\left(\frac{3}{4}\mu^4 - 3\mu, -\frac{3}{4}\mu^4, \frac{3}{4}\mu^3 - 1\right)$$

with $\mu \in \mathbb{Q}$.

One can see from (19) and (20) that,

$$Z = W = \frac{x_4}{x_5}.$$

Clearly,

$$\frac{9}{4}\lambda^3 - 1 = 1 \quad \text{and} \quad \frac{3}{4}\mu^3 - 1 = 1$$

cannot hold for any $\lambda, \mu \in \mathbb{Q}$.

Now if possible let us assume that for some λ and μ,

$$\frac{9}{4}\lambda^3 - 1 = \frac{3}{4}\mu^3 - 1.$$

That gives $3\lambda^3 = \mu^3$, which is again not possible as $\lambda, \mu \in \mathbb{Q}$.

Thus we get $Z = W = 1$ and that implies $x_4 = x_5$. Now from (12) and (13) we can write,

$$(y_1 + 2d) = \pm(y_1 - 2d).$$

This forces $y_1 = 0$ as $d \neq 0$, which is a contradiction.

Case 2: In this case we consider $y_1 = 0$. Thus our points are

$$P_1 = (x_1, 0), P_2 = (x_2, d), P_3 = (x_3, -d), P_4 = (x_4, 2d),$$

$$P_5 = (x_5, -2d), P_6 = (x_6, 3d), P_7 = (x_7, -3d).$$

One can easily check that, $k = (-x_1)^3, x_2 = x_3$, and $x_6 = x_7$. Thus,

$$d^2 = x_2^3 - x_1^3. \tag{21}$$

$$4d^2 = x_4^3 - x_1^3. \tag{22}$$

$$9d^2 = x_6^3 - x_1^3. \tag{23}$$

From (21) it is clear that one of x_1 and x_2 has to be non-zero as $d \neq 0$. So, without loss of generality we can assume that x_1 is non-zero.

Eliminating d from the equations (21), (22) and (23), we have

$$\begin{aligned} & x_6^3 - x_4^3 = 5(x_2^3 - x_1^3) \\ \Rightarrow & \left(\frac{x_4}{x_1}\right)^3 + \left(\frac{-x_6}{x_1}\right)^3 + 5\left(\frac{x_2}{x_1}\right)^3 = 5 \\ \Rightarrow & L^3 + M^3 + 5N^3 = 5 \end{aligned} \tag{24}$$

where $L = \frac{x_4}{x_1}, M = \frac{-x_6}{x_1}$, and $N = \frac{x_2}{x_1}$. Again using Bremner's result, the solution set of the equation (24) is

$$(i)\ (\nu, -\nu, 1) \text{ and } \ (ii) \left(\frac{9}{5}\nu^4 - 3\nu, -\frac{9}{5}\nu^4, \frac{9}{5}\nu^3 - 1\right),$$

where $\nu \in \mathbb{Q}$.

Note that $N \neq 1$, as $N = 1$ implies $d = 0$, which is not possible. Let x_1 be any non-zero rational number, say q. Then we have

$$x_2 = \left(\frac{9}{5}\nu^3 - 1\right) q,\ x_4 = \left(\frac{9}{5}\nu^4 - 3\nu\right) q,\ x_6 = \left(\frac{9}{5}\nu^4\right) q.$$

Substituting in the values of x_2 and x_6 into (21) and (23) respectively, we have

$$d^2 = \left[\left(\frac{9}{5}\nu^3 - 1\right)^3 - 1\right] q^3,$$

$$9d^2 = \left[\left(\frac{9}{5}\nu^4\right)^3 - 1\right] q^3.$$

Therefore we get

$$9\left[\left(\frac{9}{5}\nu^3 - 1\right)^3 - 1\right] = \left[\left(\frac{9}{5}\nu^4\right)^3 - 1\right],$$

which leads us to

$$9(9t-5)^3 = 9^3t^4 + 1000,$$

where $\nu^3 = t$.

