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Complementary Bell Numbers and p-adic Series

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Abstract

In this article, we generalize a result of Murty on the non-vanishing of complementary Bell numbers and irrationality of a p-adic series. This generalization leads to a sequence of polynomials. We partially answer the question of existence of an integral zero of those polynomials.

1 Introduction

Murty and Sumner [5] have shown that there is a sequence of integers a_k, b_k such that the following equality

$$\sum_{n=0}^{\infty} n^k n! = a_k \sum_{n=0}^{\infty} n! + b_k \tag{1}$$

holds in $\mathbb{Q}_p$. Alexander [6] has shown that a_k vanishes at most twice. In Proposition 1, we generalize Eq. (1) to show that for non-negative integers k, j there exist two sequences of integers $a_k(j), b_k(j)$ such that the following equality

$$\sum_{n=0}^{\infty} n^k (n+j)! = a_k(j)\alpha + b_k(j) \tag{2}$$

holds in $\mathbb{Q}_p$, α being the p-adic sum $\sum_{n=0}^{\infty} n!$.

It is obvious that we would like to identify non-negative integers k and j such that

$$a_k(j) = 0.$$

This would mean that the infinite sum on left-hand side of Eq. (2) is just an integer $b_k(j)$. In fact by Proposition 1 it would follow that

$$a_1(0) = 0, a_2(1) = 0$$

and

$$b_1(0) = -1, b_2(1) = 2.$$

Again by Proposition 1, it follows that

$$\sum_{n=0}^{\infty} n \cdot n! = -1$$

and

$$\sum_{n=0}^{\infty} n^2 (n+1)! = 2.$$

On the other hand, Dragovich [2] has shown that if the series

$$\sum_{n=0}^{\infty} n!$$

converges to a rational number in $\mathbb{Q}_p$ for every prime p, then the series cannot converge to the same rational number. Furthermore, the fact $a_2(j) > 0$ for every integer $j \geq 2$ leads us to conclude that for a fixed integer $j \geq 2$ if the series

$$\sum_{n=0}^{\infty} n^2 (n+j)!$$

converges to a rational number in $\mathbb{Q}_p$, then it cannot converge to a fixed rational number in $\mathbb{Q}_p$ for every prime p. In order to show that $a_k(j)$ is non-zero for selected values of k, j we need certain identities for $a_k(j)$. We derive a few of those identities in the next section.

2 Recurrence for the polynomial

We begin this section by considering the series $\sum_{n=0}^{\infty} (n+j)!$ for a fixed non-negative integer j. Observe that

$$\sum_{n=0}^{\infty} (n+j)! = \sum_{n=0}^{\infty} n! - (0! + 1! + 2! + \cdots + (j-1)!).$$

Kurepa's left factorial, $K(m)$ for a non-negative integer m is given by

$$K(m) = \begin{cases} 0, & \text{if } m = 0; \\ 0! + 1! + 2! + \cdots + (m-1)!, & \text{if } m \text{ is a positive integer.} \end{cases}$$

It then follows that

$$\sum_{n=0}^{\infty} (n+j)! = \alpha - K(j).$$

For reasons which will be clear in a moment, we define

$$\begin{aligned} a_0(x) &= 1 \\ b_0(j) &= -K(j). \end{aligned}$$

Therefore, it follows that

$$\sum_{n=0}^{\infty} (n+j)! = a_0(j)\alpha + b_0(j). \tag{3}$$

We hereby present the main result of this section.

Proposition 1. *Let $k \geq 0$ and $j \geq 0$ be fixed integers. Then there is a polynomial $a_k(x)$ and an integer $b_k(j)$ such that*

$$\sum_{n=0}^{\infty} n^k (n+j)! = a_k(j)\alpha + b_k(j)$$

where $a_k(x)$ and $b_k(j)$ are defined inductively on k as follows:

$$a_k(x) = a_{k-1}(x+1) - (x+1)a_{k-1}(x), \; k \geq 1$$

$$b_k(j) = b_{k-1}(j+1) - (j+1)b_{k-1}(j), \; k \geq 1$$

and

$$\begin{aligned} a_0(x) &= 1 \\ b_0(j) &= -K(j). \end{aligned}$$

Proof. The proof is by induction on k. The case $k = 0$ has already been worked out. So we may assume $k \geq 0$ and that the proposition holds for k.

