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Counting Non-Standard Binary Representations

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Abstract

Let $\mathcal{A}$ be a finite subset of $\mathbb{N}$ including 0 and let $f_{\mathcal{A}}(n)$ be the number of ways to write $n = \sum_{i=0}^{\infty} \epsilon_i 2^i$, where $\epsilon_i \in \mathcal{A}$. We consider asymptotics of the summatory function $s_{\mathcal{A}}(r,m)$ of $f_{\mathcal{A}}(n)$ from $m2^r$ to $m2^{r+1} - 1$, and show that $s_{\mathcal{A}}(r,m) \sim c(\mathcal{A},m)\,|\mathcal{A}|^r$ for some nonzero $c(\mathcal{A},m) \in \mathbb{Q}$.

1 Introduction

Let $f_{\mathcal{A}}(n)$ denote the number of ways to write $n = \sum_{i=0}^{\infty} \epsilon_i 2^i$, where ϵ_i belongs to the set

$$\mathcal{A} := \{0 = a_0, a_1, \ldots, a_z\},$$

with $a_i \in \mathbb{N}$ and $a_i < a_{i+1}$ for all $0 \le i \le z-1$. For more on this topic, see the author's previous work [1]. We parameterize $\mathcal{A}$ in terms of its s even elements and $(z+1) - s := t$ odd elements as follows:

$$\mathcal{A} = \{0 = 2b_1, 2b_2, \ldots, 2b_s, 2c_1 + 1, \ldots, 2c_t + 1\}.$$

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If n is even, then $\epsilon_0 = 0, 2b_2, 2b_3, \ldots,$ or $2b_s$ and

$$f_{\mathcal{A}}(n) = f_{\mathcal{A}}(n/2) + f_{\mathcal{A}}((n-2b_2)/2) + f_{\mathcal{A}}((n-2b_3)/2) + \cdots + f_{\mathcal{A}}((n-2b_s)/2).$$

Writing $n = 2\ell$, we have

$$f_{\mathcal{A}}(2\ell) = f_{\mathcal{A}}(\ell) + f_{\mathcal{A}}(\ell - b_2) + f_{\mathcal{A}}(\ell - b_3) + \cdots + f_{\mathcal{A}}(\ell - b_s),$$

so for any even n, $f_{\mathcal{A}}(n)$ satisfies a recurrence relation of order b_s.

Similarly, if $n = 2\ell + 1$ is odd, then $\epsilon_0 = 2c_1 + 1, 2c_2 + 1, \ldots,$ or $2c_t + 1$, and

$$f_{\mathcal{A}}(2\ell + 1) = f_{\mathcal{A}}(\ell - c_1) + f_{\mathcal{A}}(\ell - c_2) + \cdots + f_{\mathcal{A}}(\ell - c_t),$$

so for any odd n, $f_{\mathcal{A}}(n)$ satisfies a recurrence relation of order c_t. Dennison, Lansing, Reznick, and the author [3] gave this argument for $f_{\mathcal{A},b}(n)$, the b-ary representation of n with coefficients from $\mathcal{A}$, using residue classes mod b.

Example 1. Let $\mathcal{A} = \{0, 1, 3, 4\}$. We can write $\mathcal{A} = \{2(0), 2(0) + 1, 2(1) + 1, 2(2)\}$. Then

$$f_{\mathcal{A}}(2\ell) = f_{\mathcal{A}}(\ell) + f_{\mathcal{A}}(\ell - 2) \quad \text{and} \quad f_{\mathcal{A}}(2\ell + 1) = f_{\mathcal{A}}(\ell) + f_{\mathcal{A}}(\ell - 1). \tag{1}$$

In general, let

$$\omega_k(m) = \begin{pmatrix} f_{\mathcal{A}}(2^k m) \\ f_{\mathcal{A}}(2^k m - 1) \\ \vdots \\ f_{\mathcal{A}}(2^k m - a_z) \end{pmatrix}.$$

We shall consider the fixed $(a_z + 1) \times (a_z + 1)$ matrix $M_{\mathcal{A}}$ such that for any $k \geq 0$,

$$\omega_{k+1} = M_{\mathcal{A}} \omega_k.$$

Example 2. Returning to the set $\mathcal{A} = \{0, 1, 3, 4\}$ of Example 1 and using the equations in (1), we have

$$\omega_{k+1}(m) = \begin{pmatrix} f_{\mathcal{A}}(2^{k+1}m) \\ f_{\mathcal{A}}(2^{k+1}m - 1) \\ f_{\mathcal{A}}(2^{k+1}m - 2) \\ f_{\mathcal{A}}(2^{k+1}m - 3) \\ f_{\mathcal{A}}(2^{k+1}m - 4) \end{pmatrix} = \begin{pmatrix} f_{\mathcal{A}}(2^k m) + f_{\mathcal{A}}(2^k m - 2) \\ f_{\mathcal{A}}(2^k m - 1) + f_{\mathcal{A}}(2^k m - 2) \\ f_{\mathcal{A}}(2^k m - 1) + f_{\mathcal{A}}(2^k m - 3) \\ f_{\mathcal{A}}(2^k m - 2) + f_{\mathcal{A}}(2^k m - 3) \\ f_{\mathcal{A}}(2^k m - 2) + f_{\mathcal{A}}(2^k m - 4) \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_{\mathcal{A}}(2^k m) \\ f_{\mathcal{A}}(2^k m - 1) \\ f_{\mathcal{A}}(2^k m - 2) \\ f_{\mathcal{A}}(2^k m - 3) \\ f_{\mathcal{A}}(2^k m - 4) \end{pmatrix} \tag{2}$$

If $M_{\mathcal{A}}$ is the matrix in (2), then $\omega_{k+1}(m) = M_{\mathcal{A}} \omega_k(m)$.

