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On the Enumeration of Restricted Words over a Finite Alphabet

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Abstract

We present a method for the enumeration of restricted words over a finite alphabet. Restrictions are described through the inclusion or exclusion of suitable building blocks used to construct the words by concatenation. Our approach, which relies on the invert transform and its representation in terms of partial Bell polynomials, allows us to generalize and address in a systematic manner previous results in the subject.

1 Introduction

Motivated by a recent paper of Janjić [7], we examine the problem of enumerating words over a finite alphabet using the invert transform, partial Bell polynomials, and building blocks that allow us to effectively manage a wide variety of restrictions.

Recall that, given a sequence $x = (x_n)$, its INVERT transform $\mathscr{Y}(x) = (y_n)$ is defined by

$$1 + \sum_{n=1}^{\infty} y_n t^n = \frac{1}{1 - \sum\limits_{n=1}^{\infty} x_n t^n},$$

see, e.g., the paper by Cameron [5], and the work by Bernstein and Sloane [2]. In short, if $X(t) = \sum_{n=1}^{\infty} x_n t^n$ and $Y(t) = \sum_{n=1}^{\infty} y_n t^n$, then

$$1 + Y(t) = \frac{1}{1 - X(t)}, \text{ and so } Y(t) = \frac{X(t)}{1 - X(t)}.$$

This map is clearly invertible on the space of linear recurrence sequences with constant coefficients, and we have $\mathscr{Y}^m(x) = \frac{1}{m}\mathscr{Y}(mx)$ for every $m \in \mathbb{N}$. Indeed, inductively, if $Y_m(t)$ denotes the generating function of $\mathscr{Y}^m(x)$, then

$$Y_m(t) = \frac{Y_{m-1}(t)}{1 - Y_{m-1}(t)} = \frac{\frac{X(t)}{1-(m-1)X(t)}}{1 - \frac{X(t)}{1-(m-1)X(t)}} = \frac{X(t)}{1 - mX(t)}.$$

The following statement is straightforward.

Lemma 1. *For $b \in \mathbb{N}$, the invert transform of $x_n = b^{n-1}$ is given by $y_n = (b+1)^{n-1}$. Thus for $n > 1$, y_n counts the number of words of length $n-1$ over the alphabet $\{0, 1, \ldots, b\}$. Moreover, by applying the invert transform m times, we get that $\mathscr{Y}^m(x)$ counts the words over the alphabet $\{0, 1, \ldots, b+m-1\}$.*

Corollary 2. *Let $x = (x_n)$ be a sequence of nonnegative integers. If $x_n \le b^{n-1}$ for all $n \in \mathbb{N}$, then the elements of $\mathscr{Y}^m(x)$ count some restricted words over the alphabet $\{0, 1, \ldots, b+m-1\}$.*

This is clear since, if $x_n \le w_n$ for every n, then $\big(1 - \sum\limits_{n=1}^{\infty} x_n t^n\big)^{-1} \le \big(1 - \sum\limits_{n=1}^{\infty} w_n t^n\big)^{-1}$.

For $b = 1$, several families of restricted words were considered by Janjić in [7, Prop. 20]. Consistent with op. cit., we let f_m denote the m-th invert transform of a given sequence f_0, i.e., $f_m = \mathscr{Y}^m(f_0)$. In [7, Corollary 24, 28, and 33], Janjić discussed three examples in which f_0 is a binary sequence. He posted the problem of finding, for an arbitrary sequence f_0, either a recurrence or an explicit formula for $f_m(n)$ in terms of m and n, together with a description of the set of restricted words over the alphabet $\{0, 1, \ldots, m\}$ counted by $f_m(n)$. He also discussed two instances in which the sequence f_0 is not binary but satisfies a second order linear recurrence relation. For all these examples, Janjić gives recurrence relations for f_m obtained from the recurrence relation satisfied by f_0.

In this paper, we present a somewhat different method for the enumeration of restricted words that may be applied to a wide class of restrictions, see Section 2. Our approach relies on a family of building blocks that we utilize to construct words by concatenation, and on a representation of the invert transform in terms of partial Bell polynomials. We discuss several binary and nonbinary examples (Sections 3 and 4, respectively) and shed some light on the examples considered by Janjić [7]. For all of the examples discussed here, we give closed combinatorial expressions.

We finish the introduction by recalling that, by means of Faà di Bruno's formula, the invert transform can be written in terms of partial Bell polynomials, see e.g. [4]. Indeed, if (y_n) is the invert transform of $x = (x_n)$, then

$$y_n = \sum_{k=1}^{n} \frac{k!}{n!} B_{n,k}(1!x_1, 2!x_2, \dots) \text{ for } n \geq 1, \tag{1}$$

where $B_{n,k}$ denotes the (n,k)-th partial Bell polynomial defined as

$$B_{n,k}(z_1, \dots, z_{n-k+1}) = \sum_{\alpha \in \pi(n,k)} \frac{n!}{\alpha_1! \alpha_2! \cdots \alpha_{n-k+1}!} \left(\frac{z_1}{1!}\right)^{\alpha_1} \cdots \left(\frac{z_{n-k+1}}{(n-k+1)!}\right)^{\alpha_{n-k+1}}$$

with $\pi(n,k)$ denoting the set of multi-indices $\alpha \in \mathbb{N}_0^{n-k+1}$ such that $\alpha_1 + \alpha_2 + \cdots = k$ and $\alpha_1 + 2\alpha_2 + 3\alpha_3 + \cdots = n$. For the basic properties of these polynomials, we refer to Comtet's book [6, Section 3.3]. A large collection of Bell identities and convolution formulas is available throughout the literature, see e.g. [1, 3, 9].

