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Rectangles Of Nonvisible Lattice Points

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Abstract

A lattice point $(0,0) \neq (x,y) \in \mathbb{Z}^2$ is called *visible* (from the origin) if $\gcd(x,y)=1$ and *nonvisible* otherwise. Given positive integers a,b, define $M := M(a,b)$ and $N := N(a,b)$ to be the positive integers M and N having the least value of $\max(M,N)$ with the property that $\gcd(M-i,N-j)>1$ for all $1 \leq i \leq a$ and $1 \leq j \leq b$. We give upper and lower bounds for M,N.

1 Introduction

A lattice point $(0,0) \neq (x,y) \in \mathbb{Z}^2$ is called *visible* (from the origin) if $\gcd(x,y)=1$ and *nonvisible* otherwise (see Herzog and Stewart [2]). In other words, (r,s) is visible iff $\frac{r}{s}$ is in lowest terms.

In [4], Pighizzini and Shallit defined, for a positive integer n, the function $S(n)$, which is the least positive integer r such that there exists $m \in \{0, 1, \ldots, r\}$ with $\gcd(r-i, m-j) > 1$ for $0 \le i, j < n$. This is equivalent to finding the square of side n, nearest to the origin in the first quadrant of the real xy plane, where all its lattice points are nonvisible from the origin. It was shown in [4] that

$$S(n) < e^{(2+o(1))n^2 \log n} \qquad \text{as} \qquad n \to \infty, \tag{1}$$

and computed $S(n)$ and the corresponding m's for $n = 1, 2, 3$. This function was also studied by Wolfram [6, pp. 613, 1093] who computed $S(4)$.

Here, we generalize the function $S(n)$. Given positive integers a, b, let $(M(a, b), N(a, b))$ be a minimal pair of positive integers such that $\gcd(M - i, N - j) > 1$ for all $1 \le i \le a$ and $1 \le j \le b$. More precisely, given positive integers a, b, define $M := M(a, b)$ and $N := N(a, b)$ to be the positive integers M and N having the least value of $\max(M, N)$ with the property that $\gcd(M - i, N - j) > 1$ for all $1 \le i \le a$ and $1 \le j \le b$. This is equivalent to finding the rectangle with sides a, b, nearest to the origin in the first quadrant of real xy plane, where all its lattice points are nonvisible from the origin.

Without loss of generality, we assume that $a \ge b$. In this note, we prove the following result. We always write p for a prime number.

Theorem 1. *If $a \ge b$, we then have*

(i) $\max\{M(a, b), N(a, b)\} \le \exp((6/\pi^2 + o(1))ab \log ab)$ *as* $b \to \infty$.

(ii) $\max\{M(a, b), N(a, b)\} \le \exp(0.721521ab \log ab)$ *if* $b > 100$.

(iii) We have

$$M(a, b) \ge \exp((c_1 + o(1))b \log ab) \quad \text{and} \quad N(a, b) \ge \exp((c_1 + o(1))a \log ab),$$

where

$$c_1 = 1 - \sum_{p \ge 2} \frac{1}{p^2} = 0.547753\cdots$$

provided $b \to \infty$ in such a way that $\log \log a = o(b)$.

Taking $a = b = n$, (i) above shows that

$$S(n) \le \exp((12/\pi^2 + o(1))n^2 \log n) \qquad \text{as} \qquad n \to \infty,$$

which improves (1). We also give a lower bound for $S(n)$. We prove

Theorem 2. *For $n > 1$, we have*

$$S(n) \ge \exp(.82248 n \log n).$$

We also give an algorithm for computing M and N for a given a and b. This is stated in Section 3 and values of M and N are computed for some small values of a, b. The proof of Theorem 2 is given in Section 4.

2 Preliminaries

For a positive integer i, let p_i denote the i-th prime. Thus $p_1 = 2, p_2 = 3, \ldots$. For real $x > 1$, let

$$\pi(x) = \sum_{p \le x} 1 \quad \text{and} \quad \theta(x) = \sum_{p \le x} \log p.$$

From the prime number theorem, we have $\pi(x) \le s_1 x / \log x$ and $\theta(p_\ell) \le s_2 \ell \log \ell$ for positive constants s_1, s_2. The following results give explicit values of s_1 and s_2.