We now review some basic concepts of sequences from Section 8.1 of Lidl and Niederreiter [5] and include a matrix view of recurrence relations, following Reznick [6].

Consider a sequence $(b(n))$ such that

$$b(n) + c_{k-1}b(n-1) + c_{k-2}b(n-2) + \cdots + c_0 b(n-k) = 0 \tag{3}$$

for all $n \geq k$ and $c_i \in \mathbb{N}$. By shifting the sequence, we see that

$$b(n+k) + c_{k-1}b(n+k-1) + c_{k-2}b(n+k-2) + \cdots + c_0 b(n+k-k) = 0 \tag{4}$$

for $n \geq 0$. Then (3) is a *homogeneous k-th order linear recurrence relation*, and $(b(n))$ is a *homogeneous k-th order linear recurrence sequence*. For any sequence $(b(n))$ satisfying (3) we define the *characteristic polynomial*

$$f(x) = x^k + c_{k-1}x^{k-1} + c_{k-2}x^{k-2} + \cdots + c_0. \tag{5}$$

We can also consider a recurrence relation from the point of view of a matrix system, considering k sequences indexed as $(b_i(n))$ for $1 \leq i \leq k$ which satisfy

$$b_i(n+1) = \sum_{j=1}^{k} m_{ij} b_j(n)$$

for $n \geq 0$ and $1 \leq i \leq k$. Then

$$\begin{pmatrix} b_1(n+1) \\ \vdots \\ b_k(n+1) \end{pmatrix} = \begin{pmatrix} m_{11} & \cdots & m_{1k} \\ \vdots & & \vdots \\ m_{k1} & \cdots & m_{kk} \end{pmatrix} \begin{pmatrix} b_1(n) \\ \vdots \\ b_k(n) \end{pmatrix}$$

for $n \geq 0$. To simplify the notation, if $M = [m_{ij}]$ and

$$\mathbf{B}(n) = \begin{pmatrix} b_1(n) \\ \vdots \\ b_k(n) \end{pmatrix},$$

then $\mathbf{B}(n+1) = M\mathbf{B}(n)$ for $n \geq 0$. Thus $\mathbf{B}(n) = M^n\mathbf{B}(0)$ for $n \geq 0$, where

$$\mathbf{B}(0) = \begin{pmatrix} b_1(0) \\ \vdots \\ b_k(0) \end{pmatrix}$$

is the vector of initial conditions.

As an additional connection between these two views of linear recurrence sequences, note that for a sequence satisfying (3),

$$\begin{pmatrix} b(n+1) \\ b(n+2) \\ \vdots \\ b(n+k-1) \\ b(n+k) \end{pmatrix} = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -c_0 & -c_1 & \cdots & -c_{k-2} & -c_{k-1} \end{pmatrix} \begin{pmatrix} b(n) \\ b(n+1) \\ \vdots \\ b(n+k-2) \\ b(n+k-1) \end{pmatrix},$$

where this matrix, the *companion matrix* to g, has characteristic polynomial $(-1)^k g$.

In this matrix point of view, the *characteristic polynomial* of M is

$$g(\lambda) := \det(M - \lambda I_k).$$

By the Cayley-Hamilton Theorem, $g(M) = \mathbf{0}$, the $k \times k$ zero matrix.

If $g(x)$ is the characteristic polynomial in (5), then

$$\mathbf{0} = g(M) = M^k + c_{k-1}M^{k-1} + c_{k-2}M^{k-2} + \cdots + c_0 I_k.$$

Hence for any $n \geq 0$,

$$\mathbf{0} = M^{n+k} + c_{k-1}M^{n+k-1} + c_{k-2}M^{n+k-2} + \cdots + c_0 M^n$$

and thus

$$\begin{aligned}\mathbf{0} &= \left(M^{n+k} + c_{k-1}M^{n+k-1} + c_{k-2}M^{n+k-2} + \cdots + c_0 M^n\right)\mathbf{B}(0)\\ &= \mathbf{B}(n+k) + c_{k-1}\mathbf{B}(n+k-1) + c_{k-2}\mathbf{B}(n+k-2) + \cdots + c_0\mathbf{B}(n).\end{aligned}$$

Thus each sequence $(b_j(n))$ satisfies the original linear recurrence (4).

2 Main result

We will use the ideas of Section 1 to examine the asymptotic behavior of the summatory function $\sum_{n=m2^r}^{m2^{r+1}-1} f_{\mathcal{A}}(n)$, but we must first establish a lemma.

Lemma 3 ([4, 5.6.5 & 5.6.9])**.** *Let $M = [m_{ij}]$ be an $n \times n$ matrix with characteristic polynomial $g(\lambda)$ and eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_y$. Then*

$$\max_{1 \leq i \leq y} |\lambda_i| \leq \max_{1 \leq i \leq n} \sum_{j=1}^{n} |m_{ij}|.$$

Theorem 4. *Let $\mathcal{A}, f_{\mathcal{A}}(n), M_{\mathcal{A}}$, and $\omega_k(m)$ be as above, with the additional assumption that there exists some odd $a_i \in \mathcal{A}$. Define*

$$s_{\mathcal{A}}(r, m) = \sum_{n=m2^r}^{m2^{r+1}-1} f_{\mathcal{A}}(n).$$

Let $|\mathcal{A}|$ denote the number of elements in the set $\mathcal{A}$. Then for a fixed value of m,

$$\lim_{r \to \infty} \frac{s_{\mathcal{A}}(r, m)}{|\mathcal{A}|^r} = c(\mathcal{A}, m),$$

for some nonzero constant $c(\mathcal{A}, m) \in \mathbb{Q}$, so $s_{\mathcal{A}}(r, m) \sim c(\mathcal{A}, m)\,|\mathcal{A}|^r$.