We will prove that this equation does not have any rational solution. Putting $s = 3t$ we have the following equation:

$$9s^4 - 243s^3 + 1215s^2 - 2025s + 2125 = 0. \tag{25}$$

Using the rational root theorem, we can see that (25) does not have any rational solution. Therefore our system of equations [(9) to (15)] do not have any rational solution. Hence there do not exist seven rational points on $y^2 = x^3 + k$ with y-coordinates in arithmetic progression, i.e., $S_y(E) \leq 6$.

□

Now we will state the result related to simultaneous arithmetic progression.

Theorem 3. *Let $E : y^2 = x^3 + k$ be an elliptic curve for some $k \in \mathbb{Q}$. Then $S_{x,y}(E) \leq 3$. Moreover, there exist infinitely many such elliptic curves $E : y^2 = x^3 + k$ with $S_{x,y}(E) = 3$.*

Proof. Let d and d' be given nonzero rational numbers. Suppose $P_1 = (x_1, y_1)$ is a rational point on $E : y^2 = x^3 + k$ for some $k \in \mathbb{Q}$. Therefore, P_1 satisfies

$$y_1^2 = x_1^3 + k. \tag{26}$$

Suppose P_2 and P_3 are other rational points on E such that P_1, P_2 and P_3 form a simultaneous arithmetic progression with differences d and d' of length 3. Therefore, we let $P_2 = (x_1 + d, y_1 + d')$ and $P_3 = (x_1 - d, y_1 - d')$ and we get

$$(y_1 + d')^2 = (x_1 + d)^3 + k \tag{27}$$

and

$$(y_1 - d')^2 = (x_1 - d)^3 + k. \tag{28}$$

By subtracting (28) from (27), we get

$$4y_1d' = 6x_1^2d + 2d^3. \tag{29}$$

From (26) and (27), we get

$$2y_1d' + {d'}^2 = 3x_1^2d + 3x_1d^2 + d^3. \tag{30}$$

Therefore, from (29) and (30), we get

$$2{d'}^2 = 6x_1d^2 \iff x_1 = \frac{1}{3}\frac{{d'}^2}{d^2}. \tag{31}$$

Once we get x_1 as a function of d and d', we can get y_1 and k as a function of d and d'. Since this is true for a given non-zero rational numbers d and d', we conclude that there are infinitely many elliptic curves E of the form $y^2 = x^3 + k$ which admits $S_{x,y}(E) \geq 3$.

In order to finish the proof of the theorem, now it is enough to prove that $S_{x,y}(E) \leq 3$ for all elliptic curves $E : y^2 = x^3 + k$.

Let E be one such curve. If possible, let P_1, P_2, P_3 and P_4 be the rational points on E which form a simultaneous arithmetic progression of length 4 with some differences d and d' for the x and y coordinates respectively. Now, we need to consider several cases depending on the arrangement of coordinates of P_i's.

Case 1: Let $P_1 = (x_1 - d, y_1 - d'), P_2 = (x_1, y_1), P_3 = (x_1 + d, y_1 + d')$ and $P_4 = (x_1 + 2d, y_1 + 2d')$.

Since P_1, P_2 and P_3 are in simultaneous arithmetic progression, by the previous discussion, we conclude that $x_1 = \frac{1}{3}\frac{d'^2}{d^2}$. Since P_4 is a rational point on E, we get

$$(y_1 + 2d')^2 = (x_1 + 2d)^3 + k. \tag{32}$$

From the equations (32) and (26), we get

$$4y_1 d' + 4{d'}^2 = 6x_1^2 d + 12x_1 d^2 + 8d^3. \tag{33}$$

Now, by putting $x_1 = d'^2/3d^2$ in (33), we arrive at $d = 0$, which is a contradiction. Thus, we conclude, in this case, $S_{x,y}(E) \leq 3$.

Case 2: Let $P_1 = (x_1, y_1), P_2 = (x_1+d, y_1+d'), P_3 = (x_1+2d, y_1+3d'), P_4 = (x_1+3d, y_1+2d')$.