From its definition, it is clear that $B_{n,k}$ is homogeneous of degree k:

$$B_{n,k}(az_1, \dots, az_{n-k+1}) = a^k B_{n,k}(z_1, \dots, z_{n-k+1}).$$

Thus, since $\mathscr{Y}^m(x) = \frac{1}{m}\mathscr{Y}(mx)$ for every $m \in \mathbb{N}$, we get from (1) that the terms $y_{m,n}$ of the m-th invert transform of $x = (x_n)$ can be written as

$$y_{m,n} = \sum_{k=1}^{n} \frac{k!}{n!} m^{k-1} B_{n,k}(1!x_1, 2!x_2, \dots) \text{ for } n \geq 1.$$

As we will illustrate in the next sections, this representation of $\mathscr{Y}^m$ provides an effective tool to enumerate restricted words and leads to interesting combinatorial formulas.

2 Restricted words

For a fixed $b \in \mathbb{N}$ we consider the following family of *building blocks*:

> Let $W_b(j)$ be the set of words of length j, starting with b, followed by $j-1$ letters from the alphabet $\{0, 1, \dots, b-1\}$.

Observe that $|W_b(j)| = b^{j-1}$. Moreover, any word of length n over the alphabet $\{0, 1, \dots, b\}$, starting with b, can be made by unique concatenation of blocks in $W_b(j)$ for $j = 1, \dots, n$.

Given $b \in \mathbb{N}$ and a sequence f_0 of nonnegative integers, let $\{U_j\}$ be a fixed sequence of subsets $U_j \subseteq W_b(j)$ such that $|U_j| = f_0(j)$ for $j = 1, 2, \dots$, and let

$\mathscr{U}_{b,n}(f_0)$ be the set of words of length n over the alphabet $\{0,1,\dots,b\}$, starting with b, made by concatenation of blocks from U_j for $j=1,\dots,n$. If $f_0(j)=0$, no block from U_j is used.

Theorem 3. *Let f_0 be a sequence of nonnegative integers such that $f_0(j)\le b^{j-1}$ for every $j\in\mathbb{N}$ and some $b\in\mathbb{N}$. Then the sequence*

$$f_1(0)=1,\quad f_1(n)=\sum_{k=1}^{n}\frac{k!}{n!}B_{n,k}(1!f_0(1),2!f_0(2),\dots),$$

counts the number of words in $\mathscr{U}_{b,n}(f_0)$. Moreover, each term $\frac{k!}{n!}B_{n,k}(1!f_0(1),2!f_0(2),\dots)$ counts the number of such words made with exactly k blocks.

Proof. By definition, every word in $\mathscr{U}_{b,n}(f_0)$ can be written as a unique concatenation of blocks from a subset of $W_b(j)$ with $f_0(j)$ elements for $j=1,\dots,n$. Any of these words must start with a block from $W_b(j)$ for some j, followed by a word in $\mathscr{U}_{b,n-j}(f_0)$. Thus we have the recurrence relation

$$f_1(n)=\sum_{j=1}^{n}f_0(j)f_1(n-j)\ \text{ with }\ f_1(0)=1,$$

which at the level of generating functions implies

$$1+\sum_{n=1}^{\infty}f_1(n)t^n=\frac{1}{1-\sum\limits_{n=1}^{\infty}f_0(n)t^n}.$$

In other words, the sequence f_1 is the invert transform of f_0, which as discussed in the introduction, can be written in terms of partial Bell polynomials, as claimed. Finally, the statement about the number of words made with exactly k blocks is a direct consequence of the definition of the partial Bell polynomials. $\square$

The case of binary building blocks ($b=1$) will be discussed with more details in the next section, and the importance of greater values of b will be illustrated in Section 4.

Remark 4. We will choose b based on the type of restricted words we are interested in counting, not on the size of the alphabet. Once b is fixed, we can enlarge the alphabet by repeatedly applying the invert transform.

Given $b\in\mathbb{N}$ and a sequence f_0 of nonnegative integers, let $\{U_j\}$ be a fixed sequence of subsets $U_j\subseteq W_b(j)$ such that $|U_j|=f_0(j)$ for $j=1,2,\dots$. Every word in U_j has length j and starts with b followed by $j-1$ letters from the alphabet $\{0,1,\dots,b-1\}$. If we allow the character b to be replaced by any element of the set $\{b,\dots,b+m-1\}$ for $m\in\mathbb{N}$, then every word in U_j generates m words over the alphabet $\{0,1,\dots,b+m-1\}$. Thus each U_j gives rise to a set of words of length j, call it U_j^m, such that $|U_j^m|=m\cdot|U_j|$. We let

> $\mathscr{U}_{b,n}^m(f_0)$ be the set of words of length n over the alphabet $\{0, 1, \ldots, b+m-1\}$, *starting with* b, made by concatenation of blocks from U_j^m for $j = 1, \ldots, n$. If $f_0(j) = 0$, no block from U_j^m is used.