Lemma 3. *Let x be real and positive and ℓ be a positive integer. We have*

(i) $\pi(x) \le \frac{x}{\log x}\left(1 + \frac{1.2762}{\log x}\right)$ *for* $x > 1$.

(ii) $p_\ell \ge \ell \log \ell$ *for* $\ell \ge 1$.

(iii) $\theta(p_\ell) \le \ell(\log \ell + \log\log \ell - .75)$ *for* $\ell \ge 8$.

(iv) $\theta(x) \ge x\left(1 - \frac{1}{\log x}\right)$ *for* $x \ge 41$.

(v) $\sum_{p \le x} \frac{1}{p} \le \log\log x + 0.2615 + \frac{1}{\log^2 x}$ *for* $x > 1$.

The estimates $(ii), (iv)$ and (v) are Rosser and Schoenfeld [5, (3.12), (3.16), (3.20)], respectively. The estimate (i) is due to Dusart [1] and (iii) is derived from estimates in [1].

For given integers $j \ge r \ge 1$, let

$$r' := r'(j) := \#\{i : 1 \le i \le r \text{ and } \gcd(i, j) = 1\}.$$

Let

$$R_j := \max\left\{ r' - \frac{r\varphi(j)}{j} : 1 \le r < j \right\},$$

where $\varphi(j)$ is the Euler phi-function. It is easy to see that $R_p = 1 - 1/p$. For a real number x, let $\{x\}$ denote the fractional part of x; i.e., $\{x\} = x - \lfloor x \rfloor$. We prove the following estimate.

Lemma 4. *If $n > 100$, then*

$$\sum_{j=1}^{n} R_j \le .375 n \log n - .432 n - 10.$$

Proof. For $1 \le r < j$, we have

$$r'(j) \le r - \sum_{p|j} \left\lfloor \frac{r}{p} \right\rfloor + \sum_{pq|j} \left\lfloor \frac{r}{pq} \right\rfloor - \sum_{pqr|j} \left\lfloor \frac{r_j}{pqr} \right\rfloor + \cdots ,$$

where $p, q, r, \ldots$ are primes dividing j. Since

$$\frac{\varphi(j)}{j} = 1 - \sum_{p|j} \frac{1}{p} + \sum_{pq|j} \frac{1}{pq} - \sum_{pqr|j} \frac{1}{pqr} + \cdots ,$$

we get

$$r' - \frac{r\varphi(j)}{j} \le \sum_{p|j} \left\{ \frac{r_j}{p} \right\} - \sum_{pq|j} \left\{ \frac{r_j}{pq} \right\} + \sum_{pqr|j} \left\{ \frac{r_j}{pqr} \right\} - \cdots .$$

Since $r/s \le \lfloor r/s \rfloor + 1 - 1/s$ holds for positive integers r, s, we get

$$R_j \le \sum_{p|j} \left(1 - \frac{1}{p}\right) + \sum_{pqr|j} \left(1 - \frac{1}{pqr}\right) + \cdots$$

Let $\omega(j)$ be the number of distinct prime divisors of j and put $\omega_t = \binom{j}{t}$. Then

$$R_j \le \sum_{t \text{ odd}} \omega_t - \sum_{p|j} \frac{1}{p} = 2^{\omega(j)-1} - \sum_{p|j} \frac{1}{p}.$$

Thus, for $n > 100$, we have

$$\begin{aligned}
\sum_{j=1}^{n} R_j &\le \sum_{j=1}^{100} R_j + \frac{1}{2} \sum_{j>100}^{n} 2^{\omega(j)} - \sum_{j>100}^{b} \sum_{p|j} \frac{1}{p} \\
&= \sum_{j=1}^{100} \left(R_j - 2^{\omega(j)-1} - \sum_{p|j} \frac{1}{p} \right) + \frac{1}{2} \sum_{j=1}^{n} 2^{\omega(j)} - \sum_{j=2}^{n} \sum_{p|j} \frac{1}{p} \\
&\le -130.4778 + \frac{1}{2} \sum_{j=1}^{n} 2^{\omega(j)} - \sum_{p \le n} \left\lfloor \frac{n}{p} \right\rfloor \frac{1}{p}. \qquad (2)
\end{aligned}$$