Since these are rational points on E, we have

$$y_1^2 = x_1^3 + k. \tag{34}$$

$$(y_1 + d')^2 = (x_1 + d)^3 + k, \tag{35}$$

$$(y_1 + 3d')^2 = (x_1 + 2d)^3 + k, \tag{36}$$

and

$$(y_1 + 2d')^2 = (x_1 + 3d)^3 + k. \tag{37}$$

Eliminating y_1 and k, from the equations (34), (35) and (36), we get

$$6{d'}^2 = -3x_1^2 d + 3x_1 d^2 + 5d^3. \tag{38}$$

Again, eliminating y_1 and k, from the equations (34), (36) and (37), we have

$$6{d'}^2 = -15x_1^2 d - 57x_1 d^2 - 65d^3. \tag{39}$$

Now eliminating d' from the equations (38) and (39), we get

$$6x_1^2 + 30x_1 d + 35d^2 = 0. \tag{40}$$

Clearly, the quadratic equation (40) in x_1 does not have any rational solutions and hence we conclude, in this case, $S_{x,y}(E) \leq 3$.

The remaining 22 cases can be proved similarly and we list them in Table 1.

From Table 1, we can see that the discriminant D is not a perfect square for a rational number d in all the cases. Hence, we conclude that x_1 cannot be rational, which is a contradiction. Therefore, in all the cases, we get $S_{x,y}(E) \leq 3$. This completes the proof of Theorem 3.

□

4 Some open problems

While we obtained upper bounds for the lengths of $y-$arithmetic progressions and simultaneous arithmetic progressions on Mordell curves, we could not obtain an upper bound for the length of $x-$arithmetic progressions on Mordell curves. It would be of interest to determine such an upper bound. Similarly it would be of interest to determine upper bounds for the length of $x-$arithmetic progressions, $y-$arithmetic progressions and simultaneous arithmetic progressions on various models of elliptic curves.