Corollary 5. *Let f_0 be a sequence of nonnegative integers such that $f_0(j) \le b^{j-1}$ for every $j \in \mathbb{N}$ and some $b \in \mathbb{N}$. For $m \in \mathbb{N}$, the sequence*

$$f_m(0) = 1, \quad f_m(n) = \sum_{k=1}^{n} \frac{k!}{n!} m^{k-1} B_{n,k}(1!f_0(1), 2!f_0(2), \ldots),$$

counts the number of words in $\mathscr{U}_{b,n}^m(f_0)$. Note that f_m is the m-th invert transform of f_0.

Proof. By Theorem 3, the expression $\frac{k!}{n!} B_{n,k}(1!f_0(1), 2!f_0(2), \ldots)$ counts the number of words in $\mathscr{U}_{b,n}(f_0)$ of the form $w_{j_1} \cdots w_{j_k}$, where each w_{j_i} is a subword in U_{j_i} for $i = 1, \ldots, k$. If each of the subwords $w_{j_2}, \ldots, w_{j_k}$ is allowed to be replaced by any of the corresponding m blocks in $U_{j_2}^m, \ldots, U_{j_k}^m$, respectively, we obtain $\frac{k!}{n!} m^{k-1} B_{n,k}(1!f_0(1), 2!f_0(2), \ldots)$ words in $\mathscr{U}_{b,n}^m(f_0)$. Note that every word in $\mathscr{U}_{b,n}^m(f_0)$ can be constructed in this manner.

Adding over k from 1 to n, we obtain the total number of elements of $\mathscr{U}_{b,n}^m(f_0)$. □

3 Restricted words of binary type

For $b = 1$, the building blocks in $W_b(j)$ are words of the form $10 \cdots 0$ with $j-1$ zeros. Then, for $m \in \mathbb{N}$ and a given binary sequence f_0, the set

> $\mathscr{U}_{1,n}^m(f_0)$ consists of words of length n over the alphabet $\{0, 1, \ldots, m\}$, starting with 1, made by concatenating blocks of the form $k\,0 \cdots 0$ with $j-1$ zeros, $k \in \{1, \ldots, m\}$, where blocks of length j are used only if $f_0(j) = 1$.

For example, if f_0 is the sequence $1, 0, 1, 1, 0, 0, \ldots$, then the elements of $\mathscr{U}_{1,5}^2(f_0)$ are words of length 5 over the alphabet $\{0, 1, 2\}$, starting with 1, made by using only the building blocks of length 1, 3, and 4:

$$1, 2 \in U_1^2, \;\; 100, 200 \in U_3^2, \text{ and } 1000, 2000 \in U_4^2.$$

Moreover, the set $\mathscr{U}_{1,5}^2(f_0)$ consists of 32 such words:

11111, 11112, 11121, 11211, 12111,
11122, 11212, 11221, 12112, 12121, 12211,
11222, 12122, 12212, 12221, 12222,
10011, 10012, 10021, 10022, 11001, 11002, 12001, 12002,
11100, 11200, 12100, 12200, 10001, 10002, 11000, 12000.

Observe that the block restrictions given through the sequence f_0 carry over to the words in $\mathscr{U}^m_{1,n}(f_0)$. For instance, in the above example, the sequence f_0 induces the restrictions that $\mathscr{U}^2_{1,5}(f_0)$ contains no words with an isolated zero ($f_0(2) = 0$) and no words with more than three consecutive zeros ($f_0(j) = 0$ for $j \geq 5$).

Remark 6. By treating the first 1 as a dummy character, we get a natural bijection between $\mathscr{U}^m_{1,n+1}(f_0)$ and the set of all words of length n with the restrictions determined by f_0.

We proceed to illustrate the results from Section 2 with a few examples. In the remaining part of this section, we will freely use some basic properties of partial Bell polynomials found in Comtet's book [6, Section 3.3]. More intricate identities will be proved in the Appendix.

Example 7 (cf. [7, Cor. 33])**.** Suppose we wish to count the number of words of length n over the alphabet $\{0, 1, \dots, m\}$ avoiding runs of ℓ zeros, $\ell \geq 1$. This can be done by computing $f_m(n+1) = \left|\mathscr{U}^m_{1,n+1}(f_0)\right|$ with

$$f_0(j) = \begin{cases} 1, & \text{if } 1 \leq j \leq \ell; \\ 0, & \text{otherwise.} \end{cases}$$

By Corollary 5, we have

$$f_m(n+1) = \sum_{k=1}^{n+1} \frac{k!}{(n+1)!} m^{k-1} \, B_{n+1,k}(1!, 2!, \dots, \ell!, 0, \dots).$$

By Identity 1 in the Appendix, we have

$$B_{n+1,k}(1!, 2!, \dots, \ell!, 0, \dots) = \frac{(n+1)!}{k!} \sum_{j=0}^{\lfloor \frac{n+1-k}{\ell} \rfloor} (-1)^j \binom{k}{j} \binom{n-\ell j}{k-1},$$

and therefore,

$$f_m(n+1) = \sum_{k=1}^{n+1} \sum_{j=0}^{\lfloor \frac{n+1-k}{\ell} \rfloor} (-1)^j \binom{k}{j} \binom{n-\ell j}{k-1} m^{k-1}.$$

For various values of ℓ and m, this gives the sequences A000045, A028859, A125145, A086347, A180033, A000073, A119826, A000078, A209239, A000322 in Sloane [8].