Observe that

$$\sum_{n=0}^{\infty} n^k (n+j+1)! = \sum_{n=0}^{\infty} n^{k+1} (n+j)! + (j+1) \sum_{n=0}^{\infty} n^k (n+j)!.$$

Thus using $\sum_{n=0}^{\infty} n^k (n+j)! = a_k(j)\alpha + b_k(j)$ and then comparing the coefficient of α, term without α, we have

$$a_k(j+1) - (j+1)a_k(j) = a_{k+1}(j) \tag{4}$$

and

$$b_k(j+1) - (j+1)b_k(j) = b_{k+1}(j).$$

□

Corollary 2. *The series $\sum_{n=0}^{\infty} n^k (n+j)!$ converges to an integer whenever $a_k(j)$ vanishes.*

Corollary 3.

$$a_k(0) = a_{k+1}(-1).$$

The next proposition may remind us about a similar kind of property exhibited by Bernoulli polynomials.

Proposition 4. *The derivative of $a_k(x)$ is given by*

$$\frac{d}{dx} a_k(x) = -k a_{k-1}(x), \quad k \geq 1.$$

Proof. The proposition is easily seen to be true for $k = 1$ and we prove the proposition by induction on k. We differentiate the expression

$$a_K(x+1) - (x+1)a_K(x) = a_{K+1}(x)$$

given in Proposition 1 to obtain

$$a'_K(x+1) - a_K(x) - a'_K(x)(x+1) = a'_{K+1}(x).$$

We assume the proposition holds for $k = K$ to obtain

$$-K a_{K-1}(x+1) - a_K(x) + K a_{K-1}(x)(x+1) = a'_{K+1}(x).$$

Again we consider Proposition 1 to obtain the desired result.

□

Proposition 5. *If $c_{i,k}$ denotes the coefficients of x^i in $a_k(x)$ then*

$$-k c_{i,k-1} = (i+1) c_{i+1,k}$$

for non-negative integers i, k and $i \leq k-1$.

Proof. The proof follows by comparing constant term and the coefficient of powers of x in Proposition 4. □

We note that if k is a prime then for $i \leq k-2$

$$\gcd(i+1, k) = 1.$$

Hence $i+1$ must divide $c_{i,k-1}$ and k must divide $c_{i+1,k}$ but $a_k(0) = a_k$ and so we can write

$$a_p(x) \equiv a_p - x^p \pmod{p}.$$

We proceed for a few more congruences for $a_k(x)$. We start with a proposition that states $a_k(x)$ can be determined using a_k and the binomial coefficients.

Proposition 6. *The polynomial $a_k(x)$ is given by*

$$a_k(x) = \sum_{i=0}^{k} {}^kC_i a_i (-1)^{k-i} x^{k-i}.$$

Proof. Applying induction on Proposition 4, it follows that

$$\frac{d^i}{dx^i} a_k(x) = (-1)^i k(k-1)(k-2)\cdots(k-i+1)a_{k-i}(x).$$

We write $a_k(x)$ as

$$a_k(x) = \sum_{i=0}^{k} b_i x^i.$$

Then $\frac{d^i}{dx^i} a_k(x)$ at x equal to 0 must be $b_i i!$. Hence b_i must be

$$(-1)^i \; {}^kC_i a_{k-i}(0).$$

Using the fact that $a_{k-i}(0) = a_{k-i}$ the result follows.