Assuming $n > 100$, we have

$$\begin{aligned}
\sum_{p \le n} \left\lfloor \frac{n}{p} \right\rfloor \frac{1}{p} &\ge \sum_{p \le n} \left(\frac{n+1}{p^2} - \frac{1}{p} \right) \ge (n+1) \sum_{p \le b} \left(\frac{1}{p^2} - \frac{1}{p(n+1)} \right) \\
&\ge (n+1) \sum_{p \le 101} \left(\frac{1}{p^2} - \frac{1}{101p} \right) \ge .432(n+1). \qquad (3)
\end{aligned}$$

As in the proof of [3, Lemma 9] for $n \geq 248$, and using exact computations for $n \in [101, 247]$, we obtain

$$\sum_{j=2}^{n} 2^{\omega(j)} - 120 \leq .375 n \log n \qquad \text{for all} \qquad n > 100. \tag{4}$$

Combining the estimates (2), (3) and (4) above, we get the assertion of the lemma. □

Lemma 5. *For a positive integer n, we have*

$$\sum_{j=1}^{n} \frac{\varphi(j)}{j} \leq \frac{6n}{\pi^2} + \log n + 1. \tag{5}$$

Proof. We have

$$\sum_{j=1}^{n} \frac{\varphi(j)}{j} = \sum_{j=1}^{b} \frac{\mu(j)}{j} \left\lfloor \frac{n}{j} \right\rfloor = \sum_{j=1}^{n} \frac{\mu(j)}{j} \left(\frac{n}{j} - \left\{ \frac{n}{j} \right\} \right) = n \sum_{j=1}^{b} \frac{\mu(j)}{j^2} - \sum_{j=1}^{n} \frac{\mu(j)}{j} \left\{ \frac{n}{j} \right\}.$$

Hence, inequality (5) follows from

$$\sum_{j=1}^{n} \frac{\mu(j)}{j^2} = \sum_{j=1}^{\infty} \frac{\mu(j)}{j^2} - \sum_{j>n} \frac{\mu(j)}{j^2} < \frac{6}{\pi^2} + \sum_{j>n} \frac{1}{j^2} \leq \frac{6}{\pi^2} + \int_{n}^{\infty} \frac{du}{u^2} = \frac{6}{\pi^2} + \frac{1}{n},$$

and

$$-\sum_{j=1}^{n} \frac{\mu(j)}{j} \left\{ \frac{n}{j} \right\} \leq \sum_{j=2}^{n} \frac{1}{j} < \int_{1}^{n} \frac{du}{u} = \log n.$$

□

We now define two functions f and g on $\mathbb{N}$ with values in the positive real numbers given by

$$f(n) = \begin{cases} \sum_{j=1}^{n} \varphi(j)/j, & \text{if } n \leq 100; \\ 6n/\pi^2 + \log n + 1, & \text{if } n > 100; \end{cases}$$

and

$$g(n) = \begin{cases} \sum_{j=1}^{n} R_j, & \text{if } n \leq 100; \\ .375 n \log n - .432 n - 10, & \text{if } n > 100. \end{cases}$$

We observe from Lemmas 4 and 5 that inequalities $f(n) \leq 6n/\pi^2 + \log n + 1$ for $n \geq 1$ and $g(n) \leq .375 n \log n$ hold for all $n \geq 7$.

3 Proof of Theorem 1

3.1 Proof of the upper bounds (i) and (ii) in Theorem 1

Let a and b be positive integers with $a \geq b$. If $p \mid M$ and $p \mid N$ for each $p \leq b$, then

$$\gcd(M-i, N-j) > 1 \quad \text{for} \quad 1 \leq i \leq a, 1 \leq j \leq b \quad \text{and} \quad \gcd(i,j) \neq 1.$$

If $p \mid M$ and $N \equiv 1 \pmod{p}$ for every $b < p \leq a$, then

$$\gcd(M-i, N-1) > 1 \quad \text{for} \quad b < i \leq a.$$

Let

$$T := T(a,b) := \{(i,j) : 1 \leq i \leq a, 1 \leq j \leq b, \gcd(i,j) = 1\} \setminus \{(i,1) : b < i \leq a\},$$

and let $t = \#T$. We label the elements of $T(a,b)$ as

$$T(a,b) = \{(i_l, j_l) : 1 \leq l \leq t\}$$

in lexicographic order. Hence $(i_1, j_1) = (1,1), (i_2, j_2) = (1,2), \ldots$.