Example 8 (cf. [7, Cor. 28])**.** Consider words of length n over the alphabet $\{0, 1, \dots, m\}$ avoiding runs of zeros of odd length. By adding a 1 to the left of each of these words, we obtain the set $\mathscr{U}^m_{1,n+1}(f_0)$ with

$$f_0(j) = \begin{cases} 1, & \text{if } j \text{ is odd;} \\ 0, & \text{if } j \text{ is even.} \end{cases}$$

In this case, $f_m(n+1) = \left|\mathscr{U}^m_{1,n+1}(f_0)\right|$ is given by

$$f_m(n+1) = \sum_{k=1}^{n+1} \frac{k!}{(n+1)!} m^{k-1}\, B_{n+1,k}(1!, 0, 3!, 0, 5!, \dots).$$

It is easy to see that $B_{n+1,k}(1!, 0, 3!, 0, 5!, \dots) = 0$ unless $n+1+k$ is even. Moreover,

$$\begin{aligned} B_{n+1,k}(1!, 0, 3!, 0, 5!, 0, \dots) &= \frac{(n+1)!}{(n+1-k)!} B_{n+1-k,k}(0, 2!, 0, 4!, 0, \dots) \\ &= \frac{(n+1)!}{(\frac{n+k+1}{2})!} B_{\frac{n+k+1}{2},k}(1!, 2!, 3!, \dots) = \frac{(n+1)!}{k!}\binom{\frac{n+k-1}{2}}{k-1}. \end{aligned}$$

Hence the number of elements in $\mathscr{U}^m_{1,n+1}(f_0)$ is given by

$$f_m(n+1) = \sum_{\substack{1\le k\le n+1\\ n+k+1\equiv 0 \pmod 2}} \binom{\frac{n+k-1}{2}}{k-1} m^{k-1} = \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor} \binom{n-k}{k} m^{n-2k}.$$

For $m = 1, \dots, 10$, this gives the sequences A000045, A000129, A006190, A001076, A052918, A005668, A054413, A041025, A099371, A041041 in Sloane [8].

Example 9. For $\ell \in \mathbb{N}$ let

$$f_0(j) = \begin{cases} 0, & \text{if } j \le \ell; \\ 1, & \text{otherwise.} \end{cases}$$

Then the sequence

$$f_m(n) = \sum_{k=1}^{n} \frac{k!}{n!} m^{k-1}\, B_{n,k}(0, \dots, 0, (\ell+1)!, (\ell+2)!, \dots)$$

counts the number of words of length n over the alphabet $\{0, 1, \dots, m\}$, starting with 1, such that every nonzero element is followed by at least ℓ zeros.

Since

$$B_{n,k}(0, \dots, 0, (\ell+1)!, (\ell+2)!, \dots) = \frac{n!}{(n-\ell k)!} B_{n-\ell k,k}(1!, 2! \dots) \text{ for } \ell k < n,$$

and since $B_{n-\ell k,k}(1!, 2!, \dots) = \frac{(n-\ell k)!}{k!}\binom{n-\ell k-1}{k-1}$, we get

$$\frac{k!}{n!} B_{n,k}(0, \dots, 0, (\ell+1)!, (\ell+2)!, \dots) = \binom{n-\ell k-1}{k-1} \text{ for } \ell k < n.$$

Note that $B_{n,k}(0, \dots, 0, (\ell+1)!, (\ell+2)!, \dots) = 0$ for $\ell k \ge n$. Thus we arrive at

$$f_m(n) = \sum_{k=1}^{\lfloor \frac{n-1}{\ell} \rfloor} \binom{n-\ell k-1}{k-1} m^{k-1}.$$

In this case, all words in $\mathscr{U}_{1,n}^m(f_0)$ for $n \geq 2\ell+1$ start with a 1 followed by at least ℓ zeros, and end with ℓ zeros. By removing these $2\ell+1$ common characters, we get words of length $n-2\ell-1$ such that every nonzero element not in the last position is followed by at least ℓ zeros. In conclusion, the number

$$f_m(n+2\ell+1) = \sum_{k=1}^{\lfloor \frac{n+2\ell}{\ell} \rfloor} \binom{n-\ell(k-2)}{k-1} m^{k-1}$$

counts the number of words of length n over the alphabet $\{0,1,\dots,m\}$ having at least ℓ zeros between any two nonzero elements.

For $\ell=1$ this example was discussed by Janjić [7, Cor. 24], and for $\ell=2$ and $m=1$, the sequence $f_1(n+3)$ gives the Narayana cows sequence, A000930 in Sloane [8]. For $\ell=2$ and $m=2,\dots,4$, we get the sequences A003229, A084386, A089977. Other values of ℓ and m lead to: A003269, A052942, A143454, A003520, A143447, A143455, A005708.

We finish the section with a couple of more elaborate examples.

