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Fibonacci s-Cullen and s-Woodall Numbers

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Abstract

The m-th Cullen number C_m is a number of the form $m2^m + 1$ and the m-th Woodall number W_m has the form $m2^m - 1$. In 2003, Luca and Stănică proved that the largest Fibonacci number in the Cullen sequence is $F_4 = 3$ and that $F_1 = F_2 = 1$ are the largest Fibonacci numbers in the Woodall sequence. Very recently, the second author proved that, for any given $s > 1$, the equation $F_n = ms^m \pm 1$ has only finitely many solutions, and they are effectively computable. In this note, we shall provide the explicit form of the possible solutions.

1 Introduction

A *Cullen number* is a number of the form $m2^m+1$ (denoted by C_m), where m is a nonnegative integer. This sequence was introduced in 1905 by Father J. Cullen [2] and it was mentioned in the well-known book of Guy [5, Section **B20**]. These numbers gained great interest in 1976, when Hooley [7] showed that almost all Cullen numbers are composite. However, despite being very scarce, it is still conjectured that there are infinitely many *Cullen primes*.

In a similar way, a *Woodall number* (also called *Cullen number of the second kind*) is a positive integer of the form $m2^m - 1$ (denoted by W_m). It is also known that almost all Woodall numbers are composite. However, it is also conjectured that the set of *Woodall primes* is infinite.

These numbers can be generalized to the *s-Cullen and s-Woodall numbers* which are numbers of the form

$$C_{m,s} = ms^m + 1 \text{ and } W_{m,s} = ms^m - 1,$$

where $m \geq 1$ and $s \geq 2$. This family was introduced by Dubner [3]. A prime of the form $C_{m,s}$ is $C_{139948,151}$ an integer with 304949 digits.

Many authors have searched for special properties of Cullen and Woodall numbers and their generalizations. We refer the reader to [4, 6, 9, 10] for classical and recent results on this subject.

In 2003, Luca and Stănică [8, Theorem 3] proved that the largest Fibonacci number in the Cullen sequence is $F_4 = 3 = 1 \cdot 2^1 + 1$ and that $F_1 = F_2 = 1 = 1 \cdot 2^1 - 1$ are the largest Fibonacci numbers in the Woodall sequence.

Recall that $\nu_p(r)$ denotes the p-adic order of r, which is the exponent of the highest power of a prime p which divides r. Also, the *order (or rank) of appearance* of n in the Fibonacci sequence, denoted by $z(n)$, is defined as the smallest positive integer k, such that $n \mid F_k$ (for results on this function, see [13] and references therein). Let p be a prime number and set $e(p) := \nu_p(F_{z(p)})$.

Very recently, Marques [11] proved that if the equation

$$F_n = ms^m + \ell \tag{1}$$

has solution, with $m > 1$ and $\ell \in \{\pm 1\}$, then $m < (6.2 + 1.9e(p)) \log(3.1 + e(p))$, for some prime factor p of s. This together with the fact that $e(p) = 1$ for all prime $p < 2.8 \cdot 10^{16}$ (PrimeGrid, March 2014) implies that there is no Fibonacci number that is also a nontrivial (i.e., $m > 1$) s-Cullen number or s-Woodall number when the set of prime divisors of s is contained in $\{2, 3, 5, \ldots, 27999999999999991\}$. This is the set of the first 759997990476073 prime numbers.

In particular, the previous result ensures that for any given $s \geq 2$, there are only finitely many Fibonacci numbers which are also s-Cullen numbers or s-Woodall numbers and they are effectively computable.

In this note, we shall invoke the primitive divisor theorem to provide explicitly the possible values of m satisfying Eq. (1). More precisely,

Theorem 1. *Let $s > 1$ be an integer. Let (n, m, ℓ) be a solution of the Diophantine equation (1) with $n, m > 1$ and $\ell \in \{-1, 1\}$. Then $m = e(p)/\nu_p(s)$, for some prime factor p of s.*

In particular, we have that $m \leq e(p)$ for some prime factor p of s. Also, we can deduce [11, Corollary 3] from the above theorem. In fact, for all $p < 2.8 \cdot 10^{16}$ we have $e(p) = 1$ and then if (n, m, ℓ) is a solution, with $m > 1$, we would have the contradiction that $1 < m = e(p)/\nu_p(s) = 1/\nu_p(s)$ for some p dividing s.

2 The proof

Suppose that $n \leq 27$. Then $\max\{2s^2 - 1, m2^m - 1\} \leq ms^m + \ell = F_n \leq F_{27} = 196418$ yields $s \leq 313$ and $m \leq 13$. For this, we prepare a simple *Mathematica* program which, in a few seconds, does not return any solution with $m > 1$.

So we may suppose that $n \geq 28$. We rewrite Eq. (1) as $F_n - \ell = ms^m$. It is well-known that $F_n \pm 1 = F_a L_b$, where $2a, 2b \in \{n \pm 2, n \pm 1\}$. (This factorization depends on the class of n modulo 4. See [12, (3)] for more details.) Then the main equation becomes

$$F_a L_b = ms^m,$$

where $2a, 2b \in \{n \pm 2, n \pm 1\}$ and $|a - b| \in \{1, 2\}$. Since $a - b \in \{\pm 1, \pm 2\}$, then $\gcd(a, b) \in \{1, 2\}$ and then $\gcd(F_a, L_b) = 1, 2$ or 3. Therefore, we have $F_a = m_1 s_1^m$ and $L_b = m_2 s_2^m$, where $m_1 m_2 = m, s_1 s_2 = s$ and $\gcd(m_1, m_2), \gcd(s_1, s_2) \in \{1, 2, 3\}$. We claim that $s_1 > 1$. Suppose, to get a contradiction, that $s_1 = 1$, then $F_a = m_1$ and $L_b = m_2 s^m$. Since $2a - 4 \geq n - 6 \geq (n + 8)/2 \geq b + 3$, we arrive at the following contradiction:

$$m^2 \geq m_1^2 = F_a^2 \geq \alpha^{2a-4} \geq \alpha^{b+3} \geq 2L_b = 2m_2 s^m \geq 2^{m+1} > m^2,$$

where $\alpha = (1 + \sqrt{5})/2$. Here, we used that $F_j \geq \alpha^{j-2}$ and $L_j \leq \alpha^{j+1}$. Thus $s_1 > 1$. Since $a \geq (n - 2)/2 \geq 13$, then by the primitive divisor theorem (see [1]), there exists a primitive divisor p of F_a (i.e., $p \mid F_a$ and $p \nmid F_1 \cdots F_{a-1}$). We also have that $p \equiv \pm 1 \pmod{a}$. In particular, $p \geq a - 1$. Thus $p \mid F_a = m_1 s_1^m$. Suppose that $p \mid m_1$. In this case, one has that $a - 1 \leq p \leq m_1 \leq m$. On the other hand, we get

$$2^m \leq m_1 s_1^m = F_a \leq \alpha^{a-1} < 2^{a-1}.$$

Thus $m < a - 1$ which gives a contradiction. Therefore $p \nmid m_1$ and consequently $p \mid s_1$. This yields $\nu_p(F_a) = m\nu_p(s_1) = m\nu_p(s)$ (because $p > 3, s = s_1 s_2$ and $\gcd(s_1, s_2) \leq 3$). On the other hand, $z(p) = a$ and so $\nu_p(F_{z(p)}) = \nu_p(F_a) = m\nu_p(s)$ as desired. □

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