□

Now, we include a table containing first few polynomials $a_k(x)$.

k	$a_k(x)$
0	1
1	$-x$
2	$-1+x^2$
3	$1+3x-x^3$
4	$+2-4x-6x^2+x^4$
5	$-9-10x+10x^2+10x^3-x^5$
6	$9+54\ x+30\ x^2-20\ x^3-15\ x^4+x^6$
7	$50-63\ x-189\ x^2-70\ x^3+35\ x^4+21\ x^5-x^7$
8	$-267-400x+252x^2+504\ x^3+140\ x^4-56x^5-28x^6+x^8$
9	$413+2403\ x+1800\ x^2-756\ x^3-1134\ x^4-252\ x^5+84\ x^6$ $+36\ x^7-x^9$
10	$2180-4130\ x-12015\ x^2-6000\ x^3+1890\ x^4+2268\ x^5+$ $420\ x^6-120\ x^7-45\ x^8+x^{10}$
11	$-17731-23980\ x+22715\ x^2+44055\ x^3+16500\ x^4-4158\ x^5$ $-4158x^6-660x^7+165x^8+55x^9-x^{11}$
12	$50533+212772x+143880x^2-90860x^3-132165x^4-39600x^5$ $+8316x^6+7128x^7+990x^8-220\ x^9-66x^{10}+x^{12}$
13	$110176-656929x-1383018x^2-623480x^3+295295x^4+343629\ x^5$ $+85800x^6-15444x^7-11583x^8-1430x^9+286x^{10}+78x^{11}-x^{13}$
14	$-1966797-1542464x+4598503x^2+6454084x^3+2182180x^4$ $-826826x^5-801801x^6-171600x^7+27027x^8+18018x^9+2002x^{10}$ $-364x^{11}-91x^{12}+x^{14}$

It is easy to see from the table that

$$a_1(0)=0, a_2(1)=0$$

and so we would like to identify integers k and j such that $a_k(j)$ is zero/nonzero. We partially identify such k and j in the next section.

3 On non-vanishing of the polynomials

The next proposition helps us in concluding non-vanishing of $a_k(x)$ whenever k is a prime.

Proposition 7. *If p is a prime then $a_p(j)$ does not vanish for every j in $\mathbb{Z}$ with j incongruent to 1 modulo p.*

Proof. The proposition can be easily verified for $p=2$. For $p \geq 3$, Proposition 6 and the congruence

$$\binom{p}{i} \equiv 0 \pmod{p} \text{ for } 1 \leq i \leq p-1$$

leads us to

$$a_p(j) \equiv a_p - j^p \pmod{p}.$$

Murty [5] has shown that

$$a_p \equiv 1 \pmod{p}.$$

Considering Fermat's theorem the proposition follows.

□

Observe that $a_1(x) = -x > 0$ for $x < 0$. More generally, we have the following proposition.

Proposition 8. *$a_k(x) > (k-1)!$ for $k \geq 1$ and $x \leq -k$.*

Proof. We prove this proposition by induction on k.

Assume $a_k(x) > (k-1)!$ for $x \leq -k$ holds for some fixed $k \geq 1$ then the recurrence relation

$$a_{k+1}(x) = a_k(x+1) - (x+1)a_k(x)$$

for $a_k(x)$ gives us

$$a_{k+1}(x) > (k-1)! + (k-1)(k-1)!$$

for $x \leq -k-1$.

Hence the proposition follows.

□

With this proposition it is clear that $a_k(x)$ does not vanish for $k \geq 1$ and $x \leq -k$.

To analyse $a_k(x)$ further, we start with the following result.

Proposition 9. *For a non-negative integer m and an integer j*

$$a_{p+m}(j) \equiv a_{m+1}(j) + a_m(j) \pmod{p}.$$

Proof. For an integer j, by Proposition 6 it follows that

$$a_p(j) \equiv 1 - j \pmod{p}.$$

Applying the recurrence given in Proposition 1 and the fact

$$a_2(j) + a_1(j) = j^2 - j - 1,$$

it follows that

$$a_{p+1}(j) \equiv a_2(j) + a_1(j) \pmod{p}.$$

Again, applying the recurrence in Proposition 1 repeatedly we obtain the desired result.