We consider the system of congruences

$$\begin{aligned} M, N &\equiv 0 \pmod{p} \quad \text{for} \quad p \leq b; \\ M &\equiv 0 \pmod{p} \quad \text{and} \quad N \equiv 1 \pmod{p} \quad \text{for} \quad b < p \leq a; \end{aligned}$$

and

$$M \equiv i_\ell \pmod{p_{\pi(b)+\ell}} \quad \text{and} \quad N \equiv j_l \quad \text{mod } p_{\pi(b)+\ell}) \quad \text{for} \quad 1 \leq \ell \leq t.$$

By the Chinese remainder theorem, we get

$$\max(M,N) \leq \prod_{\ell \leq \pi(a)+t} p_\ell. \tag{6}$$

We now estimate $\pi(a)+t$. For every $1 \leq j \leq b$, write $a = jq_j + r_j$ where $0 \leq r_j < j$. By dividing a into intervals of length j, we obtain

$$\begin{aligned} t + a - b &= \sum_{j=1}^{b} (q_j \varphi(j) + r'_j) = a \sum_{j=1}^{b} \frac{\varphi(j)}{j} + \sum_{j=1}^{b} \left(r'_j - \frac{r_j \varphi(j)}{j} \right) \\ &\leq a \sum_{j=1}^{b} \frac{\varphi(j)}{j} + \sum_{j=1}^{b} R_j, \end{aligned}$$

which gives

$$t + \pi(a) \leq ab \left(\frac{\sum_{j=1}^{b} \varphi(j)/j - 1}{b} + \frac{b + \pi(a) + \sum_{j=1}^{b} R_j}{ab} \right).$$

Assume that $b > 100$. By Lemmas 4, 5, 3 (i) and the fact that $a \geq b$, we obtain

$$\begin{aligned}
& \frac{\sum_{j=1}^{b} \varphi(j)/j - 1}{b} + \frac{b + \pi(a) + \sum_{j=1}^{b} R_j}{ab} \\
\leq\ & \frac{6}{\pi^2} + \frac{\log b}{b} + \frac{b + .375b\log b - .432b - 10 + \pi(a)}{ab} \\
\leq\ & \frac{6}{\pi^2} + \frac{\log b}{b} + \frac{.568 + \frac{3}{8}\log b}{a} + \frac{a(1 + 1.2762/\log a) - 10}{ab\log a} \\
\leq\ & \frac{6}{\pi^2} + \frac{11\log b}{8b} + \frac{1}{b\log b}\left(1 + \frac{1.2762}{\log b}\right) - \frac{10}{b^2}. \qquad (7)
\end{aligned}$$

In particular,

$$t + \pi(a) \leq \left(\frac{6}{\pi^2} + o(1)\right) ab \qquad \text{when} \qquad b \to \infty. \qquad (8)$$

Additionally, since the last expression (7) is a decreasing function of b, we obtain

$$t + \pi(a) \leq .67252ab \qquad \text{for} \qquad b > 100.$$

Define $h_0(b) = .67252$ if $b > 100$ and for $b \leq 100$ let this function be defined in the following way:

$$\begin{aligned}
h_0(b) \ &:= \ \frac{\sum_{j=1}^{b} \varphi(j)/j - 1}{b} \\
& + \max_{b \leq a \leq 100} \left\{ \frac{b + \sum_{j=1}^{b} R_j + \pi(a)}{ab}, \frac{b + \sum_{j=1}^{b} R_j}{101b} + \frac{1}{b\log 101}\left(1 + \frac{1.2762}{\log 101}\right) \right\}.
\end{aligned}$$

We then obtain from $a \geq b$ and Lemma 3 (i) that $t + \pi(a) \leq h_0(b)ab$.

