The Gaussian Coefficient Revisited

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Abstract

We give a new $q$-$(1 + q)$-analogue of the Gaussian coefficient, also known as the $q$-binomial which, like the original $q$-binomial $\binom{n}{k}_q$, is symmetric in $k$ and $n - k$. We show this $q$-$\binom{n}{k}_q$ is more compact than the one discovered by Fu, Reiner, Stanton, and Thiem. Underlying our $q$-$\binom{n}{k}_q$ is a Boolean algebra decomposition of an associated poset. These ideas are extended to the Birkhoff transform of any finite poset. We end with a discussion of higher analogues of the $q$-binomial.

1 Introduction

Inspired by work of Fu, Reiner, Stanton, and Thiem [2], Cai and Readdy [1] asked the following question. Given a combinatorial $q$-analogue

$$X(q) = \sum_{w \in X} q^{a(w)},$$

where $X$ is a set of objects and $a(\cdot)$ is a statistic defined on the elements of $X$, when can one find a smaller set $Y$ and two statistics $s$ and $t$ such that

$$X(q) = \sum_{w \in Y} q^{s(w)} \cdot (1 + q)^{t(w)}.$$
Such an interpretation is called an $q$-$\binom{1+q}{k}$-analogue. Examples of $q$-$\binom{1+q}{k}$-analogues have been determined for the $q$-binomial by Fu et al. [2], and for the $q$-Stirling numbers of the first and second kinds by Cai and Readdy [1], who also gave poset and homotopy interpretations of their $q$-$\binom{1+q}{k}$-analogues.

In 1916 MacMahon [3, 4, 5] observed that the Gaussian coefficient, also known as the $q$-binomial coefficient, is given by

$$\binom{n}{k}_q = \sum_{w \in \Omega_{n,k}} q^{\text{inv}(w)}.$$ 

Here $\Omega_{n,k} = \mathcal{S}(0^{n-k}, 1^k)$ denotes all permutations of the multiset $\{0^{n-k}, 1^k\}$, that is, all words $w = w_1 \cdots w_n$ of length $n$ with $n-k$ zeroes and $k$ ones, and $\text{inv}(\cdot)$ denotes the inversion statistic defined by $\text{inv}(w_1w_2 \cdots w_n) = \{ (i, j) : 1 \leq i < j \leq n, w_i > w_j \}$. Fu et al. defined a subset $\Omega'_{n,k} \subseteq \Omega_{n,k}$ and two statistics $a$ and $b$ such that

$$\binom{n}{k}_q = \sum_{w \in \Omega'_{n,k}} q^{a(w)} \cdot (1 + q)^{b(w)}.$$ 

In this paper we will return to the original study by Fu et al. of the Gaussian coefficient. We discover a more compact $q$-$\binom{1+q}{k}$-analogue which, like the original Gaussian coefficient, is also symmetric in the variables $k$ and $n - k$. See Corollary 6 and Theorem 12. This symmetry was missing in Fu et al.’s original $q$-$\binom{1+q}{k}$-analogue. We give a Boolean algebra decomposition of the related poset $\Omega_{n,k}$. Since this poset is a distributive lattice, in the last section we extend these ideas to poset decompositions of any distributive lattice and other analogues.

## 2 A poset interpretation

In this section we consider the poset structure on 0-1-words in $\Omega_{n,k}$. For further poset terminology and background, we refer the reader to [6].

We begin by making the set of elements $\Omega_{n,k}$ into a graded poset by defining the cover relation to be

$$u \circ 01 \circ v < u \circ 10 \circ v,$$

where $\circ$ denotes concatenation of words. The word $0^{n-k}1^k$ is the minimal element and the word $1^k0^{n-k}$ is the maximal element in the poset $\Omega_{n,k}$. Furthermore, this poset is graded by the inversion statistic. This poset is simply the interval $[\hat{0}, x]$ of Young’s lattice, where the minimal element $\hat{0}$ is the empty Ferrers diagram and $x$ is the Ferrers diagram consisting of $n - k$ columns and $k$ rows.

An alternative description of the poset $\Omega_{n,k}$ is that it is isomorphic to the Birkhoff transform of the Cartesian product of two chains. Let $C_m$ denote the $m$-element chain. The poset $\Omega_{n,k}$ is isomorphic to the distributive lattice of all lower order ideals of the product $C_{n-k} \times C_k$, usually denoted by $J(C_{n-k} \times C_k)$. 