Example 10. Let

$$f_0(j) = \begin{cases} 1, & \text{if } j=1 \text{ or } j=\ell+1; \\ 0, & \text{otherwise.} \end{cases}$$

The sequence $f_m(n+1) = \left|\mathscr{U}_{1,n+1}^m(f_0)\right|$ counts the number of words of length n over the alphabet $\{0,1,\dots,m\}$ containing zeros only in substrings of length ℓ. By Corollary 5,

$$f_m(n+1) = \sum_{k=1}^{n+1} \frac{k!}{(n+1)!} m^{k-1} B_{n+1,k}(1!,0,\dots,(\ell+1)!,0\dots).$$

By Identity 2 in the Appendix, we have

$$B_{n,k}(1!,0,\dots,(\ell+1)!,0\dots) = \binom{n}{n-k+\frac{n-k}{\ell}} \frac{(n-k+\frac{n-k}{\ell})!}{(\frac{n-k}{\ell})!} \quad \text{for } n-k \equiv 0 \pmod{\ell},$$

and therefore,

$$f_m(n+1) = \sum_{\substack{1\leq k\leq n+1 \\ n+1\equiv k \pmod{\ell}}} \frac{k!}{(n+1)!} m^{k-1} \binom{n+1}{n+1-k+\frac{n+1-k}{\ell}} \frac{(n+1-k+\frac{n+1-k}{\ell})!}{(\frac{n+1-k}{\ell})!}.$$

Writing $n+1 = q\ell+r$ with $0 \leq r < \ell$, we obtain

$$\begin{aligned} f_m(n+1) &= \sum_{\substack{1\leq k\leq n+1 \\ n+1\equiv k \pmod{\ell}}} \frac{k!}{(q\ell+r)!} m^{k-1} \binom{q\ell+r}{q\ell+r-k+\frac{q\ell+r-k}{\ell}} \frac{(q\ell+r-k+\frac{q\ell+r-k}{\ell})!}{(\frac{q\ell+r-k}{\ell})!} \\ &= \sum_{i=0}^{q} \frac{(\ell i+r)!}{(q\ell+r)!} m^{\ell i+r-1} \binom{q\ell+r}{(\ell+1)(q-i)} \frac{((\ell+1)(q-i))!}{(q-i)!} = \sum_{i=0}^{q} \binom{\ell i+r}{q-i} m^{\ell i+r-1}. \end{aligned}$$

In conclusion, the number of words of length n over the alphabet $\{0, 1, \ldots, m\}$ containing zeros only in substrings of length ℓ, is given by

$$f_m(n+1) = \sum_{i=0}^{\lfloor \frac{n+1}{\ell} \rfloor} \binom{n+1-\ell i}{i} m^{n-\ell i}.$$

For $\ell = 2$ and $m = 1, \ldots, 6$, this gives the sequences A068921, A255115, A255116, A255117, A255118, A255119 in Sloane [8]. And for $m = 1$ and $\ell = 3, \ldots, 7$, we get A003269, A003520, A005708, A005709, A005710.

Example 11. For $\ell \in \mathbb{N}$ let

$$f_0(j) = \begin{cases} 0, & \text{if } j = \ell + 1; \\ 1, & \text{otherwise.} \end{cases}$$

Then the sequence

$$f_m(n+1) = \sum_{k=1}^{n+1} \frac{k!}{(n+1)!} m^{k-1}\, B_{n+1,k}(1!, \ldots, \ell!, 0, (\ell+2)!, (\ell+3)!, \ldots)$$

counts the number of words of length n over the alphabet $\{0, 1, \ldots, m\}$ avoiding runs of exactly ℓ zeros. By means of Identity 3 in the Appendix, the partial Bell polynomials can be written as

$$\begin{aligned} &B_{n+1,k}(1!, \ldots, \ell!, 0, (\ell+2)!, (\ell+3)!, \ldots) \\ &\qquad = \frac{(n+1)!}{k!} \left(\sum_{\kappa<k} (-1)^\kappa \binom{k}{\kappa} \binom{n-(\ell+1)\kappa}{k-\kappa-1} + (-1)^k \delta_{n+1,(\ell+1)k} \right), \end{aligned}$$

where $\delta_{n+1,(\ell+1)k}$ is the Kronecker delta function. Thus

$$f_m(n+1) = \sum_{k=1}^{n+1} \left(\sum_{\kappa<k} (-1)^\kappa \binom{k}{\kappa} \binom{n-(\ell+1)\kappa}{k-\kappa-1} + (-1)^k \delta_{n+1,(\ell+1)k} \right) m^{k-1}.$$

If $n + 1 = (\ell + 1)q$, then

$$f_m(n+1) = (-1)^q m^{q-1} + \sum_{\kappa=0}^{q-1} \sum_{k=\kappa+1}^{n+1-\ell\kappa} (-1)^\kappa \binom{k}{\kappa} \binom{n-(\ell+1)\kappa}{k-\kappa-1} m^{k-1},$$

and if $n + 1 \not\equiv 0$ modulo $(\ell + 1)$, then

$$f_m(n+1) = \sum_{\kappa=0}^{\lfloor \frac{n+1}{\ell+1} \rfloor} \sum_{k=\kappa+1}^{n+1-\ell\kappa} (-1)^\kappa \binom{k}{\kappa} \binom{n-(\ell+1)\kappa}{k-\kappa-1} m^{k-1}.$$

For various values of ℓ and m, this gives the sequences A005251, A120925, A255813, A255814, A255815, A049856, A108758 in Sloane [8].

4 Further (nonbinary) examples

In [7, Corollary 37], Janjić discusses 01-avoiding words over a finite alphabet by considering the m-th invert transform of the sequence $f_0(j) = j$ for $j \in \mathbb{N}$. He verifies that f_m satisfies the same recurrence relation as the sequence counting such words, but there is no other indication of how this f_0 is combinatorially related to the given restriction.

In this section, we will use our results from Section 2 to shed some light on the counting of 01-avoiding words and will discuss examples that lead to the invert transform of the triangle numbers and other more general figurate numbers.