Proof. Let $g(\lambda) \coloneqq \det(M_{\mathcal{A}} - \lambda I)$ be the characteristic polynomial of $M_{\mathcal{A}}$ with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_y$, where each λ_i has multiplicity e_i. We can write

$$g(\lambda) = \sum_{k=0}^{a_z+1} \alpha_k \lambda^k.$$

By Cayley-Hamilton, we know that $g(M_{\mathcal{A}}) = \mathbf{0}$. Thus we have

$$\mathbf{0} = g(M_{\mathcal{A}}) = \sum_{k=0}^{a_z+1} \alpha_k M_{\mathcal{A}}^k$$

and hence, for all r,

$$\mathbf{0} = \left(\sum_{k=0}^{a_z+1} \alpha_k M_{\mathcal{A}}^k \right) \omega_r(m) = \sum_{k=0}^{a_z+1} \alpha_k \omega_{r+k}(m).$$

Since

$$\omega_{r+k}(m) = \begin{pmatrix} f_{\mathcal{A}}(2^{r+k}m) \\ f_{\mathcal{A}}(2^{r+k}m-1) \\ \vdots \\ f_{\mathcal{A}}(2^{r+k}m-a_z) \end{pmatrix},$$

we have

$$\sum_{k=0}^{a_z+1} \alpha_k f(2^{r+k}m - j) = 0 \tag{6}$$

for all $0 \le j \le a_z$.

Let $I_r = \{2^r, 2^r+1, 2^r+2, \dots, 2^{r+1}-1\}$. Then $I_r = 2I_{r-1} \cup (2I_{r-1}+1)$. Thus

$$\begin{aligned} s_{\mathcal{A}}(r,m) &= \sum_{n=m2^r}^{m2^{r+1}-1} f_{\mathcal{A}}(n) \\ &= \sum_{n=m2^{r-1}}^{m2^r-1} \left(f_{\mathcal{A}}(2n) + f_{\mathcal{A}}(2n+1) \right) \\ &= \sum_{n=m2^{r-1}}^{m2^r-1} \left(f_{\mathcal{A}}(n) + f_{\mathcal{A}}(n-b_2) + \cdots + f_{\mathcal{A}}(n-b_s) + f_{\mathcal{A}}(n-c_1) + \cdots + f_{\mathcal{A}}(n-c_t) \right). \end{aligned}$$

Since

$$\sum_{n=m2^{r-1}}^{m2^r-1} f_{\mathcal{A}}(n-k) = \sum_{n=m2^{r-1}}^{m2^r-1} f_{\mathcal{A}}(n) + \sum_{j=1}^{k} \left(f_{\mathcal{A}}(m2^{r-1}-j) - f_{\mathcal{A}}(m2^r-j) \right),$$

we deduce that

$$\begin{aligned} s_{\mathcal{A}}(r,m) &= |\mathcal{A}| \sum_{n=m2^{r-1}}^{m2^r-1} f_{\mathcal{A}}(n) + h(r,m) \\ &= |\mathcal{A}| \, s_{\mathcal{A}}(r-1,m) + h(r,m), \end{aligned}$$

where

$$h(r,m) = \sum_{i=2}^{s}\sum_{j=1}^{b_i}\left(f_{\mathcal{A}}(m2^{r-1}-j) - f_{\mathcal{A}}(m2^{r}-j)\right) + \sum_{i=1}^{t}\sum_{j=1}^{c_i}\left(f_{\mathcal{A}}(m2^{r-1}-j) - f_{\mathcal{A}}(m2^{r}-j)\right)$$

and

$$\sum_{k=0}^{a_z+1}\alpha_k h(r+k,m) = 0$$

by Equation (6).

Thus we have an inhomogeneous recurrence relation for $s_{\mathcal{A}}(r,m)$ and will first consider the corresponding homogeneous recurrence relation

$$s_{\mathcal{A}}(r,m) = |\mathcal{A}|\, s_{\mathcal{A}}(r-1,m),$$

which has solution $s_{\mathcal{A}}(r,m) = c\,|\mathcal{A}|^r$. Then the solution to our inhomogeneous recurrence relation is of the form

$$s_{\mathcal{A}}(r,m) = c\,|\mathcal{A}|^r + \sum_{i=1}^{y} p_i(\lambda_i, r),$$

where

$$p_i(\lambda_i,r) = \sum_{j=1}^{e_i} c_{ij} r^{j-1}\lambda_i^r.$$

By Lemma 3, $|\lambda_i|$ is bounded above by the maximum row sum of $M_{\mathcal{A}}$, which is at most $|\mathcal{A}|-1$ since all elements of $M_{\mathcal{A}}$ are either 0 or 1 and by assumption not all elements have the same parity. Hence the $c|\mathcal{A}|^r$ term dominates $s_{\mathcal{A}}(r,m)$ as $r\to\infty$, so

$$\lim_{r\to\infty}\frac{s_{\mathcal{A}}(r,m)}{|\mathcal{A}|^r} = c.$$

Observe that

$$\sum_{k=0}^{a_z+1}\alpha_k \sum_{i=1}^{y} p_i\left(\lambda_i, r+k\right) = 0.$$