□

Proposition 10. *For a prime p such that*

$$p \equiv 2, 3 \pmod 5,$$

$a_{p+1}(j)$ does not vanish for any integer j.

Proof. By previous proposition

$$a_{p+1}(j) \equiv j^2 - j - 1 \pmod p.$$

However, for a prime $p \neq 2, 5$ considering

$$4(j^2 - j - 1) = (2j-1)^2 - 5,$$

it is clear that $a_{p+1}(j)$ is not congruent to 0 modulo p whenever the Legendre symbol

$$\binom{5}{p} = -1.$$

Now, $\binom{5}{p} = -1$ if and only if

$$p \equiv 2, 3 \pmod 5.$$

Hence the proposition follows. □

Corollary 11. *$a_8(j)$, $a_{14}(j)$, $a_{18}(j)$ and $a_{24}(j)$ does not vanish for any integer j.*

Proposition 12. *For a prime $p \equiv 2, 5, 6, 7, 8, 11 \pmod{13}$ and an integer j not divisible by p, $a_{p+2}(j)$ does not vanish.*

Proof. By Proposition 9, for $m = 2$, one has

$$a_{p+2}(j) \equiv -j(j^2 - j - 3) \pmod p.$$

Hence the proposition follows. □

Proposition 13. *If $a_p(1)$ is not divisible by p^2, then $a_p(x)$ is an irreducible polynomial over $\mathbb{Q}$.*

Proof. By Proposition 6 we have

$$a_p(x+1) \equiv a_p - (x+1)^p \pmod p.$$

Considering the congruence for a_p given by Murty [5] again it follows that

$$a_p(x+1) \equiv -x^p \pmod p.$$

Hence by Eisenstein's criterion, the result follows.

□

As a consequence of Proposition 13 it is clear that if $a_p(1)$ is not divisible by p^2, then there does not exist an integer j such that $a_p(j) = 0$. The next proposition gives us a conditional statement for deciding whether $a_p(j)$ is different from 1.

Proposition 14. *For an odd prime p, if $a_p - 1$ is not divisible by p^2, then $a_p(x) - 1$ is an irreducible polynomial.*

Proof. Following the steps of Proposition 13 we have

$$a_p(x) - 1 \equiv -x^p \pmod p.$$

Hence by Eisenstein's criterion for the irreducibility of a polynomial, the result follows.

□

Proposition 15. *For non-negative integers m, t and an integer j the following congruence holds*

$$a_{tp+m}(j) \equiv \sum_{i=o}^{t} {}^tC_i a_{m+i}(j) \pmod p. \tag{5}$$

Proof. The case $t = 0$ is obviously true. As our induction hypothesis we assume that the congruence in Eq. (5) is true for some $t \geq 0$ and by Proposition 9 it follows that

$$a_{(t+1)p+m}(j) \equiv a_{tp+m+1}(j) + a_{tp+m}(j) \pmod p. \tag{6}$$

Hence by our induction hypothesis

$$a_{(t+1)p+m}(j) \equiv \sum_{i=o}^{t} {}^tC_i a_{m+1+i}(j) + \sum_{i=o}^{t} {}^tC_i a_{m+i}(j) \pmod p \tag{7}$$

$$\equiv \sum_{i=o}^{t+1} {}^{t+1}C_i a_{m+i}(j) \pmod p. \tag{8}$$

Hence the result follows by induction. □

Proposition 16. *For non-negative integers m, i and an integer j the following congruence holds*

$$a_{p^i+m}(j) \equiv a_{m+1}(j) + i a_m(j) \pmod p.$$

Proof. The case $i = 0$ is a trivial case and the case $i = 1$ follows from Proposition 9. So we assume $i \geq 2$.