If $\pi(a) + t \leq 7$, then $\max(M, N) \leq 510510$. In fact, $b \leq a \leq 4$ in that case. Hence, we now assume that $\pi(a) + t \geq 8$. By Lemma 3 (i) and (iii) and from the fact that $a \geq b$, we have

$$\begin{aligned}
\prod_{\ell \leq \pi(a)+t} p_\ell \ &\leq \ \exp\left(abh_0(b)(\log h_0(b)ab + \log\log h_0(b)ab - .75\right) \\
&\leq \ \exp\left(abh_0(b)\log ab\left(1 + \frac{\log h_0(b) + \log\log h_0(b)ab - .75}{\log ab}\right)\right) \\
&\leq \ \exp\left(abh_0(b)\log ab\left(1 + \frac{\log h_0(b) + \log\log h_0(b)b^2 - .75}{\log b^2}\right)\right) \\
&:= \ \exp(h_1(b)ab\log b).
\end{aligned}$$

Here,

$$h_1(b) = h_0(b)\left(1 + \frac{\log h_0(b) + \log\log h_0(b)b^2 - .75}{\log b^2}\right).$$

Making $b \to \infty$, we get (i) of Theorem 1 from (8). For $b > 100$, since $h_0(b) = .67252$, we get

$$h_1(b) \le h_0(b)\left(1 + \frac{\log h_0(b) + \log\log h_0(b) \cdot 101^2 - .75}{\log 101^2}\right) \le .721521 := c_1,$$

which proves (ii) of Theorem 1. Our arguments give upper bounds for $M(a, b)$ and $N(a, b)$ in smaller ranges of b as well. That is, for $b \le 100$, we get $h_1(b) \le c_1(b)$, where the values of c_1 are given by:

b	c_1	b	c_1	b	c_1	b	c_1	b	c_1
2	9432	3	1.1429	4	.9344	5	.99964	6	.8587
7	.9074	8	.8448	9	.8279	10	.7813	11	.8186
12	.7718	13	.8034	14	.7752	15	.7608	16	.7435
17	.7689	18	.7419	19	.7646	20	.7454	≥ 21	.7463

3.2 Proof of the lower bound (iii) of Theorem 1

Let M, N satisfy the conditions of Theorem 1. For each pair (i, j) with $1 \le i \le a$ and $1 \le j \le b$, let $p_{i,j}$ be the least prime dividing $\gcd(M - i, N - j)$. We consider the set

$$\mathcal{P} = \{p_{i,j} : \ 1 \le i \le a, 1 \le j \le b\}.$$

Suppose that $p \in \mathcal{P}$. If $p \mid \gcd(M - i, N - j)$ and $p \mid \gcd(M - i', N - j')$ for some $1 \le i, i' \le a$ and $1 \le j, j' \le b$ with $(i, j) \ne (i', j')$. Then $p \mid (i - i')$ and $p \mid (j - j')$. In particular, $p \le a$. Thus, given $p \in \mathcal{P}$, let (i_0, j_0) be the least pair with $1 \le i_0 \le a$ and $1 \le j_0 \le b$ such that $p \mid \gcd(M - i, N - j)$. Then every other pair (i, j) with $1 \le i \le a$ and $1 \le j \le b$ such that $p \mid \gcd(M - i, N - j)$ has the property that $i = i_0 + up$ and $j = j_0 + vp$ for some non-negative integers u, v with $0 \le u \le \lfloor (a - 1)/p \rfloor$ and $0 \le v \le \lfloor (b - 1)/p \rfloor$. Thus, for a fixed p, the number of pairs (i, j) for which $p = p_{i,j}$ is at most

$$\left(1 + \left\lfloor \frac{a-1}{p} \right\rfloor\right)\left(1 + \left\lfloor \frac{b-1}{p} \right\rfloor\right) = 1 + \left\lfloor \frac{a-1}{p} \right\rfloor + \left\lfloor \frac{b-1}{p} \right\rfloor + \left\lfloor \frac{a-1}{p} \right\rfloor \left\lfloor \frac{b-1}{p} \right\rfloor. \quad (9)$$