Thus we can compute $\sum_{k=0}^{a_z+1}\alpha_k s_{\mathcal{A}}(r+k,m)$, and for sufficiently large r,

$$\sum_{k=0}^{a_z+1}\alpha_k s_{\mathcal{A}}(r+k,m) = c\sum_{k=0}^{a_z+1}\alpha_k\,|\mathcal{A}|^{r+k} + 0 = c\,|\mathcal{A}|^r\, g\left(|\mathcal{A}|\right).$$

Then we can solve for c to see that

$$c = c(\mathcal{A},m) := \frac{\sum_{k=0}^{a_z+1}\alpha_k s_{\mathcal{A}}(r+k,m)}{|\mathcal{A}|^r\, g\left(|\mathcal{A}|\right)}. \tag{7}$$

It remains to be shown that $c(A,m)\neq 0$, and we thank the referee for raising this point. For a particular value of n, all $|\mathcal{A}|^n$ sums of the form

$$\sum_{i=0}^{n-1}\epsilon_i 2^i,\ \epsilon_i \in \{0 = a_0 < a_1 < \cdots < a_z\}$$

have the value of the sum less than or equal to $a_z(2^n - 1)$. Thus

$$f_{\mathcal{A}}(0) + f_{\mathcal{A}}(1) + \cdots + f_{\mathcal{A}}(a_z(2^n - 1)) \geq |\mathcal{A}|^n. \tag{8}$$

Fix m. There exists $\ell \in \mathbb{N}$ such that $m2^\ell \geq a_z$. Then

$$\begin{aligned} f_{\mathcal{A}}(0) + f_{\mathcal{A}}(1) + \cdots + f_{\mathcal{A}}(a_z(2^n - 1)) &\leq f_{\mathcal{A}}(0) + f_{\mathcal{A}}(1) + \cdots + f_{\mathcal{A}}(m2^{n+\ell} - 1) \\ &= s_{\mathcal{A}}(0, m) + s_{\mathcal{A}}(1, m) + \cdots + s_{\mathcal{A}}(n + \ell - 1, m). \end{aligned} \tag{9}$$

Combining (8) and (9), we have

$$|\mathcal{A}|^n \leq s_{\mathcal{A}}(0, m) + s_{\mathcal{A}}(1, m) + \cdots + s_{\mathcal{A}}(n + \ell - 1, m).$$

We know from above that $s_{\mathcal{A}}(r, m) = (c(\mathcal{A}, m) + o(1))|\mathcal{A}|^r$. Thus

$$\begin{aligned} |\mathcal{A}|^n &\leq (c(\mathcal{A}, m) + o(1)) \left(|\mathcal{A}|^0 + |\mathcal{A}|^1 + |\mathcal{A}|^2 + \cdots + |\mathcal{A}|^{n+\ell-1}\right) \\ &< (c(\mathcal{A}, m) + o(1)) \frac{|\mathcal{A}|^{n+\ell}}{|\mathcal{A}| - 1}. \end{aligned}$$

Dividing both sides by $|\mathcal{A}|^n$, we see that

$$1 \leq (c(\mathcal{A}, m) + o(1)) \frac{|\mathcal{A}|^\ell}{|\mathcal{A}| - 1}.$$

Hence $c(\mathcal{A}, m) \neq 0$ and $s_{\mathcal{A}}(r, m) \sim c(\mathcal{A}, m)\,|\mathcal{A}|^r$. □

3 Examples

Example 5. Let $\mathcal{A} = \{0, 1, 8\}$. Then

$$f_{\mathcal{A}}(2\ell) = f_{\mathcal{A}}(\ell) + f_{\mathcal{A}}(\ell - 4) \tag{10}$$

and

$$f_{\mathcal{A}}(2\ell + 1) = f_{\mathcal{A}}(\ell), \tag{11}$$

so

$$\begin{pmatrix} f_{\mathcal{A}}(2^{k+1}m) \\ f_{\mathcal{A}}(2^{k+1}m - 1) \\ f_{\mathcal{A}}(2^{k+1}m - 2) \\ f_{\mathcal{A}}(2^{k+1}m - 3) \\ f_{\mathcal{A}}(2^{k+1}m - 4) \\ f_{\mathcal{A}}(2^{k+1}m - 5) \\ f_{\mathcal{A}}(2^{k+1}m - 6) \\ f_{\mathcal{A}}(2^{k+1}m - 7) \\ f_{\mathcal{A}}(2^{k+1}m - 8) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_{\mathcal{A}}(2^k m) \\ f_{\mathcal{A}}(2^k m - 1) \\ f_{\mathcal{A}}(2^k m - 2) \\ f_{\mathcal{A}}(2^k m - 3) \\ f_{\mathcal{A}}(2^k m - 4) \\ f_{\mathcal{A}}(2^k m - 5) \\ f_{\mathcal{A}}(2^k m - 6) \\ f_{\mathcal{A}}(2^k m - 7) \\ f_{\mathcal{A}}(2^k m - 8) \end{pmatrix}.$$