For $1 \leq j \leq p^{i-1} - 1$, considering the congruence

$$\binom{p^{i-1}}{j} \equiv 0 \pmod p$$

and $t = p^{i-1}$ in the above proposition it follows that

$$a_{p^i+m}(j) \equiv a_m(j) + a_{p^{i-1}+m}(j) \pmod p.$$

Repeating the previous step r number of times where $r \leq i-1$ we have

$$a_{p^i+m}(j) \equiv r a_m(j) + a_{p^{i-r}+m}(j) \pmod p.$$

Choosing $r = i-1$ we have

$$a_{p^i+m}(j) \equiv (i-1)a_m(j) + a_{p+m}(j) \pmod p.$$

The result follows from the above congruence and Proposition 9. □

Corollary 17. *For a positive integer i,*

$$a_{p^i} \equiv i \pmod p.$$

Proof. Choosing m, j equal to 0 the corollary follows. □

Proposition 18. *$a_{p^{zp}}(j)$ does not vanish for any integer j not divisible by p.*

Proof. We consider $i = zp$ for some non-negative integer z in the previous proposition to obtain

$$a_{p^{zp}+m}(j) \equiv a_{m+1}(j) \pmod p.$$

Choosing $m = 0$, the result follows. □

Proposition 19. *For a non-negative integer t and an integer j*

$$a_{3t}(j) \neq 0.$$

Proof. Considering $i = 2, p = 2$ in Proposition 16, it follows that

$$a_{4+m}(j) \equiv a_{1+m}(j) \pmod 2.$$

Choosing $m = 2, 5, 8, \cdots$ it is easy to see that for a positive integer t

$$a_{3t} \equiv a_3(j) \pmod 2.$$

The fact

$$a_3(j) \not\equiv 0 \pmod 2$$

leads to the desired result.

□

Proposition 20. *For a non-negative integer t*

$$a_{pt} \equiv a_{t-1} \pmod p.$$

Proof. Through Proposition 6 it is easy to see that

$$a_k(-1) = \sum_{i=0}^{k} {}^kC_i a_i$$

However, by the recurrence 1

$$a_k(-1) = a_{k-1}(0).$$

By congruence (5)

$$a_{pt+m}(j) \equiv \sum_{i=0}^{t} {}^tC_i a_{m+i}(j) \pmod{p}.$$

For $m = 0, j = 0$ above congruence reduces to

$$a_{pt}(0) \equiv \sum_{i=0}^{t} {}^tC_i a_i(0) \pmod{p}$$

and so

$$a_{pt}(0) \equiv a_{t-1} \pmod{p}.$$

□

Proposition 21. *For a positive integer i and an integer j if*

$$j \not\equiv i \pmod{p}$$

then

$$a_{p^i}(j) \neq 0.$$

Proof. Choosing $m = 0$ in Proposition 16 we have

$$a_{p^i}(j) \equiv a_1(j) + i a_0(j) \pmod{p}.$$

The result follows.

□

The next result gives us a much stronger congruence of $a_k(j)$.

Theorem 22. *For non-negative integers t, m, a positive integer n, an odd prime p and an integer j such that*

$$j \equiv 0, 1, 2 \pmod{p}$$

the following congruence

$$a_{\frac{p^p-1}{p-1} \cdot p^{n-1} t + m}(j) \equiv a_m(j) \pmod{p^n}$$

holds.