Example 12 (cf. [7, Cor. 37])**.** We start by considering words of length n over the alphabet $\{0, 1, 2\}$, starting with 2, that avoid the pattern 01. With the terminology from Section 2, this can be done with blocks in $W_b(j)$ with $b = 2$ and $j = 1, \ldots, n$. For example, for $j = 1, \ldots, 4$, we have the building blocks

$W_2(1)$	$W_2(2)$	$W_2(3)$	$W_2(4)$
2	20	200	2000
	21	*201*	*2001*
		210	*2010*
		211	*2011*
			2100
			2101
			2110
			2111

and the colored blocks are the ones we wish to avoid. Observe that, once these building blocks are removed, every word made by concatenation of the remaining blocks will be a word that starts with 2 and avoids the subword 01. Also note that the building blocks avoiding 01 are precisely those where the digits are in nonincreasing order.

Let U_j be the subset of $W_2(j)$ consisting of the j building blocks whose digits are in nonincreasing order. With this choice, the number of 01-avoiding words of length n, starting with 2, is equal to the number of elements in $\mathscr{U}_{2,n}(f_0)$ with f_0 given by $f_0(j) = j$ for $j = 1, 2, \ldots$, recall the definition before Theorem 3. By removing the starting digit 2 from each word in $\mathscr{U}_{2,n}(f_0)$, we get a natural bijection to the set of 01-avoiding words of length $n - 1$ over the alphabet $\{0, 1, 2\}$.

The number of such words is then counted by the invert transform of f_0, see Theorem 3. Moreover, the set $\mathscr{U}^m_{2,n}(f_0)$ for $m \geq 1$ contains all 01-avoiding words of length n, starting with 2, over the larger alphabet $\{0, 1, \ldots, m+1\}$. If $f_m(n) = |\mathscr{U}^m_{2,n}(f_0)|$, Corollary 5 together with Identity 4 with $r = 0$ (see Appendix) give

$$f_m(n) = \sum_{k=1}^{n} \frac{k!}{n!} m^{k-1} B_{n,k}(1! \cdot 1, 2! \cdot 2, 3! \cdot 3, \ldots) = \sum_{k=1}^{n} \binom{n+k-1}{n-k} m^{k-1} \text{ for } n > 1.$$

In conclusion, if $f_0(j) = j$ for $j \in \mathbb{N}$, then for $n > 1$ the number of 01-avoiding words of length $n-1$ over the alphabet $\{0, 1, \dots, m+1\}$ is given by

$$f_m(n) = \sum_{k=1}^{n} \binom{n+k-1}{n-k} m^{k-1}.$$

For $m = 1, \dots, 10$, this gives the sequences A001906, A001353, A004254, A001109, A004187, A001090, A018913, A004189, A004190, A004191 in Sloane [8].

Corollary 13. *It was noted in [7, Corollary 37] that the above sequence $f_m(n)$ coincides with $U_{n-1}(\frac{m+2}{2})$, where U_n is the n-th Chebyshev polynomial of the second kind. As a consequence, we obtain the identity:*

$$U_n(\tfrac{x+2}{2}) = \sum_{k=0}^{n} \binom{n+k+1}{n-k} x^k.$$

Example 14. Following the idea of the last example, we now let $b = 3$ and let U_j be the building blocks in $W_3(j)$ whose digits are in nonincreasing order. For instance, out of the 9 blocks in $W_3(3)$:

300 *301* *302* 310 311 *312* 320 321 322

there are 3 blocks that do not satisfy the restriction, so U_3 has only 6 elements. In general, for every $j \in \mathbb{N}$ there are $\binom{j+1}{2}$ such blocks, and they are precisely the building blocks that avoid the patterns 01, 02, and 12. Thus with $f_0(j) = \binom{j+1}{2}$ and U_j as above, the set $\mathscr{U}_{3,n}(f_0)$ consists of words of length n over the alphabet $\{0, 1, 2, 3\}$, starting with 3, and avoiding the patterns 01, 02, and 12. Moreover, as discussed in Section 2, the set $\mathscr{U}^m_{3,n}(f_0)$ for $m \geq 1$ consists of words over the alphabet $\{0, 1, \dots, m+2\}$, starting with 3, with the same restrictions as the words in $\mathscr{U}_{3,n}(f_0)$. Note that $\mathscr{U}^m_{3,n}(f_0)$ has the same number of elements as the set of words of length $n-1$ over $\{0, 1, \dots, m+2\}$, avoiding 01, 02, and 12.

Finally, by means of Corollary 5 and Identity 4 with $r = 1$ (see Appendix), we conclude that for $n > 1$ the sequence

$$f_m(n) = \sum_{k=1}^{n} \frac{k!}{n!} m^{k-1} B_{n,k}(1!\tbinom{2}{2}, 2!\tbinom{3}{2}, 3!\tbinom{4}{2}, \dots) = \sum_{k=1}^{n} \binom{n+2k-1}{n-k} m^{k-1}$$

gives the number of words of length $n-1$ over the alphabet $\{0, 1, \dots, m+2\}$, avoiding the patterns 01, 02, and 12.

For $m = 1, 2$, we get the sequences A052529 and A200676 in Sloane [8].

Remark 15. In each of the previous two examples, f_0 is of the form $f_0(j) = \binom{j+r}{r+1}$ for $r \geq 0$. Example 12 corresponds to $r = 0$ and Example 14 is the case when $r = 1$. Choosing $b = r+1$ for the construction of $\mathscr{U}^m_{b,n}(f_0)$, and choosing building blocks with a nonincreasing restriction on its digits, one gets words that avoid certain patterns.