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Definition 1. Let $\Omega''_{n,k}$ consist of all 0,1-words $v = v_1 v_2 \cdots v_n$ in $\Omega_{n,k}$ such that
$$v_1 \leq v_2, \ v_3 \leq v_4, \ \ldots, \ v_{2\lfloor n/2 \rfloor - 1} \leq v_{2\lfloor n/2 \rfloor}.$$

Observe that when $n$ is odd there is no condition on the last entry $w_n$. Define two maps $\phi$ and $\psi$ on $\Omega_{n,k}$ by sending the word $w = w_1 w_2 \cdots w_n$ to
$$\phi(w) = \min(w_1, w_2), \max(w_1, w_2), \min(w_3, w_4), \max(w_3, w_4), \ldots,$$
$$\psi(w) = \max(w_1, w_2), \min(w_1, w_2), \max(w_3, w_4), \min(w_3, w_4), \ldots.$$

The map $\phi$ sorts the entries in positions 1 and 2, 3 and 4, and so on. If $n$ is odd, the entry $w_n$ remains in the same position. Similarly, the map $\psi$ sorts in reverse order each pair of positions. Note that the map $\phi$ maps $\Omega_{n,k}$ surjectively onto the set $\Omega''_{n,k}$.

We have the following Boolean algebra decomposition of the poset $\Omega_{n,k}$.

**Theorem 2.** The distributive lattice $\Omega_{n,k}$ has the Boolean algebra decomposition
$$\Omega_{n,k} = \bigcup_{v \in \Omega''_{n,k}} [v, \psi(v)].$$

**Proof.** Observe that the maps $\phi$ and $\psi$ satisfy the inequalities $\phi(w) \leq w \leq \psi(w)$. Furthermore, the fiber of the map $\phi : \Omega_{n,k} \to \Omega''_{n,k}$ is isomorphic to a Boolean algebra, that is, $\phi^{-1}(v) \cong [v, \psi(v)]$. \hfill \Box

For $v \in \Omega''_{n,k}$ define the statistic
$$\text{asc}_{\text{odd}}(v) = |\{i : v_i < v_{i+1}, i \text{ odd}\}|,$$
that is, $\text{asc}_{\text{odd}}(\cdot)$ enumerates the number of ascents in odd positions.

**Corollary 3.** The $q$-binomial is given by
$$\left[ \begin{array}{c} n \\ k \end{array} \right]_q = \sum_{v \in \Omega''_{n,k}} q^{\text{inv}(v)} \cdot (1 + q)^{\text{asc}_{\text{odd}}(v)}. \quad (1)$$

**Proof.** It is enough to observe that the sum of the inversion statistic over the elements in the fiber $\phi^{-1}(v) = [v, \psi(v)]$ for $v \in \Omega''_{n,k}$ is given by $q^{\text{inv}(v)} \cdot (1 + q)^{\text{asc}_{\text{odd}}(v)}$. \hfill \Box

A geometric way to understand this $q$-(1 + $q$)-interpretation is to consider lattice paths from the origin $(0, 0)$ to $(n - k, k)$ which only use east steps $(1, 0)$ and north steps $(0, 1)$. Color the squares of this $(n - k) \times k$ board as a chessboard, where the square incident to the origin is colored white. The map $\phi$ in the proof of Theorem 2 corresponds to taking a lattice path where every time there is a north step followed by an east step that turns around a white square, we exchange these two steps. The statistic $\text{asc}_{\text{odd}}$ enumerates the number of times an east step is followed by a north step when this pair of steps borders a white square.

Let $\text{er}(n, k)$ denote the cardinality of the set $\Omega''_{n,k}$. Then we have
Proposition 4. The cardinalities \( er(n, k) \) satisfy the recursion
\[
er(n, k) = er(n - 2, k - 2) + er(n - 2, k - 1) + er(n - 2, k) \quad \text{for } 0 \leq k \leq n \text{ and } n \geq 2,
\]
with the boundary conditions \( er(0, 0) = er(1, 0) = er(1, 1) = 1 \) and \( er(n, k) = 0 \) whenever \( k > n, k < 0 \) or \( n < 0 \).

Proof. A word in \( \Omega''_{n,k} \) begins with either 00, 01 or 11, yielding the three cases of the recursion. \( \square \)

Directly we obtain the generating polynomial.

Theorem 5. The generating polynomial for \( er(n, k) \) is given by
\[
\sum_{k=0}^{n} er(n, k) \cdot x^k = (1 + x + x^2)^{\lfloor n/2 \rfloor} \cdot (1 + x)^{n-2} \cdot \lfloor n/2 \rfloor.
\]

We end with a statement concerning the symmetry of the \( q-(1 + q) \)-binomial.

Corollary 6. The set of defining elements for the \( q-(1 + q) \)-binomial satisfy the following symmetric relation:
\[
|\Omega'_{n,k}| = |\Omega'_{n,n-k}|.
\]

Proof. This follows from the fact that the generating polynomial for \( er(n, k) \) is a product of palindromic polynomials, and thus is itself a palindromic polynomial. \( \square \)

3 Analysis of the Fu–Reiner–Stanton–Thiem interpretation

A weak partition is a finite non-decreasing sequence of non-negative integers. A weak partition \( \lambda = (\lambda_1, \ldots, \lambda_{n-k}) \) with \( n - k \) parts and each part at most \( k \) where \( \lambda_1 \leq \cdots \leq \lambda_{n-k} \) corresponds to a Ferrers diagram lying inside an \((n-k) \times k\) rectangle with column \( i \) having height \( \lambda_i \). These weak partitions are in direct correspondence with the set \( \Omega_{n,k} \).