Proof. We consider three cases: $j \equiv 0 \pmod p$, $j \equiv 1 \pmod p$ and $j \equiv 2 \pmod p$
Case 1. For an integer $j \equiv 0 \pmod p$ and a positive integer r, it is easy to see that

$$\binom{\frac{p^p-1}{p-1} \cdot p^{n-1}}{r}(-j)^r = \frac{\frac{p^p-1}{p-1}(-j)^r \cdot p^{n-1}}{r}\binom{\frac{p^p-1}{p-1} \cdot p^{n-1} - 1}{r-1} \equiv 0 \pmod{p^n}.$$

For an odd prime p, following a slightly different notation, Alexander [6] has proved that

$$a_{\frac{p^p-1}{p-1} \cdot p^{n-1} t+m} \equiv a_m \pmod{p^n},\ t, m \text{ being non-negative integers.}$$

Hence by Proposition (6), it follows that for an integer $j \equiv 0 \pmod p$

$$a_{\frac{p^p-1}{p-1}}(j) \equiv 1 \pmod{p^n}. \tag{9}$$

For simplicity we denote $\frac{p^p-1}{p-1} \cdot p^{n-1}$ by k.
Case 2. In this case we are expressing $a_k(j)$ in terms of $a_{k+1}(j-1)$ and $a_k(j-1)$ and then deriving the required congruence.

Replacing j by $j-1$, Eq. (4) can be written as

$$a_k(j) = a_{k+1}(j-1) + j a_k(j-1). \tag{10}$$

For an integer $j \equiv 1 \pmod p$ and $k = \frac{p^p-1}{p-1} \cdot p^{n-1}$, $n \geq 1$

$$a_{k+1}(j-1) \equiv a_{k+1} - (j-1)(k+1)a_k \pmod{p^n}.$$

Again using the result of Alexander [6], it follows that

$$a_{k+1}(j-1) \equiv -(j-1) \equiv a_1(j-1) \pmod{p^n}.$$

Hence, it follows that

$$a_k(j) \equiv 1 \pmod{p^n} \text{ for } j \equiv 1 \pmod p \tag{11}$$

Case 3. In this case, we express $a_k(j)$ in term of $a_{k+2}(j-2)$ and other similar terms. Replacing k by $k+1$ and j by $j-1$, Eq. (10) can be written as

$$a_{k+1}(j-1) = a_{k+2}(j-2) + (j-1)a_{k+1}(j-2) \text{ and} \tag{12}$$

replacing j by $j-1$, Eq. (10) can be written as

$$a_k(j-1) = a_{k+1}(j-2) + (j-1)a_k(j-2) \tag{13}$$

Eliminating $a_k(j-1), a_{k+1}(j-1)$ from Eqs. (10), (12), and (13), it follows that

$$a_k(j) = a_{k+2}(j-2) + (j-1)a_{k+1}(j-2) + j\{a_{k+1}(j-2) + (j-1)a_k(j-2)\}. \tag{14}$$

But for an integer $j \equiv 2 \pmod p$, by Proposition 6 it is easy to see that

$$a_{k+2}(j-2) \equiv a_{k+2} + (k+2)(2-j)a_{k+1} + \frac{(k+2)(k+1)}{2}(j-2)^2 a_k \pmod{p^n}$$

Again, considering the result of Alexander it would follow that

$$a_{k+2}(j-2) \equiv -1 + (j-2)^2 \equiv a_2(j-2) \pmod{p^n}. \tag{15}$$

Also it is easy to obtain that

$$a_{k+1}(j-2) \equiv a_0(j-2) \pmod{p^n}. \tag{16}$$

Applying Eqs. (14),(15),(16) it would follow that for an integer $j \equiv 2 \pmod p$,

$$a_k(j) \equiv 1 \pmod{p^n}. \tag{17}$$

Therefore, using Eq. (4) the result follows. □

Corollary 23. *For non-negative integers t, m and a positive integer n, the following congruence*

$$a_{13\cdot 3^{n-1}t+m}(j) \equiv a_m(j) \pmod{3^n}$$

holds.

Proof. The proof follows by considering $p = 3$ in previous proposition. □

Remark: The polynomials $a_k(x)$ were previously analyzed by Wannemacker [8]. He has verified numerically that $a_k(x)$ is irreducible over $\mathbb{Z}$ for all $6 \leq k \leq 200$. He conjectured that $a_k(x)$ is irreducible over $\mathbb{Z}$ for all $k \geq 6$.

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