Example 16. Let $r \in \mathbb{N}$ and $f_0(j) = \binom{j+r}{r+1}$ for $j \geq 1$. Then for $n > 1$, $f_m(n) = |\mathscr{U}^m_{r+1,n}(f_0)|$ counts the number of words of length $n-1$ over the alphabet $\{0, 1, \ldots, m+r+1\}$, avoiding all patterns of the form a_1a_2 with $a_1, a_2 \in \{0, \ldots, r+1\}$ and $a_1 < a_2$. Applying Corollary 5 together with Identity 4 from the Appendix, we then get

$$f_m(n) = \sum_{k=1}^{n} \binom{n + (r+1)k - 1}{n-k} m^{k-1}.$$

For $m = 1$ and $r = 2, \ldots, 5$, this gives the sequences A055991, A079675, A099242, A099253 in Sloane [8].

We finish this section by revisiting another example considered by Janjić [7, Cor. 41]. While our final formula does not seem to give information that could have not been obtained from the recurrence relation given in [7], the goal here is to provide more combinatorial insight and to illustrate the versatility of the building block approach introduced in Section 2.

Example 17 (cf. [7, Cor. 41])**.** For $q \in \mathbb{N}$ consider the set of ii-avoiding words of length n over the alphabet $\{0, 1, \ldots, q\}$ for $i \in \{0, 1, \ldots, q-1\}$. This set of words can be completely described with building blocks in $W_q(j)$ for $j = 1, \ldots, n+1$. The elements of $W_q(j)$ are strings of length j, starting with q, followed by $j-1$ letters from the alphabet $\{0, 1, \ldots, q-1\}$. Clearly, there is only one word in $W_q(1)$, and there are q words in $W_q(2)$, $q0, q1, \ldots, q(q-1)$, none of which is of the form ii. Moreover, for $j > 2$ there are exactly $q \cdot (q-1)^{j-2}$ ii-avoiding words in $W_q(j)$. Thus, if f_0 is defined by

$$f_0(j) = \begin{cases} 1, & \text{if } j = 1; \\ q(q-1)^{j-2}, & \text{if } j \geq 2; \end{cases}$$

then, by Theorem 3, the sequence $f_1(n+1) = \sum_{k=1}^{n+1} \frac{k!}{(n+1)!} B_{n+1,k}(1!f_0(1), 2!f_0(2), \ldots)$ gives the number of ii-avoiding words of length n over $\{0, 1, \ldots, q\}$ for $0 \leq i < q$.

As before, Corollary 5 gives a corresponding formula for the m-th invert transform, but before we work it out, let us simplify the partial Bell polynomial:

$$\begin{aligned} B_{n+1,k}(1!f_0(1), 2!f_0(2), \ldots) &= B_{n+1,k}(1!, 2!q, 3!q(q-1), 4!q(q-1)^2, \ldots) \\ &= \sum_{\ell=0}^{k} \frac{(n+1)!}{(n+1-k)!\ell!} B_{n+1-k,k-\ell}(q, 2!q(q-1), 3!q(q-1)^2, \ldots) \\ &= \sum_{\ell=0}^{k} \frac{(n+1)!}{(n+1-k)!\ell!} q^{k-\ell}(q-1)^{n+1+\ell-2k} B_{n+1-k,k-\ell}(1!, 2!, \ldots), \end{aligned}$$

which for $k \leq n$ implies

$$\frac{k!}{(n+1)!} B_{n+1,k}(1!f_0(1), 2!f_0(2), \ldots) = \sum_{\ell=0}^{k-1} \binom{k}{\ell} \binom{n-k}{k-\ell-1} q^{k-\ell}(q-1)^{n+1+\ell-2k}.$$

Finally, by Corollary 5, we conclude that the number of ii-avoiding words of length n over the alphabet $\{0, 1, \ldots, q+m-1\}$ for $i \in \{0, 1, \ldots, q-1\}$, is given by

$$f_m(n+1) = m^n + \sum_{k=1}^{n} \sum_{\ell=0}^{k-1} \binom{k}{\ell} \binom{n-k}{k-\ell-1} q^{k-\ell}(q-1)^{n+1+\ell-2k} m^{k-1}.$$

For various values of m and q, this formula leads to the sequences A000045, A001333, A055099, A123347, A180035 in Sloane [8].

5 Appendix: Bell polynomial identities

Identity 1. For $\ell \in \mathbb{N}$ we have

$$B_{n,k}(1!, 2!, \ldots, \ell!, 0, \ldots) = \frac{n!}{k!} \sum_{j=0}^{\lfloor \frac{n-k}{\ell} \rfloor} (-1)^j \binom{k}{j} \binom{n-\ell j-1}{k-1}.$$

A proof of this identity can be found in [1, Theorem 4.1 combined with Theorem 2.1].

Identity 2. For $\ell \in \mathbb{N}$ with $\ell > 1$, we have

$$B_{n,k}(1!, 0, \ldots, (\ell+1)!, 0 \ldots) = \binom{n}{n-k+\frac{n-k}{\ell}} \frac{(n-k+\frac{n-k}{\ell})!}{(\frac{n-k}{\ell})!}$$

whenever $n-k \equiv 0 \pmod{\ell}$.