If $M_{\mathcal{A}}$ is the matrix above, then $\omega_{k+1}(m) = M_{\mathcal{A}}\omega_k(m)$. The characteristic polynomial of $M_{\mathcal{A}}$ is

$$g(x) = 1 - 3x + 3x^2 - 3x^3 + 6x^4 - 6x^5 + 3x^6 - 3x^7 + 3x^8 - x^9. \tag{12}$$

We then compute

$$\begin{aligned} s_{\mathcal{A}}(3,1) &- 3s_{\mathcal{A}}(4,1) + 3s_{\mathcal{A}}(5,1) - 3s_{\mathcal{A}}(6,1) + 6s_{\mathcal{A}}(7,1) - 6s_{\mathcal{A}}(8,1) \\ &+ 3s_{\mathcal{A}}(9,1) - 3s_{\mathcal{A}}(10,1) + 3s_{\mathcal{A}}(11,1) - s_{\mathcal{A}}(12,1) \\ &= -59184 \end{aligned}$$

Using the formula from Theorem 4, we see that

$$c(\mathcal{A},1) = \frac{-59184}{g(3)\cdot 27} = \frac{-59184}{-5408\cdot 27} = \frac{137}{338}.$$

Example 6. Let $\mathcal{A} = \{0,1,3\}$. Then

$$f_{\mathcal{A}}(2\ell) = f_{\mathcal{A}}(\ell) \tag{13}$$

and

$$f_{\mathcal{A}}(2\ell+1) = f_{\mathcal{A}}(\ell) + f_{\mathcal{A}}(\ell-1), \tag{14}$$

so

$$\begin{pmatrix} f_{\mathcal{A}}(2^{k+1}m) \\ f_{\mathcal{A}}(2^{k+1}m-1) \\ f_{\mathcal{A}}(2^{k+1}m-2) \\ f_{\mathcal{A}}(2^{k+1}m-3) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} f_{\mathcal{A}}(2^{k}m) \\ f_{\mathcal{A}}(2^{k}m-1) \\ f_{\mathcal{A}}(2^{k}m-2) \\ f_{\mathcal{A}}(2^{k}m-3) \end{pmatrix}.$$

Hence $M_{\mathcal{A}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ satisfies $\omega_{k+1}(m) = M_{\mathcal{A}}\omega_k(m)$. The characteristic polynomial of $M_{\mathcal{A}}$ is

$$g(x) = (x-1)^2(x^2 - x - 1). \tag{15}$$

Let F_k denote the k-th Fibonacci number. Then

$$f_{\mathcal{A}}(2^k - 1) = F_{k+1} \tag{16}$$

for all $k \geq 0$. This can be shown by using induction and Equations (13) and (14).

Considering the summatory function with $m = 1$ and using Equations (13),(14), and (16), we see that

$$\begin{aligned}
s_{\mathcal{A}}(r,1) &= \sum_{n=2^r}^{2^{r+1}-1} f_{\mathcal{A}}(n) \\
&= \sum_{n=2^{r-1}}^{2^r-1} \left(f_{\mathcal{A}}(2n) + f_{\mathcal{A}}(2n+1)\right) \\
&= \sum_{n=2^{r-1}}^{2^r-1} \left(f_{\mathcal{A}}(n) + f_{\mathcal{A}}(n) + f_{\mathcal{A}}(n-1)\right) \\
&= 2s_{\mathcal{A}}(r-1,1) + \sum_{n=2^{r-1}}^{2^r-1} f_{\mathcal{A}}(n-1) \\
&= 2s_{\mathcal{A}}(r-1,1) + \sum_{n=2^{r-1}}^{2^r-1} f_{\mathcal{A}}(n) + f_{\mathcal{A}}(2^{r-1}-1) - f_{\mathcal{A}}(2^r-1) \\
&= 3s_{\mathcal{A}}(r-1,1) + f_{\mathcal{A}}(2^{r-1}-1) - f_{\mathcal{A}}(2^r-1) \\
&= 3s_{\mathcal{A}}(r-1,1) + F_r - F_{r+1} \\
&= 3s_{\mathcal{A}}(r-1,1) - F_{r-1}.
\end{aligned}$$

This is an inhomogeneous recurrence relation for $s_{\mathcal{A}}(r,1)$. We first consider the corresponding homogeneous recurrence relation $s_{\mathcal{A}}(r,1) = 3s_{\mathcal{A}}(r-1,1)$, which has solution

$$s_{\mathcal{A}}(r,1) = d_1 3^r,$$

for some d_1 in $\mathbb{Q}$. Recall that the characteristic polynomial $g(x)$ of $M_{\mathcal{A}}$ has roots $1, \phi$, and $\bar{\phi}$, where the first has multiplicity 2 and the others have multiplicity 1. Hence the solution to the inhomogeneous recurrence relation is

$$s_{\mathcal{A}}(r,1) = d_1 3^r + d_2\phi^r + d_3\bar{\phi}^r + d_4(1)^r + d_5 r(1)^r, \tag{17}$$

where $d_2, d_3, d_4, d_5 \in \mathbb{Q}$. Observe that the $d_1 3^r$ summand will dominate as $r \to \infty$, so

$$\lim_{r\to\infty} \frac{s_{\mathcal{A}}(r,1)}{3^r} = d_1$$

and $s_{\mathcal{A}}(r,1) \sim d_1 3^r$.