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Table 1

Cases	Points	Final Equation	Discriminant(D)
Case 3	$(x_1, y_1), (x_1 + d, y_1 + 2d')$ $(x_1 + 2d, y_1 + d'), (x_1 + 3d, y_1 + 3d')$	$3x_1^2 + 9x_1d + 8d^2 = 0$	$-15d^2$
Case 4	$(x_1, y_1), (x_1 + d, y_1 + 2d')$ $(x_1 + 2d, y_1 + 3d'), (x_1 + 3d, y_1 + d')$	$24x_1^2 + 84x_1d + 86d^2 = 0$	$-1200d^2$
Case 5	$(x_1, y_1), (x_1 + d, y_1 + 3d')$ $(x_1 + 2d, y_1 + d'), (x_1 + 3d, y_1 + 2d')$	$3x_1^2 + 14x_1d + 28d^2 = 0$	$-140d^2$
Case 6	$(x_1, y_1), (x_1 + d, y_1 + 3d')$ $(x_1 + 2d, y_1 + 2d'), (x_1 + 3d, y_1 + d')$	$6x_1^2 + 24x_1d + 29d^2 = 0$	$-120d^2$
Case 7	$(x_1, y_1 + d'), (x_1 + d, y_1 + 2d'),$ $(x_1 + 2d, y_1), (x_1 + 3d, y_1 + 3d')$	$3x_1^2 - 3x_1d - 8d^2 = 0$	$105d^2$
Case 8	$(x_1, y_1 + d'), (x_1 + d, y_1 + 2d'),$ $(x_1 + 2d, y_1 + 3d'), (x_1 + 3d, y_1)$	$6x_1^2 + 12x_1d + 11d^2 = 0$	$-120d^2$
Case 9	$(x_1, y_1 + d'), (x_1 + d, y_1)$ $(x_1 + 2d, y_1 + 2d'), (x_1 + 3d, y_1 + 3d')$	$6x_1^2 + 6x_1d - d^2 = 0$	$60d^2$
Case 10	$(x_1, y_1 + d'), (x_1 + d, y_1)$ $(x_1 + 2d, y_1 + 3d'), (x_1 + 3d, y_1 + 2d')$	$12x_1^2 + 36x_1d + 37d^2 = 0$	$-480d^2$
Case 11	$(x_1, y_1 + d'), (x_1 + d, y_1 + 3d')$ $(x_1 + 2d, y_1), (x_1 + 3d, y_1 + 2d')$	$15x_1^2 + 45x_1d + 44d^2 = 0$	$-615d^2$
Case 12	$(x_1, y_1 + d'), (x_1 + d, y_1 + 3d')$ $(x_1 + 2d, y_1 + 2d'), (x_1 + 3d, y_1)$	$12x_1^2 + 30x_1d + 25d^2 = 0$	$-300d^2$
Case 13	$(x_1, y_1 + 2d'), (x_1 + d, y_1)$ $(x_1 + 2d, y_1 + d'), (x_1 + 3d, y_1 + 3d')$	$12x_1^2 + 30x_1d + 25d^2 = 0$	$-300d^2$
Case 14	$(x_1, y_1 + 2d'), (x_1 + d, y_1)$ $(x_1 + 2d, y_1 + 3d'), (x_1 + 3d, y_1 + d')$	$15x_1^2 + 45x_1d + 44d^2 = 0$	$-615d^2$
Case 15	$(x_1, y_1 + 2d'), (x_1 + d, y_1 + d')$ $(x_1 + 2d, y_1), (x_1 + 3d, y_1 + 3d')$	$6x_1^2 + 12x_1d + 11d^2 = 0$	$-120d^2$
Case 16	$(x_1, y_1 + 2d'), (x_1 + d, y_1 + d')$ $(x_1 + 2d, y_1 + 3d'), (x_1 + 3d, y_1)$	$3x_1^2 - 3x_1d - 8d^2 = 0$	$105d^2$
Case 17	$(x_1, y_1 + 2d'), (x_1 + d, y_1 + 3d')$ $(x_1 + 2d, y_1), (x_1 + 3d, y_1 + d')$	$12x_1^2 + 36x_1d + 37d^2 = 0$	$-480d^2$
Case 18	$(x_1, y_1 + 2d'), (x_1 + d, y_1 + 3d')$ $(x_1 + 2d, y_1 + d'), (x_1 + 3d, y_1)$	$6x_1^2 + 6x_1d - d^2 = 0$	$60d^2$
Case 19	$(x_1, y_1 + 3d'), (x_1 + d, y_1)$ $(x_1 + 2d, y_1 + d'), (x_1 + 3d, y_1 + 2d')$	$6x_1^2 + 24x_1d + 29d^2 = 0$	$-120d^2$
Case 20	$(x_1, y_1 + 3d'), (x_1 + d, y_1)$ $(x_1 + 2d, y_1 + 2d'), (x_1 + 3d, y_1 + d')$	$3x_1^2 + 21x_1d + 28d^2 = 0$	$105d^2$
Case 21	$(x_1, y_1 + 3d'), (x_1 + d, y_1 + d')$ $(x_1 + 2d, y_1), (x_1 + 3d, y_1 + 2d')$	$12x_1^2 + 42x_1d + 43d^2 = 0$	$-300d^2$
Case 22	$(x_1, y_1 + 3d'), (x_1 + d, y_1 + d')$ $(x_1 + 2d, y_1 + 2d'), (x_1 + 3d, y_1)$	$9x_1^2 + 27x_1d + 24d^2 = 0$	$-135d^2$
Case 23	$(x_1, y_1 + 3d'), (x_1 + d, y_1 + 2d')$ $(x_1 + 2d, y_1), (x_1 + 3d, y_1 + d')$	$6x_1^2 + 30x_1d + 35d^2 = 0$	$60d^2$
Case 24	$(x_1, y_1 + 3d'), (x_1 + d, y_1 + 2d')$ $(x_1 + 2d, y_1 + d'), (x_1 + 3d, y_1)$	$d = 0$	

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