Putting also

$$T = T(a, b) = \{(i, j) : 1 \le i \le a, \ 1 \le j \le b\},$$

and summing up the above inequality (9) over all the possible primes $p \in \mathcal{P}$, we get that

$$\#T = ab \le \sum_{p \in \mathcal{P}} \left(1 + \frac{a + b}{p} + \frac{ab}{p^2}\right) \le \#\mathcal{P} + (a + b)\sum_{p \le a} \frac{1}{p} + ab \sum_{p \le a} \frac{1}{p^2}. \quad (10)$$

By the prime number theorem, in the right, the second sum is

$$(a + b)(\log\log a + O(1)) = o(ab)$$

because of the assumption that $\log\log t = o(b)$ as $b \to \infty$. Put

$$c_2 = \sum_{p \geq 2} \frac{1}{p^2} = 1 - c_1$$

and $P = \#\mathcal{P}$. We then get that

$$ab \leq P + (c_2 + o(1))ab \quad \text{or} \quad P \geq (c_1 + o(1))ab \qquad (b \to \infty).$$

Now it is clear that

$$\begin{aligned} M^a &> \prod_{1 \leq i \leq a} (M - i) \geq \prod_{p \in \mathcal{P}} p \\ &\geq \prod_{k \leq P} p_k = \exp((1 + o(1))P \log P) = \exp\left((c_1 + o(1))ab \log ab\right), \end{aligned}$$

implying the desired inequality (iii) on M. A similar argument proves the inequality for N. Hence, part (iii) of Theorem 1 is proved. □

4 Proof of Theorem 2

We now prove Theorem 2 by computing $M(a, a)$ for $a > 1$. We follow the same arguments as in Section 3.2 with $a = b$ and arrive at

$$\#T = a^2 \leq \#\mathcal{P} + 2 \sum_{p \leq a} \left\lfloor \frac{a-1}{p} \right\rfloor + \sum_{p \leq a} \left\lfloor \frac{a-1}{p} \right\rfloor^2,$$

giving

$$\#\mathcal{P} \geq a^2 - 2 \sum_{p \leq a} \left\lfloor \frac{a-1}{p} \right\rfloor - \sum_{p \leq a} \left\lfloor \frac{a-1}{p} \right\rfloor^2 \geq a^2 - 2a \sum_{p \leq a} \frac{1}{p} - a^2 \sum_{p \leq a} \frac{1}{p^2}, \tag{11}$$

and

$$M^a > \prod_{p \in \mathcal{P}} p \geq \prod_{i=1}^{\#\mathcal{P}} p_i = \exp(\theta(p_{\#\mathcal{P}})). \tag{12}$$

Let $a \leq 100$. We explicitly compute the integral part of the middle term of (11), which we call it P_a, and compute $(\prod_{i=1}^{P_a} p_i)^{\frac{1}{a}}$ to get a lower bound of M giving the assertion for $a \leq 100$. In fact we get $M \geq \exp(a \log a)$ for $a \geq 2$. Now we take $a \geq 101$. Then from Lemma 3 (v) and

$$\sum_{p \geq a} \frac{1}{p^2} \leq \zeta(2) - \sum_{i=1}^{100} \frac{1}{i^2} + \sum_{p \leq 100} \frac{1}{p^2} \leq .4604,$$

we get

$$\begin{aligned}\#\mathcal{P} \geq& a^2 - .4604a^2 - 2a\left(\log\log a + .2615 + \frac{1}{\log^2 a}\right)\\ \geq& a^2\left\{.5396 - \frac{2\log\log a + .523 + \frac{2}{\log^2 a}}{a}\right\} \geq .5032a^2\end{aligned}$$

since $a \geq 101$. This together with (12) and Lemma 3 (ii) and (iv) gives

$$\begin{aligned}M^a >& \exp\left(.5032a^2\log(.5032a^2)(1 - \frac{1}{\log(.5032a^2)})\right)\\ >& \exp\left(.5032a^2(\log a)(2 + \frac{\log .5032}{\log a})(1 - \frac{1}{\log(.5032a^2)})\right)\\ >& \exp(.82248a^2\log a)\end{aligned}$$

since $a \geq 101$. The proof is now complete. □

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