Proof. Using [6, Eqs. (3n) and (3n'), Section 3.3], we have

$$\begin{aligned}
B_{n,k}(1!, 0, \ldots, (\ell+1)!, 0 \ldots) &= \sum_{\kappa=0}^{k} \sum_{\nu=\kappa}^{n} \binom{n}{\nu} B_{\nu,\kappa}(0 \ldots, (\ell+1)!, 0 \ldots) B_{n-\nu,k-\kappa}(1, 0, 0, \ldots) \\
&= \sum_{\kappa=0}^{k} \binom{n}{n-k+\kappa} B_{n-k+\kappa,\kappa}(0 \ldots, (\ell+1)!, 0 \ldots) \\
&= \binom{n}{n-k+\frac{n-k}{\ell}} \frac{(n-k+\frac{n-k}{\ell})!}{(\frac{n-k}{\ell})!} \quad \text{for } n-k \equiv 0 \pmod{\ell}.
\end{aligned}$$

□

Identity 3. For $\ell \in \mathbb{N}$,

$$\begin{aligned}
&B_{n,k}(1!, \ldots, \ell!, 0, (\ell+2)!, (\ell+3)!, \ldots) \\
&\qquad = \frac{n!}{k!} \left(\sum_{\kappa=0}^{k-1} (-1)^\kappa \binom{k}{\kappa} \binom{n-(\ell+1)\kappa-1}{k-\kappa-1} + (-1)^k \delta_{n,(\ell+1)k} \right),
\end{aligned}$$

where $\delta_{n,(\ell+1)k}$ is the Kronecker delta function.

Proof. Let $\bar{x}_\ell$, $\bar{e}_\ell$, and $\bar{x}$ be defined as

$$\bar{x}_\ell = (1!, \dots, \ell!, 0, (\ell+2)!, \dots), \quad \bar{e}_\ell = (0, \dots, 0, (\ell+1)!, 0, \dots), \text{ and } \bar{x} = \bar{e}_\ell + \bar{x}_\ell.$$

By means of [6, Eqs. (3n) and (3n'), Section 3.3], we then have

$$\begin{aligned}
B_{n,k}(\bar{x}_\ell) &= \sum_{\kappa \le k,\, \nu \le n} \binom{n}{\nu} B_{\nu,\kappa}(-\bar{e}_\ell) B_{n-\nu,k-\kappa}(\bar{x}) \\
&= \sum_{\kappa \le k} \binom{n}{(\ell+1)\kappa} \frac{((\ell+1)\kappa)!}{\kappa!} (-1)^\kappa B_{n-(\ell+1)\kappa,k-\kappa}(\bar{x}) \\
&= \frac{n!}{k!} \sum_{\kappa=0}^{k-1} (-1)^\kappa \binom{k}{\kappa} \binom{n-(\ell+1)\kappa-1}{k-\kappa-1} + (-1)^k \frac{n!}{k!(n-(\ell+1)k)!} B_{n-(\ell+1)\kappa,0}(\bar{x}) \\
&= \frac{n!}{k!} \left(\sum_{\kappa=0}^{k-1} (-1)^\kappa \binom{k}{\kappa} \binom{n-(\ell+1)\kappa-1}{k-\kappa-1} + (-1)^k \delta_{n,(\ell+1)k} \right).
\end{aligned}$$

□

Identity 4. For $r \ge 0$ let $t_j(r) = \binom{j+r}{r+1}$. Then

$$B_{n,k}(1!t_1(r), 2!t_2(r), \cdots) = \frac{n!}{k!} \binom{n+(r+1)k-1}{n-k}.$$

Proof. For $k = 1$ and all n we have

$$B_{n,1}(1!t_1(r), 2!t_2(r), \cdots) = \frac{n!}{1!} \binom{n+r}{r+1} = \frac{n!}{1!} \binom{n+r}{n-1}.$$

Now we proceed by induction on k. For $k > 1$,

$$\begin{aligned}
B_{n,k}(1!t_1(r), 2!t_2(r), \cdots) &= \frac{1}{k} \sum_{j=0}^{n} \binom{n}{j} j! \binom{j+r}{r+1} B_{n-j,k-1}(1!t_1(r), 2!t_2(r), 3!t_3(r), \cdots) \\
&= \frac{1}{k} \sum_{j=1}^{n} \binom{n}{j} j! \binom{j+r}{r+1} \frac{(n-j)!}{(k-1)!} \binom{n-j+(r+1)(k-1)-1}{n-j-k+1} \\
&= \frac{n!}{k!} \sum_{j=1}^{n-k+1} \binom{j+r}{j-1} \binom{n-j+(r+1)(k-1)-1}{n-j-k+1} \\
&= \frac{n!}{k!} \sum_{j=0}^{n-k} \binom{j+r+1}{j} \binom{n-j+(r+1)(k-1)-2}{n-k-j} \\
&= \frac{n!}{k!} \sum_{j=0}^{n-k} (-1)^{n-k} \binom{-r-2}{j} \binom{-(r+1)(k-1)-k+1}{n-k-j}.
\end{aligned}$$

Using Chu-Vandermonde's identity, we arrive at

$$B_{n,k}(1!t_1(r), 2!t_2(r), \cdots) = (-1)^{n-k}\frac{n!}{k!}\binom{-(r+1)k-k}{n-k} = \frac{n!}{k!}\binom{n+(r+1)k-1}{n-k}.$$

□

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