Using Equations (15) and (17), we can compute d_1 as

$$\begin{aligned}
s_{\mathcal{A}}(r+2,1) - s_{\mathcal{A}}(r+1,1) - s_{\mathcal{A}}(r,1) &= d_1 3^r(3^2-3-1) + d_2\phi^r(\phi^2-\phi-1) \\
&\quad + d_3\bar{\phi}^r(\bar{\phi}^2-\bar{\phi}-1) + d_4(1^2-1-1) \\
&\quad + d_5(r+2-(r+1)-r) \\
&= d_1 3^r \cdot 5 - d_4 - d_5(r-1).
\end{aligned}$$

Plugging in $r = 2$, $r = 1$, and $r = 0$ and computing sums, we see that $d_1 = 4/5$. Hence

$$\lim_{r\to\infty} \frac{s_{\mathcal{A}}(r,1)}{3^r} = \frac{4}{5}$$

and $s_{\mathcal{A}}(r,1) \sim \frac{4}{5} \cdot 3^r$.

Example 7. Let $\tilde{\mathcal{A}} = \{0,2,3\}$. Then

$$f_{\tilde{\mathcal{A}}}(2\ell) = f_{\tilde{\mathcal{A}}}(\ell) + f_{\tilde{\mathcal{A}}}(\ell - 1) \tag{18}$$

and

$$f_{\tilde{\mathcal{A}}}(2\ell + 1) = f_{\tilde{\mathcal{A}}}(\ell - 1), \tag{19}$$

so

$$\begin{pmatrix} f_{\tilde{\mathcal{A}}}(2^{k+1}m) \\ f_{\tilde{\mathcal{A}}}(2^{k+1}m-1) \\ f_{\tilde{\mathcal{A}}}(2^{k+1}m-2) \\ f_{\tilde{\mathcal{A}}}(2^{k+1}m-3) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_{\tilde{\mathcal{A}}}(2^{k}m) \\ f_{\tilde{\mathcal{A}}}(2^{k}m-1) \\ f_{\tilde{\mathcal{A}}}(2^{k}m-2) \\ f_{\tilde{\mathcal{A}}}(2^{k}m-3) \end{pmatrix}.$$

Hence $M_{\tilde{\mathcal{A}}} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ satisfies $\omega_{k+1}(m) = M_{\tilde{\mathcal{A}}}\omega_k(m)$. The characteristic polynomial of $M_{\tilde{\mathcal{A}}}$ is

$$g(x) = (x-1)^2(x^2 - x - 1). \tag{20}$$

Let F_k denote the k-th Fibonacci number. Then

$$f_{\tilde{\mathcal{A}}}(2^k - 1) = F_{k-1} \tag{21}$$

for all $k \geq 1$. This can be shown by using induction and Equations (18) and (19) to prove that $f_{\tilde{\mathcal{A}}}(2^k - 2) = F_k$ for all $k \geq 2$ and observing that Equation (19) gives $f_{\tilde{\mathcal{A}}}(2^k - 1) = f_{\tilde{\mathcal{A}}}(2^{k-1} - 2)$.

Considering the summatory function with $m = 1$ and using Equations (18),(19), and (21) and manipulations similar to those in Example 6, we see that

$$s_{\tilde{\mathcal{A}}}(r,1) = 3s_{\tilde{\mathcal{A}}}(r-1,1) - 2F_{r-3}.$$

Again, the corresponding homogeneous recurrence relation has solution

$$s_{\tilde{\mathcal{A}}}(r,1) = d_1 3^r,$$

for some d_1 in $\mathbb{Q}$, and we can use Equation (20) to see that the solution to the inhomogeneous recurrence relation is

$$s_{\tilde{\mathcal{A}}}(r,1) = d_1 3^r + d_2\phi^r + d_3\bar{\phi}^r + d_4(1)^r + d_5 r(1)^r, \tag{22}$$

where $d_2, d_3, d_4, d_5 \in \mathbb{Q}$. Observe that the $d_1 3^r$ summand will dominate as $r \to \infty$, so

$$\lim_{r\to\infty} \frac{s_{\tilde{\mathcal{A}}}(r,1)}{3^r} = d_1$$

and $s_{\tilde{\mathcal{A}}}(r,1) \sim d_1 3^r$.

Using Equations (20) and (22), we can compute d_1 as

$$s_{\tilde{\mathcal{A}}}(r+2,1) - s_{\tilde{\mathcal{A}}}(r+1,1) - s_{\tilde{\mathcal{A}}}(r,1) = d_1 3^r \cdot 5 - d_4 - d_5(r-1).$$

Plugging in $r = 2$, $r = 1$, and $r = 0$ and computing sums, we see that $d_1 = 2/5$. Hence

$$\lim_{r\to\infty} \frac{s_{\tilde{\mathcal{A}}}(r,1)}{3^r} = \frac{2}{5}$$

and $s_{\tilde{\mathcal{A}}}(r,1) \sim \frac{2}{5} \cdot 3^r$.

In Example 6, we had $\mathcal{A} = \{0,1,3\}$, and in Example 7, we had $\tilde{\mathcal{A}} = \{0,2,3\} = \{3-3, 3-1, 3-0\}$. We found $c(\mathcal{A},1)$ in Example 6 and $c(\tilde{\mathcal{A}},1)$ in Example 7 and can observe that they have the same denominator.

Given a set $\mathcal{A} = \{0, a_1, \dots, a_z\}$, let $\tilde{\mathcal{A}}$ be

$$\tilde{\mathcal{A}} := \{0, a_z - a_{z-1}, \dots, a_z - a_1, a_z\}.$$

The following chart displays the value $c(\mathcal{A},1)$ for various sets $\mathcal{A}$ and their corresponding sets $\tilde{\mathcal{A}}$, where $s_{\mathcal{A}}(r,1) \sim c(\mathcal{A},1)|\mathcal{A}|^r$. Note that in all cases the denominator of $c(\mathcal{A},1)$ is the same as that of $c(\tilde{\mathcal{A}},1)$. The following theorem will show that this holds for all $\mathcal{A}$.