Fu, Reiner, Stanton, and Thiem used a pairing algorithm to determine a subset \( \Omega'_{n,k} \subseteq \Omega_{n,k} \) of 0-1-sequences to define their \( q-(1 + q) \)-analogue of the \( q \)-binomial; see [2, Proposition 6.1]. This translates into the following statement. The set \( \Omega'_{n,k} \) is in bijection with weak partitions into \( n - k \) parts with each part at most \( k \) such that

(a) if \( k \) is even, each odd part has even multiplicity,

(b) if \( k \) is odd, each even part (including 0) has even multiplicity.

Definition 7. Let \( \text{frst}(n, k) \) be the cardinality of the set \( \Omega'_{n,k} \).
Lemma 8. The quantity $\text{frst}(n, k)$ counts the number of weak partitions into $n - k$ parts where each part is at most $k$ and each odd part has even multiplicity.

Proof. When $k$ is even there is nothing to prove. When $k$ is odd, by considering the complement of weak partitions with respect to the rectangle of size $(n - k) \times k$, we obtain a bijective proof. The same complement proof also shows the case when $k$ is even holds.

Theorem 9. The first-coefficients satisfy the recursion

$$\text{frst}(n, k) =\begin{cases} \text{frst}(n - 1, k - 1) + \text{frst}(n - 1, k), & \text{if } k \text{ is even;} \\ \text{frst}(n - 2, k - 2) + \text{frst}(n - 2, k - 1) + \text{frst}(n - 2, k), & \text{if } k \text{ is odd;} \end{cases}$$

where $0 \leq k \leq n$ and $n \geq 2$ with the boundary conditions $\text{frst}(0, 0) = \text{frst}(1, 0) = \text{frst}(1, 1) = 1$ and $\text{frst}(n, k) = 0$ whenever $k > n$, $k < 0$ or $n < 0$.

Proof. We use the characterization in Lemma 8. When $k$ is even there are two cases. If the last part is $k$, remove it to obtain a weak partition counted by $\text{frst}(n - 1, k)$. If the last part is less than $k$, then the weak partition is counted by $\text{frst}(n - 1, k - 1)$.

When $k$ is odd there are three cases. If the last two parts are equal to $k$, then removing these two parts yields a weak partition counted by $\text{frst}(n - 2, k)$. Note that we cannot have the last part equal to $k$ and the next to last part less than $k$ since $k$ is odd. If the last part is equal to $k - 1$, we can remove it to obtain a weak partition counted by $\text{frst}(n - 2, k - 1)$. Finally, if the last part is less than or equal to $k - 2$, the weak partition is counted by $\text{frst}(n - 2, k - 2)$.

Remark 10. For $k$ odd we have the shorter recursion $\text{frst}(n, k) = \text{frst}(n - 1, k - 1) + \text{frst}(n - 2, k)$. However, we use the longer recursion in the proof of Theorem 12.

Lemma 11. The inequality $\text{frst}(n, k) \leq \text{frst}(n + 1, k + 1)$ holds.

Proof. The weak partitions which lie inside the rectangle $(n - k) \times k$ and satisfy the conditions of Lemma 8 are included among the weak partitions which lie inside the larger rectangle $(n - k) \times (k + 1)$ and satisfy the same conditions.

Theorem 12. For all $0 \leq k \leq n$ the inequality $|\Omega''_{n,k}| = \text{er}(n, k) \leq \text{frst}(n, k) = |\Omega'_{n,k}|$ holds.

Proof. We proceed by induction on $n$. The induction base is $n \leq 3$. Furthermore, the inequality holds when $k$ is $0, 1, n - 1$ and $n$. When $k$ is odd we have that

$$\text{er}(n, k) = \text{er}(n - 2, k - 2) + \text{er}(n - 2, k - 1) + \text{er}(n - 2, k) \leq \text{frst}(n - 2, k - 2) + \text{frst}(n - 2, k - 1) + \text{frst}(n - 2, k) = \text{frst}(n, k).$$
Similarly, when \( k \) is even we have

\[
\begin{align*}
\text{er}(n, k) &= \text{er}(n - 2, k - 2) + \text{er}(n - 2, k - 1) + \text{er}(n - 2, k) \\
&\leq \text{frst}(n - 2, k - 2) + \text{frst}(n - 2, k - 1) + \text{frst}(n - 2, k) \\
&\leq \text{frst}(n - 1, k - 1) + \text{frst}(n - 2, k - 1) + \text{frst}(n - 2, k) \\
&= \text{frst}(n - 1, k - 1) + \text{frst}(n - 1, k) \\
&= \text{frst}(n, k),
\end{align*}
\]

where the second inequality follows from Lemma 11. These two cases complete the induction hypothesis.