$\mathcal{A}$	$c(\mathcal{A},1)$	$\mathrm{N}(c(\mathcal{A},1))$	$\tilde{\mathcal{A}}$	$c(\tilde{\mathcal{A}},1)$	$\mathrm{N}(c(\tilde{\mathcal{A}},1))$
$\{0,1,2,4\}$	$\frac{7}{11}$	0.636	$\{0,2,3,4\}$	$\frac{3}{11}$	0.273
$\{0,1,3,4\}$	$\frac{1}{2}$	0.500	$\{0,1,3,4\}$	$\frac{1}{2}$	0.500
$\{0,2,3,6\}$	$\frac{33}{149}$	0.221	$\{0,3,4,6\}$	$\frac{21}{149}$	0.141
$\{0,1,6,9\}$	$\frac{6345}{28670}$	0.221	$\{0,3,8,9\}$	$\frac{2007}{28670}$	0.070
$\{0,1,7,9\}$	$\frac{2069}{10235}$	0.202	$\{0,2,8,9\}$	$\frac{1023}{10235}$	0.100
$\{0,4,5,6,9\}$	$\frac{4044}{83753}$	0.048	$\{0,3,4,5,9\}$	$\frac{6716}{83753}$	0.080

Table 1: $c(\mathcal{A},1)$ for various sets $\mathcal{A}$ and $\tilde{\mathcal{A}}$

Theorem 8. *Let $\mathcal{A}, f_{\mathcal{A}}(n), M_{\mathcal{A}} = [m_{\alpha,\beta}]$, and $\tilde{\mathcal{A}}$ be as above, with $0 \le \alpha, \beta \le a_z$. Let $M_{\tilde{\mathcal{A}}} = \left[m'_{\alpha,\beta}\right]$ be the $(a_z+1) \times (a_z+1)$ matrix such that*

$$\begin{pmatrix} f_{\tilde{\mathcal{A}}}(2n) \\ f_{\tilde{\mathcal{A}}}(2n-1) \\ \vdots \\ f_{\tilde{\mathcal{A}}}(2n-a_z) \end{pmatrix} = M_{\tilde{\mathcal{A}}} \begin{pmatrix} f_{\tilde{\mathcal{A}}}(n) \\ f_{\tilde{\mathcal{A}}}(n-1) \\ \vdots \\ f_{\tilde{\mathcal{A}}}(n-a_z) \end{pmatrix}.$$

Then $m_{\alpha,\beta} = m'_{a_z-\alpha,a_z-\beta}$.

Proof. Recall that we can write

$$\mathcal{A} := \{0, 2b_2, \dots, 2b_s, 2c_1+1, \dots, 2c_t+1\},$$

so that

$$f_{\mathcal{A}}(2n-2j) = f_{\mathcal{A}}(n-j) + f_{\mathcal{A}}(n-j-b_2) + \cdots + f_{\mathcal{A}}(n-j-b_s)$$

and

$$f_{\mathcal{A}}(2n-2j-1) = f_{\mathcal{A}}(n-j-c_1-1) + \cdots + f_{\mathcal{A}}(n-j-c_t-1)$$

for j sufficiently large.

Then $m_{\alpha,\beta} = 1$ if and only if $f_{\mathcal{A}}(n-\beta)$ is a summand in the recursive sum that expresses $f_{\mathcal{A}}(2n-\alpha)$, which happens if and only if $2n-\alpha = 2(n-\beta)+K$, where $K \in \mathcal{A}$, and this is equivalent to $2\beta-\alpha$ belonging to $\mathcal{A}$.

Now $m'_{a_z-\alpha,a_z-\beta} = 1$ if and only if $f_{\tilde{\mathcal{A}}}(n-(a_z-\beta))$ is a summand in the recursive sum that expresses $f_{\tilde{\mathcal{A}}}(2n-(a_z-\alpha))$, which happens if and only if $2n-(a_z-\alpha) = 2(n-(a_z-\beta))+\tilde{K}$, where $\tilde{K} \in \tilde{\mathcal{A}}$. This means that $a_z+\alpha-2\beta = \tilde{K}$, which gives $2\beta-\alpha \in \mathcal{A}$. □

Thus $M_{\mathcal{A}} = S^{-1} M_{\tilde{\mathcal{A}}} S$, where

$$S = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

so $M_{\mathcal{A}}$ and $M_{\tilde{\mathcal{A}}}$ are similar matrices and thus have the same characteristic polynomial, [4, 1.3.3]. Hence the denominator in (7) for $\mathcal{A}$ is equal to the denominator in (7) for $\tilde{\mathcal{A}}$.

4 Open questions

A nicer formula for $c(\mathcal{A}, m)$ than that given in Equation (7) is desired and seems likely. To that end, we have computed values of $c(\mathcal{A})$ for a variety of sets $\mathcal{A}$ but have not been able to detect any patterns. Table 2 shows $c(\mathcal{A},1)$ for all sets of the form $\mathcal{A}=\{0,1,t\}$, where $2\le t\le 17$, and we have obtained the following bounds on $c(\mathcal{A},1)$ for sets $\mathcal{A}$ of this form.

Let $t\in\mathbb{N}$ with $t>1$ and $\mathcal{A}=\{0,1,t\}$. Choose k such that $2^k<t\le 2^{k+1}$. Recall that $f_{\mathcal{A}}(s)$ is the number of ways to write s in the form

$$s=\sum_{i=0}^{\infty}\epsilon_i 2^i,\ \text{ where } \epsilon_i\in\{0,1,t\}.$$

Then

$$s_{\mathcal{A}}(r,1)=\sum_{n=2^r}^{2^{r+1}-1} f_{\mathcal{A}}(n)\sim c(\mathcal{A},1)3^r,$$

as shown in Theorem 4. Thus

$$\begin{aligned}\sum_{s=1}^{2^n-1} f_{\mathcal{A}}(s) &= \sum_{j=0}^{n-1}\sum_{s=2^j}^{2^{j+1}-1} f_{\mathcal{A}}(s)\sim\sum_{j=0}^{n-1} c(\mathcal{A},1)3^j\\ &= c(\mathcal{A},1)\left(\frac{3^n-1}{2}\right)\approx\frac{1}{2}c(\mathcal{A},1)3^n.\end{aligned}$$

Consider choosing $\epsilon_i\in\{0,1,t\}$ for $0\le i\le n-k-3$ and $\epsilon_i\in\{0,1\}$ for $n-k-2\le i\le n-2$. Then

$$\begin{aligned}\sum_{i=0}^{n-2}\epsilon_i 2^i &\le t+t\cdot 2+t\cdot 2^2+\cdots+t\cdot 2^{n-k-3}+2^{n-k-2}+2^{n-k-1}+\cdots+2^{n-2}\\ &< t\cdot 2^{n-k-2}+2^{n-1}-1\\ &\le 2^{k+1}\cdot 2^{n-k-2}+2^{n-1}-1\\ &= 2^n-1\\ &< 2^n.\end{aligned}$$

There are $3^{n-k-2}\cdot 2^{k+1}$ such sums, and each of them is counted in $\sum_{s=1}^{2^n-1} f_{\mathcal{A}}(s)$. Thus

$$\frac{1}{2}c(\mathcal{A},1)3^n\ge 3^{n-k-2}\cdot 2^{k+1}=3^n\cdot\frac{2^{k+1}}{3^{k+2}},$$

and so $c(\mathcal{A},1)\ge\left(\frac{2}{3}\right)^{k+2}$.

Now suppose there exists some $i_0\ge n-k$ such that $\epsilon_{i_0}=t$. Then

$$\sum_{i=0}^{\infty}\epsilon_i 2^i\ge t\cdot 2^{i_0}\ge t\cdot 2^{n-k}>2^k 2^{n-k}=2^n.$$

Thus the sums counted in $\sum_{s=1}^{2^n-1} f_{\mathcal{A}}(s)$ all have the property that $\epsilon_i\in\{0,1\}$ for $n-k\le i\le n-1$, and there are $3^{n-k}\cdot 2^k$ such sums. Hence $3^{n-k}\cdot 2^k\ge\frac{1}{2}c(\mathcal{A},1)\cdot 3^n$ and $\frac{2^{k+1}}{3^k}\ge c(\mathcal{A},1)$.

Combining the above, we see that

$$\frac{2^{k+1}}{3^k}\cdot\frac{2}{9}\le c(\mathcal{A},1)\le\frac{2^{k+1}}{3^k}.$$

$\mathcal{A}$	$c(\mathcal{A},1)$	$\mathrm{N}(c(\mathcal{A},1))$	$\mathcal{A}$	$c(\mathcal{A},1)$	$\mathrm{N}(c(\mathcal{A},1))$
$\{0,1,2\}$	1	1.000	$\{0,1,3\}$	$\frac{4}{5}$	0.800
$\{0,1,4\}$	$\frac{5}{8}$	0.625	$\{0,1,5\}$	$\frac{14}{25}$	0.560
$\{0,1,6\}$	$\frac{35}{71}$	0.493	$\{0,1,7\}$	$\frac{176}{391}$	0.450
$\{0,1,8\}$	$\frac{137}{338}$	0.405	$\{0,1,9\}$	$\frac{1448}{3775}$	0.384
$\{0,1,10\}$	$\frac{1990}{5527}$	0.360	$\{0,1,11\}$	$\frac{3223}{9476}$	0.340
$\{0,1,12\}$	$\frac{2020}{6283}$	0.322	$\{0,1,13\}$	$\frac{47228}{154123}$	0.306
$\{0,1,14\}$	$\frac{35624}{122411}$	0.291	$\{0,1,15\}$	$\frac{699224}{2501653}$	0.280
$\{0,1,16\}$	$\frac{68281}{256000}$	0.267	$\{0,1,17\}$	$\frac{38132531}{146988000}$	0.259

Table 2: $c(\mathcal{A},1)$ for all sets of the form $\mathcal{A} = \{0,1,t\}$, where $2 \le t \le 17$

.

To compare these bounds with Table 2, note that if $8 < t \le 15$, then $k = 3$, and we have

$$0.132 \le c(\mathcal{A},1) \le 0.593$$

for $\mathcal{A} = \{0,1,t\}$, with t in this range.

We have also computed $c(\mathcal{A},1)$ for some sets with $|\mathcal{A}| = 4$ and $|\mathcal{A}| = 5$, and that data is contained in Table 1. Larger sets have not been considered because computations become increasingly tedious as the cardinality of $\mathcal{A}$ grows.

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