See Table 1 to compare the values of \( \text{frst}(n, k) \) and \( \text{er}(n, k) \) for \( n \leq 10 \).

### 4 Concluding remarks

Is it possible to find a \( q-(1 + q) \)-analogue of the Gaussian coefficient which has the smallest possible index set? We believe that our analogue is the smallest, but cannot offer a proof of a minimality. Perhaps a more tractable question is to prove that the Boolean algebra decomposition of \( \Omega_{n,k} \) is minimal.

We can extend these ideas involving a Boolean algebra decomposition to any distributive lattice. Let \( P \) be a finite poset and let \( A \) be an antichain of \( P \) such that there is no cover relation in \( A \), that is, there is no pair of elements \( u, v \in A \) such that \( u \prec v \). We obtain a Boolean algebra decomposition of the Birkhoff transform \( J(P) \) by defining

\[
J''(P) = \{ I \in J(P) : \text{the ideal } I \text{ has no maximal elements in the antichain } A \}.
\]
The two maps $\phi$ and $\psi$ are now defined as
\[
\phi(I) = I - \{ a \in A : \text{the element } a \text{ is maximal in } I \},
\]
\[
\psi(I) = I \cup \{ a \in A : I \cup \{ a \} \in J(P) \}.
\]

We have the following decomposition theorem.

**Theorem 13.** For $P$ any finite poset the distributive lattice $J(P)$ has the Boolean algebra decomposition

\[
J(P) = \bigcup_{I \in J'(P)} [I, \psi(I)].
\]

Yet again, how can we select the antichain $A$ such that the above decomposition $A$ has the fewest possible terms? Furthermore, would this give the smallest Boolean algebra decomposition?

Another way to extend the ideas of Theorem 2 is as follows. Define $\Omega_{n,k}^r$ to be the set of all words $v \in \Omega_{n,k}$ satisfying the inequalities
\[
v_1 \leq v_2 \leq \cdots \leq v_r,
\]
\[
v_{r+1} \leq v_{r+2} \leq \cdots \leq v_{2r},
\]
\[
\ldots,
\]
\[
v_{r \cdot \lfloor n/r \rfloor - r+1} \leq v_{r \cdot \lfloor n/r \rfloor - r+2} \leq \cdots \leq v_{r \cdot \lfloor n/r \rfloor}.
\]

For $1 \leq i \leq \lfloor r/2 \rfloor$ define the statistics $b_i(v)$ for $v \in \Omega_{n,k}^r$ to be
\[
b_i(v) = |\{ j \in [\lfloor n/r \rfloor] : v_{rj-r+1} + v_{rj-r+2} + \cdots + v_{rj} \in \{ i, r-i \} \}|.
\]

**Theorem 14.** The distributive lattice $\Omega_{n,k}$ has the decomposition

\[
\Omega_{n,k} = \bigcup_{v \in \Omega_{n,k}^r} \Omega_{r,1}^{b_1(v)} \times \Omega_{r,2}^{b_2(v)} \times \cdots \times \Omega_{r/2}^{b_{r/2}(v)}.
\]

**Corollary 15.** The $q$-binomial is given by
\[
\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{v \in \Omega_{n,k}^r} q^{\text{inv}(v)} \cdot \begin{bmatrix} r \\ 1 \end{bmatrix}_q^{b_1(v)} \cdot \begin{bmatrix} r \\ 2 \end{bmatrix}_q^{b_2(v)} \cdots \begin{bmatrix} r \\ r/2 \end{bmatrix}_q^{b_{r/2}(v)}.
\]

The least complicated case is when $r = 3$, where only one term appears in the above poset product. This term is $\Omega_{3,1}$ which is the three element chain $C_3$. The associated Gaussian coefficient is $1 + q + q^2$. Thus Corollary 15 could be called a $q$-$(1 + q + q^2)$-analogue in the case of $r = 3$. As an example, we have
\[
\begin{bmatrix} 6 \\ 3 \end{bmatrix}_q = 1 + q \cdot (1 + q + q^2)^2 + q^4 \cdot (1 + q + q^2)^2 + q^9.
\]

On a poset level this is a decomposition of $J(C_3 \times C_3)$ into two one-element posets of rank 0 and rank 9, and two copies of $C_3 \times C_3$, where one has its minimal element of rank 1 and the other of rank 4.
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References

[1] Y. Cai and M. Readdy, \(q\)-Stirling numbers: A new view, preprint, \\


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